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COX'S PARTICLE IN MAGNETIC AND ELECTRIC FIELDS AGAINST THE BACKGROUND OF EUCLIDEAN AND SPHERICAL GEOMETRIES

The generalized relativistic Klein–Fock–Gordon equation for Cox's non-point scalar particle with intrinsic structure is solved in the presence of external uniform magnetic and electric fields in the Minkowski space. Similar problems in the non-relativistic approximation in a closed spherical Riemann 3-space are examined. The complete separation of the variables in the system of special cylindric coordinates in a curved model is performed. In the presence of a magnetic field, the quantum problem in the radial variable is solved exactly, and the wave functions and the corresponding energy levels are found: the quantum motion in the z-direction is described by a one-dimensional Schrödinger-like equation in an effective potential, which turns out to be too difficult for the analytical treatment. In the presence of an electric field against the background of the curved model, the situation is similar: the radial equation is solved exactly in hypergeometric functions, but the equation in the z-variable can be examined only qualitatively.

Keywords: Cox's particle, generalized Schrödinger equation, magnetic field, electric field, Minkowski space, Riemann space.

1. Solutions of Cox's equation in a magnetic field in the Minkowski space-time

In 1982, W. Cox [1] proposed a special wave equation for a scalar particle with a larger set of tensor components, than the usual Proca approach includes: the approach was based on the use of a scalar, 4-vector, antisymmetric or (irreducible) symmetric tensor, thus starting with a 20-component wave function. Such an extension of the field variables allows one to describe a spin zero particle with additional intrinsic structure, which must manifest itself in the presence of external electromagnetic fields.

The first aim of the present paper is to elaborate several simple situations with electromagnetic fields, when the generalized solutions for Cox's scalar particle can be found. Such exact solutions are constructed in the presence of uniform electric and magnetic fields. In particular, the non-trivial additional structure of a particle modifies the frequency of a quantum oscillator arising effectively in the presence of an external magnetic field.

In addition, there arises additional question about the interaction of Cox's particle with a nontrivial geometrical background. We analyze the behavior of such a particle in the Riemann space with constant positive curvature.

The extension of the two problems (in the presence of a magnetic or electric field) to the case of a spherical space is examined. In the presence of a magnetic field, the quantum problem in the radial variable is solved exactly; the quantum motion in the z-direction is described by a 1-dimensional Schrödinger-like equation in an effective potential, which turns out to be too difficult for the analytical treatment. In the presence of an electric field, the situation is similar. The general conclusion can be done: the effects of the large-scale structure of the Universe depends greatly on the form of basic equations for an elementary particle; any modifications of them lead to new physical phenomena due to the non-Euclidean-geometry background.

First, let us consider the system of Cox's equations [1] in the Minkowski space. We will use a Proca-like generalized system obtained after the elimination of two second-rank tensors from the initial system of Cox's equations (note that $\mu = mc$)

$$\left(\mu\delta_\alpha^\beta + \lambda F_\alpha^\beta\right)\Phi_\beta = \left(i\hbar\partial_\alpha - \frac{e}{c}A_\alpha\right)\Phi,$$
$$\left(\frac{1}{\sqrt{-g}}i\hbar\partial_\alpha\sqrt{-g} - \frac{e}{c}A_\alpha\right)\Phi^\alpha = \mu\Phi, \tag{1}$$

where the additional non-zero parameter λ is associated with the intrinsic structure of the particle. In a uniform magnetic field described in the cylindrical coordinates as

$$dS^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dz^2, \quad \sqrt{-g} = r,$$
$$A_\phi = -\frac{Br^2}{2}, \quad F_{r\phi} = \partial_r A_\phi - \partial_\phi A_r = -Br, \tag{2}$$

the first equation in (1) takes the form

$$\mu\Phi_0 = i\frac{\hbar}{c}\partial_t\Phi, \quad \mu\Phi_z = i\hbar\partial_z\Phi,$$
$$\mu\Phi_r + \lambda B\frac{1}{r}\Phi_\phi = i\hbar\partial_r\Phi,$$
$$\mu\Phi_\phi - \lambda Br\Phi_r = \left(i\hbar\partial_\phi - \frac{e}{c}A_\phi\right)\Phi.$$

From two last equations, we obtain (let $\gamma = \lambda B/\mu$)

$$\Phi_r = \frac{\hbar/Mc}{1+\gamma^2}\left[i\partial_r - \gamma\frac{1}{r}\left(i\partial_\phi + \frac{eB}{2\hbar c}r^2\right)\right]\Phi,$$
$$\Phi_\phi = \frac{\hbar/Mc}{1+\gamma^2}\left[\gamma ri\partial_r + \left(i\partial_\phi + \frac{eB}{2\hbar c}r^2\right)\right]\Phi.$$

The second equation in (1) reads

$$i\frac{1}{c}\partial_t\Phi_0 - i\left(\partial_r + \frac{1}{r}\right)\Phi_r -$$
$$-\frac{1}{r^2}\left(i\partial_\phi + \frac{eB}{2\hbar c}r^2\right)\Phi_\phi - i\partial_z\Phi_z = \frac{Mc}{\hbar}\Phi. \tag{3}$$

In Eq. (3), we substitute the expressions for the components of the vector:

$$i\frac{1}{c}\partial_t\frac{i\hbar}{Mc^2}\partial_t\Phi - i\left(\partial_r + \frac{1}{r}\right)\frac{\hbar/Mc}{1+\gamma^2}\times$$
$$\times\left[i\partial_r - \gamma\frac{1}{r}\left(i\partial_\phi + \frac{eB}{2\hbar c}r^2\right)\right]\Phi -$$
$$-\frac{1}{r^2}\left(i\partial_\phi + \frac{eB}{2\hbar c}r^2\right)\frac{\hbar/Mc}{1+\gamma^2}\times$$
$$\times\left[\gamma ri\partial_r + \left(i\partial_\phi + \frac{eB}{2\hbar c}r^2\right)\right]\Phi -$$
$$-i\partial_z\frac{i\hbar}{Mc}\partial_z\Phi = \frac{Mc}{\hbar}\Phi, \tag{4}$$

so producing an extended equation for the scalar component Φ. Let us introduce the representation for the wave function $\Phi = e^{-iEt/\hbar}e^{im\phi}e^{ikz}R(r)$. Then we derive the radial equation (since λ is purely imaginary, we will make a formal change $i\gamma \Longrightarrow \gamma$; we will apply the notation $eB/2\hbar c = b$)

$$\left[(1-\gamma^2)\left(\frac{E^2}{\hbar^2c^2} - \frac{M^2c^2}{\hbar^2} - k^2\right) + \right.$$
$$+\left(\partial_r + \frac{1}{r}\right)\left(\partial_r - \frac{\gamma}{r}\left(m - br^2\right)\right) +$$
$$\left. +\frac{1}{r^2}\left(m - br^2\right)\left(\gamma r\partial_r - \left(m - br^2\right)\right)\right]R = 0. \tag{5}$$

In addition, we will use the notation

$$(1-\gamma^2)\left(\frac{E^2}{\hbar^2c^2} - \frac{M^2c^2}{\hbar^2} - k^2\right) = w^2.$$

Then the radial equation takes the form

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + w'^2 - \frac{\left(m - br^2\right)^2}{r^2}\right]R = 0,$$
$$w^2 + 2\gamma b = w'^2. \tag{6}$$

We use the well-known spectrum resulting from the solution of Eq. (6):

$$w'^2 = 4b\left(n + \frac{m + \mid m \mid + 1}{2}\right),$$

so that

$$w^2 = 4b\left(n + \frac{m + \mid m \mid + 1}{2}\right) - 2\gamma b. \tag{7}$$

Further, we find

$$\frac{E^2}{\hbar^2c^2} - \frac{M^2c^2}{\hbar^2} - k^2 =$$
$$= \frac{1}{(1-\gamma^2)}\left[4b\left(n + \frac{m + \mid m \mid + 1}{2}\right) - 2\gamma b\right]. \tag{8}$$

This represents the spectrum of a relativistic particle modified by its intrinsic (Cox's) structure. In principle, this formula provides us with a possibility to test experimentally Cox's particle intrinsic structure.

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2. Solutions of Cox's Equation in the Presence of an Electric Field in the Minkowski Space-Time

We start again with the system of first-order equations in the form (1) in the cylindrical coordinates (2):

$$A_0 = -Ez, \quad F_0^z = -E, \quad F_z^0 = -E.$$

The first equation in (1) reads

$$\mu\Phi_0 - \lambda E\Phi_z = \left(i\frac{\hbar}{c}\partial_t + \frac{eE}{c}z\right)\Phi,$$

$$\mu\Phi_z - \lambda E\Phi_0 = i\hbar\partial_z\Phi, \quad \mu\Phi_r = i\hbar\partial_r\Phi,$$

$$\mu\Phi_\phi = i\hbar\partial_\phi\Phi.$$

From whence (we use the notation $\gamma = \lambda E/\mu$), it follows:

$$\Phi_0 = \frac{\hbar/Mc}{1-\gamma^2}\left(\left(i\frac{1}{c}\partial_t + \frac{eE}{\hbar c}z\right) + \gamma i\partial_z\right)\Phi,$$
$$\Phi_z = \frac{\hbar/Mc}{1-\gamma^2}\left(i\partial_z + \gamma\left(i\frac{1}{c}\partial_t + \frac{eE}{\hbar c}z\right)\right)\Phi, \tag{9}$$
$$\Phi_r = (\hbar/Mc)i\partial_r\Phi, \quad \Phi_\phi = (\hbar/Mc)i\partial_\phi\Phi.$$

In turn, the second equation in (1) looks as

$$\left(\frac{i}{c}\partial_t + \frac{eE}{c\hbar}z\right)\Phi_0 - i\left(\partial_r + \frac{1}{r}\right)\Phi_r -$$
$$- \frac{i\partial_\phi}{r^2}\Phi_\phi - i\partial_z\Phi_z = \frac{Mc}{\hbar}\Phi. \tag{10}$$

Then, using (9), we obtain the equation for Φ:

$$\left(i\frac{1}{c}\partial_t + \frac{eE}{c\hbar}z\right)\frac{1}{1-\gamma^2}\left(i\frac{1}{c}\partial_t + \frac{eE}{\hbar c}z + \gamma i\partial_z\right)\Phi -$$
$$- i\partial_z\frac{1}{1-\gamma^2}\left(i\partial_z + \gamma\left(i\frac{1}{c}\partial_t + \frac{eE}{\hbar c}z\right)\right)\Phi -$$
$$- i\left(\partial_r + \frac{1}{r}\right)i\partial_r\Phi - \frac{i\partial_\phi}{r^2}i\partial_\phi\Phi = \frac{M^2c^2}{\hbar^2}\Phi. \tag{11}$$

From (11) with regard for the representation for the wave function $\Phi = e^{-iE't/\hbar}e^{im\phi}R(r)Z(z)$, we arrive at (instead of the electric field amplitude, it is convenient to introduce a parameter $\nu = eE/\hbar c$; instead of the energy E', we will use the parameter $\epsilon = E'/\hbar c$)

$$\frac{1}{1-\gamma^2}\left[(\epsilon+\nu z)\left(\epsilon+\nu z+\gamma i\frac{\partial}{\partial z}\right) -\right.$$
$$\left. - i\frac{\partial}{\partial z}\left(i\frac{\partial}{\partial z} + \gamma(\epsilon+\nu z)\right)\right]RZ - \frac{M^2c^2}{\hbar^2}RZ +$$
$$+\left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right]RZ = 0. \tag{12}$$

The variables are separated:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + w_\perp^2 - \frac{m^2}{r^2}\right)R(r) = 0, \tag{13}$$
$$\frac{1}{1-\gamma^2}\left[\left(\frac{\epsilon}{\hbar c} + \nu z\right)\left(\left(\frac{\epsilon}{\hbar c} + \frac{eE}{\hbar c}z\right) + \gamma i\frac{d}{dz}\right) -\right.$$
$$\left. - i\frac{d}{dz}\left(i\frac{d}{dz} + \gamma\left(\frac{\epsilon}{\hbar c} + \nu z\right)\right)\right]Z(z) -$$
$$- \frac{M^2c^2}{\hbar^2}Z(z) = +w_\perp^2 Z(z)\,. \tag{14}$$

Let us make the formal change $i\gamma \Longrightarrow \gamma$ and use the notation

$$\gamma\nu + (1+\gamma^2)(w_\perp^2 + \frac{M^2c^2}{\hbar^2}) = w_\perp'^2.$$

Then Eq. (14) reads

$$\left[\frac{d^2}{dz^2} + (\epsilon+\nu z)^2 - w_\perp'^2\right]Z = 0, \tag{15}$$

which is an equation of the same structure for an ordinary scalar relativistic particle in the uniform electric field.

3. The Schrödinger Equation in a Magnetic Field. Minkowski Space

In cylindrical coordinates, the uniform magnetic field is described by

$$A_\phi = -\frac{Br^2}{2}, \quad F_{r\phi} = -Br,$$

$$B_3 = -Br, \quad B^3 = -Br^{-1}, \quad B_iB^i = B^2.$$

We start with the Schrödinger wave equation for Cox's [1] particle in the form [2]

$$D_t\Psi = \frac{1}{2M}\left(\overset{\circ}{D}_1\overset{*}{D}_1 + \overset{\circ}{D}_2\, r^{-2}\,\overset{*}{D}_2 + \overset{\circ}{D}_3\overset{*}{D}_3\right)\Psi, \tag{16}$$

where

$$D_1 = i\hbar\partial_r, \ \ D_2 = i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2}, \ \ D_3 = i\hbar\partial_z,$$

$$\overset{\circ}{D}_1 = i\hbar\left(\partial_r + \frac{1}{r}\right), \ \ \overset{\circ}{D}_2 = i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2},$$

$$\overset{\circ}{D}_3 = i\hbar\partial_z, \quad \overset{*}{D}_1 = \frac{1}{1+\Gamma^2 B^2}\left(D_1 - \Gamma B_3 D^2\right) =$$
$$= \frac{1}{1+\Gamma^2 B^2}\left(i\hbar\partial_r - \frac{\Gamma B}{r}\left(i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2}\right)\right),$$
$$\overset{*}{D}_2 = \frac{1}{1+\Gamma^2 B^2}\left(D_2 + \Gamma B_3 D^1\right) =$$
$$= \frac{1}{1+\Gamma^2 B^2}\left(\left(i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2}\right) + i\hbar\Gamma B r\partial_r\right),$$
$$\overset{*}{D}_3 = \frac{\left(D_3 + \Gamma^2 B^3\ B_3 D_3\right)}{1+\Gamma^2 B^2} = i\hbar\partial_z. \qquad (17)$$

Below, we will use the notation

$$\frac{eB}{2\hbar c} = b, \qquad \Gamma B = \gamma.$$

We compute

$$\frac{1}{2M}\overset{\circ}{D}_1\overset{*}{D}_1 = -\frac{\hbar^2}{2M(1+\gamma^2)}\times$$
$$\times\left[\partial_r^2 + \frac{1}{r}\partial_r - \frac{\gamma}{r}\partial_r\partial_\phi + i\gamma b r\partial_r + 2i\gamma b\right],$$
$$\frac{1}{2M}\overset{\circ}{D}_2\overset{*}{D}_2 = -\frac{\hbar^2}{2M\left(1+\gamma^2\right)}\times \qquad (18)$$
$$\times\left[\frac{1}{r^2}\left(\partial_\phi - ibr^2\right)^2 + \gamma\left(\partial_\phi - ibr^2\right)\frac{1}{r}\partial_r\right],$$
$$\frac{1}{2M}\overset{\circ}{D}_3\overset{*}{D}_3 = -\frac{\hbar^2}{2M\left(1+\gamma^2\right)}\left(1+\gamma^2\right)\partial_z^2.$$

By using the representation for the wave function

$$\Psi = e^{-iEt/\hbar}e^{im\phi}e^{ikz}R(r), \quad \epsilon = \frac{2mE}{\hbar^2}\left(1+\gamma^2\right), \quad (19)$$

we obtain the radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 2i\gamma b + \epsilon - \right.$$
$$\left. - \frac{\left(m - br^2\right)^2}{r^2} - \left(1+\gamma^2\right)k^2\right]R = 0. \qquad (20)$$

For physical reasons [3], the parameter γ must be purely imaginary: $\gamma = -i\eta$. So, we obtain the radial equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 2\eta b + \epsilon - \right.$$
$$\left. - \frac{\left(m - br^2\right)^2}{r^2} - \left(1-\eta^2\right)k^2\right]R = 0. \qquad (21)$$

With the notation $\epsilon - (1-\eta^2)k^2 + 2\eta b = \epsilon'$, equation (21) can be written as

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m - br^2)^2}{r^2} + \epsilon'\right] = 0. \qquad (22)$$

This equation coincides with the differential equation arising in the usual problem of a Schrödinger particle in a magnetic field. Its solutions are known. We present here only the expression for the energy spectrum (turning from ϵ' to ϵ)

$$\epsilon = 4b\left(n + \frac{m + \mid m \mid +1}{2}\right) + \left(1-\eta^2\right)k^2 - 2\eta b. \quad (23)$$

By translating (23) to the ordinary units, we find

$$E = \frac{p^2}{2M} + \frac{1}{1-\eta^2}\frac{eB}{Mc}\hbar\times$$
$$\times\left(n + \frac{m + \mid m \mid +1}{2}\right) - \frac{\eta}{2}\frac{1}{1-\eta^2}\frac{eB}{Mc}\hbar. \qquad (24)$$

With the notation
$$\eta = \Gamma B, \quad \Gamma^* = \Gamma, \quad \omega = \frac{eB}{Mc},$$

the formula for the energy levels can be written as

$$E = \frac{p^2}{2M} + \frac{\omega\hbar}{1-(\Gamma B)^2}\times$$
$$\times\left(n + \frac{m + \mid m \mid +1}{2}\right) - \frac{\omega\hbar}{1-(\Gamma B)^2}\frac{\Gamma B}{2}. \qquad (25)$$

Thus, the intrinsic structure of Cox's particle modifies the frequency of a quantum oscillator

$$\omega \Longrightarrow \tilde{\omega} = \frac{\omega}{1-\Gamma^2 B^2}, \quad \omega = \frac{eB}{Mc}. \qquad (26)$$

4. Cox's Particle in a Magnetic Field in the Spherical Riemann Space

In the cylindric coordinate system of a spherical Riemann space (for little r and z, the metric coincides with the known one in the Minkowski space)

$$dS^2 = dt^2 - \cos^2 z\left(dr^2 + \sin^2 r\, d\phi^2\right) - dz^2,$$
$$\sqrt{-g} = \sin r\cos^2 z, \ r \in [0,\pi], \ z \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right];$$

we use dimensionless coordinates, $\frac{r}{\rho} \to r, \frac{z}{\rho} \to z$; ρ stands for the curvature radius of the spherical space).

An analog of the uniform magnetic field is given by the relations [4–6]:

$$A_\phi = B\rho^2(\cos r - 1), \quad F_{r\phi} = B\rho \sin r,$$
$$B_3 = B\rho \sin r, \quad B^3 = \frac{B}{\rho \sin r \cos^4 z},$$
$$B_i B^i = \frac{B^2}{\cos^4 z}.$$

The question about the source for such a magnetic field was not considered. However, even without any clarity concerning the source, we can study the behavior of classical and quantum-mechanical particles in this field (see [4–6]). It should be noted that, in contrast to the flat space, the invariant of the magnetic field $B^i B_i$ in the curved model depends on the coordinate z and exhibits a singular behavior at the points $z = \pm\pi/2$. These points are special in the geometrical meaning as well, because g_{rr} and $g_{\phi\phi}$ tend to zero as $z \to \pm\pi/2$. These values of z will give singular points in the resulting differential equations. These explanations are also relevant in the case of an electric field in the spherical model, which is studied in Section 8.

We start with an extended Schrödinger equation in the form [2]

$$D_t \Psi = \frac{1}{2M\rho^2}\left[\overset{\circ}{D}_1 \frac{1}{\cos^2 z} \overset{*}{D}_1 + \right.$$
$$\left. + \overset{\circ}{D}_2 \frac{1}{\sin^2 r \cos^2 z} \overset{*}{D}_2 + \overset{\circ}{D}_3 \overset{*}{D}_3\right] \Psi, \tag{27}$$

where

$$D_1 = i\hbar\partial_r, \quad D_2 = i\hbar\partial_\phi - \frac{e}{c} B\rho^2(\cos r - 1),$$
$$D_3 = i\hbar\partial_z, \quad \overset{\circ}{D}_1 = i\hbar\left(\partial_r + \frac{\cos r}{\sin r}\right),$$
$$\overset{\circ}{D}_2 = i\hbar\partial_\phi - \frac{e}{c} B\rho^2(\cos r - 1), \quad \overset{\circ}{D}_3 = i\hbar\left(\partial_z - 2\frac{\sin z}{\cos z}\right),$$
$$\overset{*}{D}_1 = \frac{(D_1 - \Gamma B_3 D^2)}{1 + \Gamma^2 B^2 \cos^{-4} z} = \frac{1}{1 + \Gamma^2 B^2 \cos^{-4} z} \times$$
$$\times \left[i\hbar\partial_r + \Gamma B \sin r \cos^2 z \left(i\hbar\partial_\phi - \frac{e}{c} B\rho^2(\cos r - 1)\right)\right],$$
$$\overset{*}{D}_2 = \frac{(D_2 + \Gamma B_3 D^1)}{1 + \Gamma^2 B^2 \cos^{-4} z} = \frac{1}{1 + \Gamma^2 B^2 \cos^{-4} z} \times$$
$$\times \left[i\hbar\partial_\phi - \frac{e}{c} B\rho^2(\cos r - 1) - i\hbar \frac{\Gamma B \sin r}{\cos^2 z} \partial_r\right],$$
$$\overset{*}{D}_3 = \frac{1}{1 + \Gamma^2 B^2 \cos^{-4} z}(D_3 + \Gamma^2 B^3 B_3 D_3) = i\hbar\partial_z.$$

With the notation

$$\frac{eB\rho^2}{\hbar c} = b, \quad \frac{\Gamma B}{\cos^2 z} = \gamma(z),$$

we obtain

$$\frac{1}{2M\rho^2} \overset{\circ}{D}_1 g^{11} \overset{*}{D}_1 = \frac{\hbar^2}{2M\rho^2 \cos^2 z\,(1 + \gamma^2(z))} \times$$
$$\times \left[\partial_r^2 + \frac{\cos r}{\sin r}\partial_r + i\gamma(z) b \frac{\cos r - 1}{\sin r}\partial_r + \right.$$
$$\left. + \frac{\gamma(z)}{\sin r}\partial_r\partial_\phi - i\gamma(z) b\right],$$
$$\frac{1}{2M\rho^2} \overset{\circ}{D}_2 g^{22} \overset{*}{D}_2 = \frac{\hbar^2}{2M\rho^2 \sin^2 r \cos^2 z\,(1 + \gamma^2(z))} \times$$
$$\times [\partial_\phi + ib(\cos r - 1)] \times$$
$$\times [\partial_\phi + ib(\cos r - 1) - \gamma(z) \sin r \partial_r],$$
$$\frac{1}{2M\rho^2} \overset{\circ}{D}_3 g^{33} \overset{*}{D}_3 = \frac{\hbar^2}{2M\rho^2}\left(\partial_z - 2\frac{\sin z}{\cos z}\right)\partial_z. \tag{28}$$

Using the representation

$$\Psi = e^{-iEt/\hbar} e^{im\phi} Z(z) R(r), \quad \epsilon = E/(\hbar^2/2M\rho^2),$$

we reduce the Schrödinger equation to the form (for physical reasons [1, 3], the function $\gamma(z)$ must be imaginary: $i\gamma(z) \Longrightarrow \gamma(z)$)

$$\left[\frac{1}{\cos^2 z\,(1 - \gamma^2(z))} \times \right.$$
$$\times \left(\partial_r^2 + \frac{\cos r}{\sin r}\partial_r - \frac{[m + b(\cos r - 1)]^2}{\sin^2 r} - b\gamma(z)\right) +$$
$$\left. + \epsilon + \left(\partial_z - 2\frac{\sin z}{\cos z}\right)\partial_z\right] R(r) Z(z) = 0.$$

Separating the variables

$$\frac{1}{R(r)}\left(\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} - \frac{[m + b(\cos r - 1)]^2}{\sin^2 r}\right) R(r) +$$
$$+ \frac{1}{Z(z)} \cos^2 z\,(1 - \gamma^2(z)) \left(-\frac{b\gamma(z)}{\cos^2 z\,(1 - \gamma^2(z))} + \epsilon + \right.$$
$$\left. + \left(\partial_z - 2\frac{\sin z}{\cos z}\right)\partial_z\right) Z(z) = 0, \tag{29}$$

we have

$$\left[\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} - \right.$$
$$\left. - \frac{[m + b(\cos r - 1)]^2}{\sin^2 r} + \Lambda\right] R(r) = 0, \tag{30}$$
$$\left(\frac{d^2}{dz^2} - 2\frac{\sin z}{\cos z}\frac{d}{dz} + \epsilon - \frac{b\gamma(z)}{\cos^2 z\,(1 - \gamma^2(z))} - \right.$$
$$\left. - \frac{\Lambda}{\cos^2 z\,(1 - \gamma^2(z))}\right) Z(z) = 0; \tag{31}$$

the last equation can be reduced to the form (note that $\gamma = B\Gamma$)

$$\left(\frac{d^2}{dz^2} - 2\frac{\sin z}{\cos z}\frac{d}{dz} + \epsilon - \frac{b\gamma + \Lambda \cos^2 z}{\cos^4 z - \gamma^2}\right) Z(z) = 0. \tag{32}$$

5. Analysis of the Equation in Variable z

Excluding the term with the first derivative in (32), we obtain the equation with the effective potential

$$Z(z)=\frac{1}{\cos z}f(z),\quad \left(\frac{d^2}{dz^2}+\epsilon+1-U(z)\right)f(z)=0,$$
$$U(z)=\frac{b\gamma+\Lambda\cos^2 z}{\cos^4 z-\gamma^2},\quad U(z=0)=\frac{b\gamma+\Lambda}{1-\gamma^2},\qquad (33)$$
$$U(z=\pm\frac{\pi}{2})=-\frac{b}{\gamma}.$$

The expression for the effective force looks

$$F_z=-\frac{dU}{dz}=-2\cos z\sin z\times$$
$$\times\frac{\Lambda\cos^4 z+2b\gamma\cos^2 z+\gamma^2\Lambda}{(\cos^4 z-\gamma^2)^2};\qquad (34)$$

the points of vanishing force (or of a local extremum) are $z=0$ and the roots of the quadratic equation

$$\Lambda\cos^4 z+2b\gamma\cos^2 z+\gamma^2\Lambda=0\Longrightarrow$$
$$\Longrightarrow\left(\cos^2 z\right)|_{1,2}=-\frac{b}{\Lambda}\gamma\pm\sqrt{\left(\frac{b^2}{\Lambda^2}-1\right)\gamma^2}.\qquad (35)$$

Due to the inequality $\Lambda^2>b^2$ (see Section 6), the quantity under the square root is negative. Therefore, in the physical region of the variable z, we have no other force-vanishing points in addition to $z=0$. In the new variable $\cos^2 z=y$, the differential equation (32) takes the form

$$\left[\frac{d^2}{dy^2}+\left(\frac{3}{2}\frac{1}{y}+\frac{1}{2}\frac{1}{y-1}\right)\frac{d}{dy}-\frac{\epsilon}{4y(y-1)}+\right.$$
$$\left.+\frac{b\gamma+\Lambda y}{(y-\gamma)(y+\gamma)4y(y-1)}\right]Z(y)=0.\qquad (36)$$

The behavior near five singular points can be found straightforwardly:

$y\sim 1(z\to 0)$

$$\left[\frac{d^2}{dy^2}+\left(\frac{1}{2}\frac{1}{y-1}\right)\frac{d}{y}-\right.$$
$$\left.-\frac{\epsilon}{4(y-1)}+\frac{b\gamma+\Lambda}{(1-\gamma^2)\,4(y-1)}\right]Z(y)=0,\qquad (37)$$

$$Z(y)=\exp[\pm\sqrt{A(y-1)}],\quad A=\epsilon-\frac{b\gamma+\Lambda}{1-\gamma^2};$$

$y\sim 0$

$$\left(\frac{d^2}{dy^2}+\frac{3}{2y}\frac{d}{dy}+\frac{\epsilon}{4y}+\frac{b}{4\gamma y}\right)Z(y)=0,\qquad (38)$$
$$Z(y)=\frac{\exp[\pm\sqrt{Cy}]}{\sqrt{y}},\quad C=-\epsilon-\frac{b}{\gamma};$$

$y\sim\infty$

$$\left(\frac{d^2}{dy^2}+\frac{2}{y}\frac{d}{dy}-\frac{\epsilon}{4y^2}\right)Z(z)=0,\qquad (39)$$
$$Z=y^D,\quad D=\frac{-1\pm\sqrt{\epsilon+1}}{2};$$

$y\sim+\gamma$

$$\left[\frac{d^2}{dy^2}+\frac{1}{2}\left(\frac{3}{\gamma}+\frac{1}{\gamma-1}\right)\frac{d}{dy}+\right.$$
$$\left.+\frac{\Lambda+b}{8\gamma(1-\gamma)}\frac{1}{y-\gamma}\right]Z(y)=0$$

or

$$\left[(y-\gamma)\frac{d^2}{dy^2}+M(y-\gamma)\frac{d}{dy}+N\right]Z(y)=0,$$
$$M=\frac{1}{2}\left(\frac{3}{\gamma}+\frac{1}{\gamma-1}\right),\quad N=\frac{\Lambda+b}{8\gamma(1-\gamma)}.$$

Changing the variable $-M(y-\gamma)=x$, we get

$$\left(x\frac{d^2}{dx^2}-x\frac{d}{dx}-\alpha\right)Z=0,\quad \alpha=\frac{N}{M},$$

which is a confluent hypergeometric equation of the special form

$$\left(x\frac{d^2}{dx^2}+(c-x)\frac{d}{dx}-a\right)Z=0,$$
$$c=0,\quad a=\frac{N}{M}=\frac{\Lambda+b}{4(3-4\gamma)}.$$

Its general solution looks as $Z=c_1M(a+1,2,y)+$ $+\,c_2U(a+1,2,y)$.

Now, let us consider the case $y\sim-\gamma$:

$$\left[\frac{d^2}{dy^2}\frac{1}{2}\left(\frac{3}{-\gamma}+\frac{1}{-\gamma-1}\right)\frac{d}{dy}+\right.$$
$$\left.+\frac{\Lambda-b}{8(-\gamma)(1+\gamma)}\frac{1}{y+\gamma}\right]Z(y)=0$$

or, shorter,

$$\left[(y-\gamma)\frac{d^2}{dy^2}+M'(y-\gamma)\frac{d}{dy}+N'\right]Z(y)=0,$$

$$-M'(y+\gamma) = x, \ \left(x\frac{d^2}{dx^2} - x\frac{d}{dx} - \alpha'\right)Z=0, \ a' = \frac{N'}{M'},$$

which is a confluent hypergeometric equation of the special form

$$\left(x\frac{d^2}{dx^2} + (c-x)\frac{d}{dx} - a\right) Z = 0 = 0,$$
$$c = 0, \quad \alpha = \frac{N'}{M'} = \frac{\Lambda - b}{4(3+4\gamma)};$$

its general solution is $Z = c_1 M(a+1, 2, y) + c_2 U(a+ +1, 2, y)$.

The further analytical treatment of the differential equation (36) is very difficult because of the complexity of this equation.

6. Solutions of the Radial Equation

Now, let us consider the radial equation (30)

$$\left[\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} - \right.$$
$$\left. - \frac{[m + b(\cos r - 1)]^2}{\sin^2 r} + \Lambda\right] R(r) = 0;$$

for definiteness, let the magnetic field be directed in the negative direction of the axis z: $b = -B$, $B > 0$. Then

$$\frac{d^2}{dr^2}R + \frac{1}{\tan r}\frac{dR}{dr} - \frac{[m + B(1-\cos r)]^2}{\sin^2 r}R + \lambda R = 0.$$

With the new variable $1 - \cos r = 2z$, we obtain

$$\left[z(1-z)\frac{d^2}{dz^2} + (1-2z)\frac{d}{dz} - \right.$$
$$\left. - \frac{1}{4}\left(\frac{m^2}{z} - 4B^2 + \frac{(m+2B)^2}{1-z}\right) + \Lambda\right] R = 0. \quad (40)$$

With the substitution $R = z^a(1-z)^b\, F$, for

$$a = \pm\frac{m}{2}, \quad b = \pm\frac{m+2B}{2},$$

Eq. (40) yields

$$z(1-z)F'' + [(2a+1) - 2(a+b+1)z]\, F' -$$

$$- \left[a(a+1) + 2ab + b(b+1) - B^2 - \Lambda\right] F = 0, \quad (41)$$

which can be identified with the hypergeometric-type equation

$$z(1-z)F + [\gamma - (\alpha+\beta+1)z]F' - \alpha\beta F = 0.$$

We will search for the solutions describing the bound states; in this case, the parameters a and b should be positive:

$$z = \sin^2\frac{r}{2}, \ \ z \in [0, +1], \ \ r \in [0, +\pi],$$
$$R = \left(\sin\frac{r}{2}\right)^{+|m|} \left(\cos\frac{r}{2}\right)^{+|m+2B|} F\left(\alpha, \beta, \gamma; -\sin^2\frac{r}{2}\right); \quad (42)$$

for (α, β, γ), we have

$$\gamma = + \mid m \mid +1, \ \ a = +\frac{\mid m \mid}{2}, \ \ b = +\frac{\mid m+2B \mid}{2},$$
$$\alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} + \Lambda}, \quad (43)$$
$$\beta = a + b + \frac{1}{2} + \sqrt{B^2 + \frac{1}{4} + \Lambda}.$$

The polynomial condition is

$$\alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} + \Lambda} = -n = 0, -1, ...$$

From whence, we obtain the quantization rule

$$\Lambda + \frac{1}{4} = -B^2 + \left(a + b + \frac{1}{2} + n\right)^2$$

and the corresponding radial functions

$$R = \left(\sin\frac{r}{2}\right)^{+|m|} \left(\cos\frac{r}{2}\right)^{+|m+2B|} F \times$$
$$\times \left(-n, \mid m \mid + \mid m+2B \mid +1+n, \mid m \mid +1 - \sin^2\frac{r}{2}\right). \quad (44)$$

While examining the boundary properties of the functions, we should consider features of the parametrized spherical space S_3, the sphere

$$u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1,$$

by the coordinates (r, ϕ, z). In particular,

$$r = 0, \quad u_1 = 0, \quad u_2 = 0, \quad u_3 = \sin z,$$

$$u_0 = + \cos z, \quad z \in [-\pi/2, +\pi/2],$$

$$r = \pi, \quad u_1 = 0, \quad u_2 = 0, \quad u_3 = \sin z,$$

$$u_0 = - \cos z, \quad z \in [-\pi/2, +\pi/2];$$

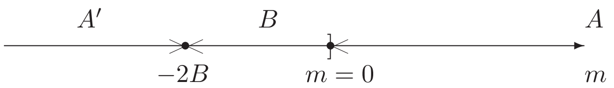

Fig. 1. Orientation $B>0$

this means that the complete curve $u_0^2 + u_3^2 = 1$ in S_3 is given by two parts:

$$\{(r=0, \phi \text{ is mute}, z) + (r=\pi, \phi \text{ is mute}, z)\}.$$

Correspondingly, we must require

$$R(0)=0, \quad R(\pi)=0 \qquad (45)$$

at non-zero m.

First, let us consider the case $m=0$:

$$\underline{m=0}, \quad a=0, \quad b=+B,$$
$$\Lambda+\frac{1}{4} = 2B(n+1/2)+(n+1/2)^2 \Longrightarrow \Lambda > B,$$
$$R=\left(\cos\frac{r}{2}\right)^{+2B} Y\left(-n, 2B+n+1, 1;\ -\sin^2\frac{r}{2}\right), \qquad (46)$$
$$R_{r\to 0}=1, \quad R_{r\to\pi}=0.$$

For $m=0$, the function Ψ does not depend on ϕ, being continuous and single-valued. From (46), we obtain

$$\Lambda = B + 2Bn + n^2 + n > B. \qquad (47)$$

Now, let us consider the case

$$\underline{m>0}, \quad a=+\frac{m}{2}, \quad b=+\frac{m+2B}{2},$$
$$\Lambda+\frac{1}{4} = 2B(n+m+1/2)+$$
$$+(n+m+1/2)^2 \Longrightarrow \Lambda > B,$$
$$R=\left(\sin\frac{r}{2}\right)^{+m}\left(\cos\frac{r}{2}\right)^{m+2B}\times \qquad (48)$$
$$\times F\left(-n, 2B+2m+n+1, m+1;\ -\sin^2\frac{r}{2}\right),$$
$$R_{r\to 0}=0, \quad R_{r\to\pi}=0.$$

It follows from (48) that

$$\Lambda = B + 2B(n+m) + (n+m)^2 + (n+m) > B.$$

Now, let us consider the possibility

$$\underline{m<-2B}, \quad a=-\frac{m}{2}, \quad b=-\frac{m+2B}{2}>0,$$
$$\Lambda+\frac{1}{4} = -2B(n-m+1/2)+(n-m+1/2)^2 =$$
$$=(n-m+1/2)((n-m+1/2-2B)>0,$$
$$R=\left(\sin\frac{r}{2}\right)^{-m}\left(\cos\frac{r}{2}\right)^{-(m+2B)} F\times \qquad (49)$$
$$\times\left(-n, -2m-2B+1+n, -m+1; -\sin^2\frac{r}{2}\right),$$
$$R_{r\to 0}=0, \quad R_{r\to\pi}=0.$$

One of the relations (49) yields

$$\Lambda = B - 2B(n-m) - 2B + (n-m)^2 + (n-m) >$$
$$> B + m(n-m) + m + (n-m)^2 + (n-m)] =$$
$$= B + n^2 + n - 2nm > B.$$

Now, let us examine the case $\underline{-2B<m<0}$,

$$a=-\frac{m}{2}>0, \quad b=\frac{m+2B}{2}>0,$$
$$\Lambda+\frac{1}{4} = 2B(n+1/2)+(n+1/2)^2 > B,$$
$$R=\left(\sin\frac{r}{2}\right)^{-m}\left(\cos\frac{r}{2}\right)^{m+2B}\times \qquad (50)$$
$$\times F\left(-n, 2B+n+1, -m+1;\ -\sin^2\frac{r}{2}\right),$$
$$R_{r\to 0}=0, \quad R_{r\to\pi}=0.$$

From the expression for Λ, we derive the restriction

$$\Lambda = B + 2Bn + n^2 + n > B. \qquad (51)$$

Now, the last possibility is

$$\underline{m=-2B}, \quad a=+B, \quad b=0,$$
$$\Lambda+\frac{1}{4} = 2B(n+1/2)+(n+1/2)^2,$$
$$R=\sin^{2B}\frac{r}{2} Y\left(-n, 2B+n+1, -B+1; -\sin^2\frac{r}{2}\right), \qquad (52)$$
$$R_{r\to+0}=0, \quad R_{r\to+\pi}=1.$$

These solutions are discontinuous, because the wave functions depend on ϕ as $r\to\pi$.

Let us collect the results together (see also Fig. 1).

$$\underline{(A), \ m>0,}$$
$$\Lambda+\frac{1}{4} = (n+1/2+m)(n+1/2+m+2B);$$

$(A'),\ \ m < -2B,$

$$\Lambda + \frac{1}{4} = (n + 1/2 - m)(n + 1/2 - m - 2B);$$

$(B),\ \ -2B < m \le 0,$

$$\Lambda + \frac{1}{4} = (n + 1/2)(n + 1/2 - 2B).$$

In the usual measure units, these formulas read

$(A),\ \ m > 0,$

$$\rho^2\Lambda_0 + \frac{1}{4} = +2\frac{eB}{\hbar c}\rho^2(n+m+1/2) + (n+m+1/2)^2;$$

$(A'),\ \ m < -2\frac{eB}{\hbar c}\rho^2,$

$$\rho^2\Lambda_0 + \frac{1}{4} = -2\frac{eB}{\hbar c}\rho^2(n-m+1/2) + (n-m+1/2)^2;$$

$(B),\ \ -2\frac{eB}{\hbar c}\rho^2 < m \le 0,$

$$\rho^2\Lambda_0 + \frac{1}{4} = 2\frac{eB}{\hbar c}\rho^2(n+1/2) + (n+1/2)^2.$$

The transition to the limit in the Minkowski space is attained accordingly to ($\rho \to \infty$)

$$\begin{aligned} &\underline{m < 0,} \quad \Lambda_0 = 2\frac{eB}{\hbar c}(n+1/2); \\ &\underline{m \ge 0,} \quad \Lambda_0 = +2\frac{eB}{\hbar c}(n+m+1/2), \end{aligned} \qquad (53)$$

where

$$\lim_{\rho\to\infty}\Lambda_0 = \frac{2M}{\hbar^2}\left(E - \frac{P^2}{2M}\right).$$

Thus, we have the well-known result

$$E - \frac{P^2}{2M} = \frac{eB\hbar}{Mc}\left(\frac{m + \mid m \mid}{2} + n + 1/2\right). \qquad (54)$$

7. Cox's particle in an Electric Field. Minkowski Space

The Schrödinger equation for Cox's particle in an electric field has the form [2]

$$\left(D_t - c\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1+\Gamma^2 E_i E^i)}\right)\Psi = -\frac{1}{2M}\overset{\circ}{D}_k \times \\ \times g^{kj}\left[D_j + \frac{\Gamma^2 E_j (E^i D_i) + \mu\Gamma E_j}{1 + \Gamma^2 E_i E^i}\right]\Psi. \qquad (55)$$

We use the notation:

$$A_0 = -eEz, \quad E_i = (F_{01}, F_{02}, F_{03}),$$

$$g^{11}E_1 = E^1, \quad g^{22}E_2 = E^2, \quad g^{33}E_3 = E^3,$$

$$i\hbar\partial_t - eA_0 = D_t, \quad i\hbar\partial_k = D_k,$$

$$\frac{i\hbar}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} = \overset{\circ}{D}_k\,.$$

In the cylindric coordinates, the field is characterized by

$$E_3 = E, \quad E^3 = -E, \quad E_3E^3 = -E^2.$$

First, we get (let $\Gamma E = \gamma$)

$$\left(D_t - c\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1+\Gamma^2 E_i E^i)}\right) = \\ = i\hbar\partial_t + eEz + c\,\frac{\gamma^2\mu + \gamma D_3}{2\,(1-\gamma^2)}. \qquad (56)$$

Next, we consider the Hamiltonian

$$H = \frac{1}{2M}\left[\overset{\circ}{D}_1 D_1 + \overset{\circ}{D}_2\frac{1}{r^2}D_2 + \\ + \overset{\circ}{D}_3\left(D_3 + \frac{\mu\gamma}{1-\gamma^2}\right)\right]. \qquad (57)$$

In the explicit form, the extended Schrödinger equation looks as follows (to allow for imaginary γ, we make the formal change $i\gamma \to \gamma$):

$$\left(i\hbar\partial_t + eEz - \frac{Mc^2\gamma^2}{2\,(1+\gamma^2)} + \frac{\gamma}{2\,(1+\gamma^2)}\hbar c\partial_z\right)\Psi = \\ = -\frac{\hbar^2}{2M}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{\partial_\phi^2}{r^2} + \partial_z^2 - \frac{(mc/\hbar)\gamma}{1+\gamma^2}\partial_z\right)\Psi. \qquad (58)$$

With the representation $\Psi = e^{-iWt/\hbar}e^{im\phi}Z(z)R(t)$ and the notation

$$\frac{M^2c^2}{\hbar^2} = \frac{1}{\lambda^2}, \quad \frac{2M}{\hbar^2}W = w, \quad \frac{2M}{\hbar^2}eEz = \nu,$$

we obtain

$$\frac{1}{Z(z)}\left(\frac{d^2}{dz^2} + \nu\,z + w - \frac{1}{\lambda^2}\frac{\gamma^2}{1+\gamma^2}\right)Z(z) + \\ + \frac{1}{R(r)}\left(\frac{d^2}{dr^2} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}\right)R(r). \qquad (59)$$

After the separation of the variables ($w_\perp > 0$ stands for the separation constant), we have

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + w_\perp\right) R(r) = 0, \tag{60}$$

$$\left(\frac{d^2}{dz^2} + \nu z + w'\right) Z(z) = 0,$$
$$w' = w - w_\perp + \frac{1}{\lambda^2}\frac{\gamma^2}{1+\gamma^2}. \tag{61}$$

In fact, (60) and (61) coincide with the well-known ones for an ordinary particle in a uniform electric field. The equation in the variable z looks as a 1-dimensional Schrödinger equation with the potential $U(z) = -\nu z$, $\nu > 0$:

$$\left(\frac{d^2}{dz^2} + w' + \nu z\right) Z(z) = 0. \tag{62}$$

The form of the curve $U(z)$ indicates that any particle moving from the right must be reflected by this barrier in a vicinity of the point $z_0 = -\frac{w'}{\nu}$ (we assume that the electric force acts in the positive direction of the axis z).

The solution of Eq. (62) can be presented in term of the Airy function. Indeed, in (62), let us change the variable

$$\nu z + w' = ax, \quad \left(\frac{d^2}{dx^2} + \frac{a^3}{\nu^2}x\right) Z(x) = 0;$$

let it be (for definiteness, $\nu > 0$)

$$\frac{a^3}{\nu^2} = -1, \quad a = -\nu^{2/3},$$
$$x = \frac{\nu z + w'}{-\nu^{2/3}} = -\nu^{1/3} z - \frac{w'}{\nu^{2/3}}. \tag{63}$$

Then we arrive at the Airy equation

$$\left(\frac{d^2}{dx^2} - x\right) Z(x) = 0; \tag{64}$$

to the turning point $z_0 = -w'/\nu$, there corresponds the value $x_0 = 0$. Moreover, Eq. (64) can be related to the Bessel equation. Indeed, let us introduce the variable

$$\xi = \frac{2}{3}x^{3/2}, \qquad x = \frac{3}{2}\xi^{2/3}, \tag{65}$$

then the Airy equation gives

$$\left(\frac{1}{3\xi}\frac{d}{d\xi} + \frac{d^2}{d\xi^2} - 1\right) Z = 0.$$

Applying the substitution $Z = \xi^{1/3} f(\xi)$, we arrive at the Bessel equation [8]

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - 1 - \frac{1/9}{\xi^2}\right) f(\xi) = 0 \tag{66}$$

with two linearly independent solutions

$$f_1(\xi) = J_{+1/3}(i\xi), \qquad f_2(\xi) = J_{-1/3}(i\xi). \tag{67}$$

Thus, the general solutions of the Airy equation can be constructed as linear combinations of

$$Z_1(x) = \xi^{1/3} J_{+1/3}(i\xi), \quad Z_2(x) = \xi^{1/3} J_{-1/3}(i\xi),$$

where

$$i\xi = i\frac{2\sqrt{\nu}}{3}(z + \frac{w'}{\nu})^{3/2}. \tag{68}$$

With the use of the known relation [8]

$$J_\mu(y) = \frac{(y/2)^\mu}{\Gamma(\mu+1)} e^{-iy}_1 F_1\left(\mu + \frac{1}{2}, 2\mu+1, 2iy\right)$$

and with the notation $y = i\xi$, $\mu = +1/3, -1/3$, we express two independent solutions of the Schrödinger equation as

$$Z_1 = \xi^{1/3} J_{+1/3}(i\xi) = \xi^{+1/3}\frac{(i\xi/2)^\mu}{\Gamma(\mu+1)} e^{\xi} \times$$
$$\times {}_1F_1\left(+\mu + \frac{1}{2}, +2\mu+1, -2\xi\right),$$
$$Z_2 = \xi^{1/3} J_{-1/3}(i\xi) = \xi^{1/3}\frac{(i\xi/2)^{-\mu}}{\Gamma(-\mu+1)} e^{\xi} \times$$
$$\times {}_1F_1\left(-\mu + \frac{1}{2}, -2\mu+1, -2\xi\right).$$

8. Cox's particle in Electric Field in the Spherical Model

In the cylindrical coordinate system,

$$dS^2 = dt^2 - \cos^2 z\left(dr^2 + \sin^2 r d\phi^2\right) - dz^2,$$

the external electric field along the axis z is given by

$$A_0 = -E\rho \tan z, \quad E_3 = \frac{E}{\cos^2 z},$$
$$E^3 = -\frac{E}{\cos^2 z}, \quad E_3 E^3 = -\frac{E^2}{\cos^4 z}. \tag{69}$$

Note that the infinite values for E^3 at $z = \pm\pi/2$ have no metrical sense, and everything is correct in the resulted differential equation in the z-variable (see (75)).

Below, we use the operators (and the dimensionless coordinate $tc/\rho \to t$)

$$i\frac{\hbar c}{\rho}\partial_t - eA_0 = D_t,\quad i\frac{\hbar}{\rho}\partial_k = D_k,\quad \frac{i\hbar/\rho}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} = \overset{\circ}{D}_k\,.$$

We start with the extended form of the Schrödinger equation [2]

$$\left(D_t - c\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1+\Gamma^2 E_i E^i)}\right)\Psi = \\ = \frac{\overset{\circ}{D}_k\,(-g^{kj})}{2M\rho^2}\left(D_j + \frac{\Gamma^2 E_j (E^i D_i) + \mu\Gamma E_j}{1+\Gamma^2 E_i E^i}\right)\Psi. \quad (70)$$

After the needed calculation, we get representation for the wave equation

$$\frac{\hbar c}{\rho}\left(i\partial_t + \frac{eE\rho}{\hbar c/\rho}\tan z + \right. \\ \left. + \frac{1}{2}\frac{Mc\rho}{\hbar}\frac{\gamma^2(z)}{1-\gamma^2(z)} + \frac{1}{2}\frac{\gamma(z)}{1-\gamma^2(z)}i\partial_z\right)\Psi = \\ = -\frac{\hbar^2}{2M\rho^2}\left[\frac{1}{\cos^2 z}\left(\partial_r^2 + \frac{\cos r}{\sin r}\,\partial_r + \frac{\partial_\phi^2}{\sin^2 r}\right) + \right. \\ \left. + \left(\partial_z - 2\frac{\sin z}{\cos z}\right)\left(\frac{1-2\gamma^2(z)}{1-\gamma^2(z)}\partial_z - \frac{Mc\rho}{\hbar}\frac{i\gamma(z)}{1-\gamma^2(z)}\right)\right]\Psi.$$

Note that two terms proportional to $i\gamma(z)\partial_z$ compensate each other. In addition in view of physical reasons, we perform the formal change $i\gamma \to \gamma$:

$$\frac{\hbar c}{\rho}\left(i\partial_t + \frac{eE\rho}{\hbar c/\rho}\tan z - \frac{1}{2}\frac{Mc\rho}{\hbar}\frac{\gamma^2(z)}{1+\gamma^2(z)}\right)\Psi = \\ = -\frac{\hbar^2}{2M\rho^2}\left[\frac{1}{\cos^2 z}\left(\partial_r^2 + \frac{\cos r}{\sin r}\partial_r + \frac{\partial_\phi^2}{\sin^2 r}\right) + \right. \\ + \left(\partial_z - 2\,\frac{\sin z}{\cos z}\right)\left(\frac{1+2\gamma^2(z)}{1+\gamma^2(z)}\partial_z\right) - \\ \left. - \frac{Mc\rho}{\hbar}\frac{\partial}{\partial z}\frac{\gamma(z)}{1+\gamma^2(z)} + \frac{Mc\rho}{\hbar}\frac{\gamma(z)}{1+\gamma^2(z)}2\frac{\sin z}{\cos z}\right]\Psi. \quad (71)$$

With the representation

$$\Psi = e^{-iwt}e^{im\phi}R(r)Z(z), \quad w = \frac{W\rho}{\hbar c},$$

and the notation

$$W = w\frac{\hbar c}{\rho}\frac{1}{\hbar^2/2M\rho^2} = 2w\frac{M\rho c}{\hbar}, \quad \nu = eE\rho\frac{1}{\hbar^2/2M\rho^2},$$

$$\frac{1}{2}Mc^2\frac{1}{\hbar^2/2M\rho^2} = \frac{M^2\rho^2 c^2}{\hbar^2} = \mu^2,$$

we arrive at

$$\cos^2 z\left(W + \nu\tan z - \mu^2\frac{\gamma^2(z)}{1+\gamma^2(z)}\right)R(r)Z(z) + \\ + \left(\partial_r^2 + \frac{\cos r}{\sin r}\partial_r - \frac{m^2}{\sin^2 r}\right)R(r)Z(z) + \\ + \cos^2 z\left[\left(\partial_z - 2\frac{\sin z}{\cos z}\right)\left(\frac{1+2\gamma^2(z)}{1+\gamma^2(z)}\partial_z\right) - \right. \\ \left. - \mu\frac{\partial}{\partial z}\frac{\gamma(z)}{1+\gamma^2(z)} + \mu\frac{\gamma(z)}{1+\gamma^2(z)}2\frac{\sin z}{\cos z}\right]R(r)Z(z) = 0. \quad (72)$$

After the separation of the variable, we obtain

$$\left(\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} - \frac{m^2}{\sin^2 r} + \Lambda\right)R(r) = 0 \quad (73)$$

and

$$\left[\left(\frac{d}{dz} - 2\frac{\sin z}{\cos z}\right)\left(\frac{1+2\gamma^2(z)}{1+\gamma^2(z)}\frac{d}{dz}\right) - \right. \\ - \mu\frac{d}{dz}\frac{\gamma(z)}{1+\gamma^2(z)} + \mu\frac{\gamma(z)}{1+\gamma^2(z)}2\frac{\sin z}{\cos z} + W + \\ \left. + \nu\tan z - \mu^2\frac{\gamma^2(z)}{1+\gamma^2(z)} - \frac{\Lambda}{\cos^2 z}\right]Z(z) = 0. \quad (74)$$

Remember that $\gamma(z) = \gamma\,\cos^{-2} z$. The last equation can be translated to the form

$$\left(\frac{\cos^4 z + 2\gamma^2}{\cos^4 z + \gamma^2}\frac{d^2}{dz^2} - 2\frac{\sin z}{\cos z}\times \right. \\ \times\frac{\gamma^2\cos^4 z + 2\gamma^4 + \cos^8 z}{(\cos^4 z + \gamma^2)^2}\frac{d}{dz} - \mu\gamma\frac{\cos^2 z}{\cos^4 z + \gamma^2}\frac{d}{dz} + \\ + 4\mu\gamma^3\frac{\sin z\cos z}{(\cos^4 z + \gamma^2)^2} + W + \nu\tan z - \\ \left. - \frac{\mu^2\gamma^2}{\cos^4 z + \gamma^2} - \frac{\Lambda}{\cos^2 z}\right)Z(z) = 0. \quad (75)$$

We could not proceed further with this differential equation because of its complexity.

9. Conclusion

The generalized relativistic Klein–Fock–Gordon equation for Cox's scalar particle with intrinsic structure has been investigated and solved in the presence of the external uniform magnetic and electric fields in the Minkowski space.

Similar problems in the non-relativistic approximation for the case of a closed spherical Riemann 3-space have been examined as well. The complete separation of the variables in the system of special cylindric coordinates in the curved model has performed for both cases. In the presence of a magnetic field, the quantum problem in the radial variable has been solved exactly, and the wave functions and the corresponding energy levels have been found: the quantum motion in the z-direction is described by a 1-dimensional Schrödinger-like equation with an effective potential, which turns out to be too difficult for the analytical treatment. In the presence of an electric field against the background of the curved model, the situation is similar: the radial equation is solved exactly in hypergeometric functions, whereas the equation in the z-variable can be treated only qualitatively.

So, the additional Cox's intrinsic structure of a spin-zero particle turns out to be very sensitive to the external geometrical background of a space-time model.

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ЧАСТИНКА КОКСА
У МАГНIТНОМУ I ЕЛЕКТРИЧНОМУ
ПОЛЯХ НА ТЛI ЕВКЛIДОВОЇ
I СФЕРИЧНОЇ ГЕОМЕТРIЇ

Р е з ю м е

Узагальнене релятивiстське рiвняння Клейна–Фока–Гордона для неточечної скалярної частинки Кокса з внутрiшньою структурою розв'язано в присутностi зовнiшнiх однорiдних магнiтного й електричного полiв у випадку простору Мiнковського. Аналогiчнi завдання розглянуто в нерелятивiстському наближеннi для випадку замкнутого сферичного тривимiрного простору Рiмана. Виконано повне роздiлення змiнних у спецiальнiй системi цилiндричних координат в обох випадках. У присутностi магнiтного поля квантова задача для радiальної змiнної вирiшено точно, знайдено хвильовi функцiї i вiдповiднi рiвнi енергiї. Квантовий рух у z-напрямку описується одновимiрним рiвнянням типу Шредiнгера в ефективному потенцiалi, яке виявляється занадто складним для аналiтичного розв'язання. У присутностi електричного поля на тлi викривленої моделi ситуацiя аналогiчна: радiальне рiвняння вирiшено точно в гiпергеометричних функцiях, рiвняння в z-змiнної може бути дослiджено тiльки якiсно.