S.B. GRIGORYEV, A.B. LEONOV
Dnipropetrovsk National University
(72, Gagarina Ave., Dnipropetrovsk 49010, Ukraine; e-mail: arkadiyleonov@yahoo.com)

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EINSTEIN EQUATIONS IN THE CASE OF STATIC CYLINDRICAL SYMMETRY AND THE DIAGONAL STRESS-ENERGY TENSOR WITH MUTUALLY PROPORTIONAL COMPONENTS

The Einstein equations with the stress-energy tensor in the form of a diagonal matrix with mutually proportional components are studied in the static cylindrically symmetric case. Several known exact solutions fall into this case (static electric field, some perfect fluid solutions, and solution with the cosmological constant). Coefficients of proportionality in the stress-energy tensor serve as parameters that allow studying a more general case (as well as obtaining new solutions for particular values of these coefficients). The initial system of equations is simplified and transformed into a system of two first-order ordinary differential equations. An exact solution is found for a broad set of parameters. The equilibrium points of the system of equations are considered, and the qualitative behavior of the solutions near the hyperbolic equilibrium points is studied.

Keywords: Einstein equations, cylindrical symmetry, stress-energy tensor.

1. Introduction

The Einstein equations are considered in the case of cylindrical symmetry without rotation together with a stress-energy tensor T^μ_ν in the form of a diagonal matrix with mutually proportional diagonal elements

$$T^\mu_\nu = \begin{pmatrix} lp & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -mp & 0 \\ 0 & 0 & 0 & -np \end{pmatrix}, \quad (1)$$

where p is an unknown function, and l, m, and n are some constants (not necessary discrete).

We chose such form of the stress-energy tensor, because several typical cases fall into this category, namely: the vacuum solution, solution with a static electric field, stationary perfect fluid with the equation of state $\epsilon = \alpha p$, and the case of the cosmological constant. Therefore, all these cases can be studied simultaneously.

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The static cylindrically symmetric case has been studied in numerous works. The vacuum solution can be found in [1]. A number of solutions have been obtained in the case of the electromagnetic field [2–5].

The case of the perfect fluid has been reduced to the second-order linear differential equation in [7]. Starting with the metric tensor in different forms and making various assumptions, several different solutions have been obtained in [7–11]. In the case of a perfect fluid with the equation of state $\epsilon = \alpha p$, the general solution has been found in [13]. A stationary rotating perfect fluid has been studied in [12].

The static cylindrically symmetric case with the cosmological constant has been studied in [14]. A nonlinear conformally invariant scalar field has been considered in [15].

The embedding of the cylindrically symmetric configurations in the external spacetime gained attention in recent years [16–18]. For a discussion of the cylindrically symmetric cosmological solution, see [19], the collapse of a cylindrically symmetric configurations of matter has been studied in several works (see, e.g., [20] and references therein).

The study of the gravitomagnetic effects under cylindrical symmetry and their possible connection to the astrophysical phenomena can be found in [21].

The investigation of cylindrically symmetric gravitational waves and related effects has a long history, starting from the work of Einstein and Rosen. The modern interest in them arises from the gravitation quantization and the problem of the energy of a gravitational field. The literature on these questions is extensive, but we point only to one particular work [22].

We reduce the Einstein equations to the system of two first-order differential equations and find the general solution for a quite general set of parameters that characterize the stress-energy tensor (in all cases with $n = -1$). In the case of a static electric field ($l = -1$, $m = -1$, $n = -1$), this solution reduces to that obtained by Raychaudhuri (see [5, 6]).

In the other cases, the system of equations can be studied at least qualitatively. The general review of the qualitative analysis of dynamical systems can be found in [30], and work [25] is particularly devoted to the qualitative analysis of a system of two equations.

Every solution of the system of first-order differential equations (in general, nonlinear ones) can be thought as a curve in the phase space of the system. Knowing the structure of the phase space, one can describe the behavior of all solutions (including the solutions with basically different behavior) for all initial conditions. Since the structure of the phase space is largely determined by the equilibrium points of the system, one can extract information about the behavior of the solutions without knowing their explicit form. The case where the system consists of only two equations is practically the simplest one, has the advantage to be easily visualized and, thus, is the most desirable.

Different typical cases and corresponding values of the parameters in the stress-energy tensor, where $k = \frac{1}{2}(l + m + n + 1)$; the vacuum case can be obtained by setting $l = 0$, $m = 0$, $n = 0$, and $k = 0$, bypassing the definition of k

	l	m	n	k
Static electric field	-1	-1	-1	-1
Stationary perfect fluid	α	1	1	$\frac{3+\alpha}{2}$
Cosmological constant	-1	1	1	1

In [23, 24], the qualitative analysis had been applied to the Einstein equations in the case of the spherical symmetry together with the stress-energy tensor with mutually proportional components. The stress-energy tensor had been parametrized by two parameters – different values of these parameters correspond to different fields: scalar field, perfect fluid, *etc.* Then, the Einstein equations had been reduced to the autonomous system of two ordinary differential equations, and the phase space of this system had been studied in detail. It had been shown that different metrics, corresponding to the different values of parameters in the stress-energy tensor, demonstrate the same qualitative behavior. The sets of parameters that give rise to the metrics with similar qualitative behavior had been identified.

Qualitative analysis has been used to study spacetime singularities in the presence of scalar fields [26]. The example of the qualitative analysis of cosmological models (in the Brans–Dicke theory) can be found in [27].

We now find the equilibrium points of the system of equations. It turns out that the system has a nonhyperbolic equilibrium point and, if a certain relation holds between parameters l, m, and n ($k^2 = m(n+1)$ with $k = \frac{1}{2}(l + m + n + 1)$, see below), acquires a set of hyperbolic equilibrium points. We determine their type in what follows.

2. Derivation of a System of Equations

We use the signature $(+,-,-,-)$ and the system of geometric units, in which $c = G = 1$. The Einstein equations are written in the following form:

$$R^{\mu}{}_{\nu} - \frac{1}{2}\,\delta^{\mu}{}_{\nu} R = 8\pi T^{\mu}{}_{\nu}. \quad (2)$$

We begin with the cylindrically symmetric static metric [28]

$$ds^2 = e^{2U}dt^2 - e^{2K-2U}(d\rho^2 + dz^2) - W^2 e^{-2U} d\varphi^2. \quad (3)$$

In accord with the usual convention, ρ is the "radial" coordinate, and the coordinate "z" runs along the axis of symmetry. Since we consider a static case, the unknown functions U, K, and W depend only on ρ. The Einstein equations for the metric (3) are

$$e^{2U-2K}\left(2U''+2U'\frac{W'}{W}-U'^2-K''-\frac{W''}{W}\right)=8\pi lp, \quad (4)$$

$$e^{2U-2K}\left(U'^2-K'\frac{W'}{W}\right)=-8\pi p, \quad (5)$$

$$-e^{2U-2K}\left(K''+U'^2\right)=-8\pi mp, \quad (6)$$

$$-e^{2U-2K}\left(U'^2-K'\frac{W'}{W}+\frac{W''}{W}\right)=-8\pi np. \quad (7)$$

Derivatives with respect to ρ are denoted by primes.

We multiply each equation by e^{2K-2U} and denote $\Pi=8\pi pe^{2K-2U}$. Then, multiplying the two last equations by -1, we obtain

$$2U''+2U'\frac{W'}{W}-U'^2-K''-\frac{W''}{W}=l\Pi, \quad (8)$$

$$U'^2-K'\frac{W'}{W}=-\Pi, \quad (9)$$

$$K''+U'^2=m\Pi, \quad (10)$$

$$U'^2-K'\frac{W'}{W}+\frac{W''}{W}=n\Pi. \quad (11)$$

Subtracting Eq. (9) from Eq. (11), we have

$$2U''+2U'\frac{W'}{W}-U'^2-K''-\frac{W''}{W}=l\Pi, \quad (12)$$

$$U'^2-K'\frac{W'}{W}=-\Pi, \quad (13)$$

$$K''+U'^2=m\Pi, \quad (14)$$

$$\frac{W''}{W}=(n+1)\Pi. \quad (15)$$

In view of Eqs. (14) and (15), we simplify (12) to the form

$$2U''+2U'\frac{W'}{W}=(l+m+n+1)\Pi, \quad (16)$$

$$U'^2-K'\frac{W'}{W}=-\Pi, \quad (17)$$

$$K''+U'^2=m\Pi, \quad (18)$$

$$\frac{W''}{W}=(n+1)\Pi. \quad (19)$$

We note that, in fact, this whole system consists of only derivatives of some functions. At first, we denote $U'=\Omega$, $K'=\Phi$. If we also denote $\frac{W'}{W}=\Psi$, then $\frac{W''}{W}=\Psi'+\Psi^2$. So, we obtain

$$2\Omega'+2\Omega\Psi=(l+m+n+1)\Pi, \quad (20)$$

$$\Omega^2-\Phi\Psi=-\Pi, \quad (21)$$

$$\Phi'+\Omega^2=m\Pi, \quad (22)$$

$$\Psi'+\Psi^2=(n+1)\Pi. \quad (23)$$

Let us use the second of these equations to get rid of Π in the other equations:

$$2\Omega'+2\Omega\Psi=(l+m+n+1)(\Phi\Psi-\Omega^2), \quad (24)$$

$$\Phi'+\Omega^2=m(\Phi\Psi-\Omega^2), \quad (25)$$

$$\Psi'+\Psi^2=(n+1)(\Phi\Psi-\Omega^2). \quad (26)$$

Thus, denoting $k=\frac{1}{2}(l+m+n+1)$, we finally arrive at a usable system of equations

$$\Omega'=-\Omega\Psi+k(\Phi\Psi-\Omega^2), \quad (27)$$

$$\Phi'=-\Omega^2+m(\Phi\Psi-\Omega^2), \quad (28)$$

$$\Psi'=-\Psi^2+(n+1)(\Phi\Psi-\Omega^2). \quad (29)$$

The function Π related to the "pressure" p as $\Pi=8\pi pe^{2K-2U}$ can be calculated from the solution of the system as $\Pi=\Phi\Psi-\Omega^2$.

3. Reduction to a System of Two Equations

System (27) can be integrated once in two slightly different ways.

Consider the substitution

$$\Omega=kF, \quad (30)$$

$$\Phi=G+mF, \quad (31)$$

$$\Psi=H+(n+1)F, \quad (32)$$

where F, G, and H are some new unknown functions.

The inverse transformation is

$$F=\frac{1}{k}\,\Omega, \quad (33)$$

$$G = \Phi - \frac{m}{k}\,\Omega, \quad (34)$$

$$H = \Psi - \frac{n+1}{k}\,\Omega. \quad (35)$$

Using it in system (27) leads to the following system for F, G, and H:

$$F' = (m(n+1) - k^2)F^2 + mFH+$$

$$+(G-F)(H+(n+1)F), \quad (36)$$

$$G' = (m(n+1) - k^2)F^2 + mFH, \quad (37)$$

$$H' = -H(H+(n+1)F). \quad (38)$$

Let us subtract Eq. (37) from (36):

$$(F-G)' = (G-F)(H+(n+1)F), \quad (39)$$

$$H' = -H(H+(n+1)F). \quad (40)$$

We obtain immediately the integrable system

$$\frac{(F-G)'}{F-G} = -(H+(n+1)F), \quad (41)$$

$$\frac{H'}{H} = -(H+(n+1)F) \quad (42)$$

or

$$\frac{(F-G)'}{F-G} = \frac{H'}{H}. \quad (43)$$

Then

$$F - G = C_H H, \quad (44)$$

where C_H is a constant of integration.

Thus, system (36)–(38) reduces to

$$F' = -k^2F^2 + (mF - C_H H)(H+(n+1)F), \quad (45)$$

$$H' = -H(H+(n+1)F). \quad (46)$$

There is a different version of substitution (30)–(32):

$$\Omega = F + kH, \quad (47)$$

$$\Phi = G + mH, \quad (48)$$

$$\Psi = (n+1)H. \quad (49)$$

With the inverse transformation

$$F = \Omega - \frac{k}{n+1}\Psi, \quad (50)$$

$$G = \Phi - \frac{m}{n+1}\Psi, \quad (51)$$

$$H = \frac{1}{n+1}\Psi. \quad (52)$$

Repeat the same steps as in the case of (30)–(32). All transformations are alike, and we obtain the system

$$F' = -(n+1)FH, \quad (53)$$

$$H' = -C_F(n+1)FH + m(n+1)H^2 - (F+kH)^2 \quad (54)$$

together with the result of integration

$$C_F F = G - H, \quad (55)$$

where C_F is a constant of integration in the case of substitution (47)–(49).

4. A Special Case of the System of Three Equations: $k = 0$, $n = -1$

The first substitution works in all cases where $k \neq 0$, and the second works if $n \neq -1$. It leaves the case, in which $k = 0$ and $n = -1$. In this case, the system of equations can be integrated explicitly.

Setting $k = 0$ and $n = -1$ in (27) gives us the system

$$\Omega' = -\Omega\Psi, \quad (56)$$

$$\Phi' = -\Omega^2 + m(\Phi\Psi - \Omega^2), \quad (57)$$

$$\Psi' = -\Psi^2. \quad (58)$$

The definition of k implies that $l = -m$, and we deal with a stress-energy tensor of the form $T^\mu_\nu = \mathrm{diag}(-mp, -p, -mp, p)$.

Integrating the third equation, we have

$$\Psi = \frac{1}{\rho + C_1}, \quad (59)$$

where C_1 is a constant of integration.

The constant C_1 only affects the position of the axis of symmetry with regards to the coordinate ρ (it is not fixed by metric (3) that we chose). Hence, we may set $C_1 = 0$, so $\Psi = \frac{1}{\rho}$.

Integrating the rest of equations, we obtain

$$\Omega = \frac{C_2}{\rho}, \tag{60}$$

$$\Phi = \frac{C_2^2}{\rho} + C_3\rho^m, \tag{61}$$

$$\Psi = \frac{1}{\rho}, \tag{62}$$

where C_2 and C_3 are constants of integration.

Integrating once more, we obtain the functions directly related to the components of the metric tensor:

$$U = \ln(C_4\rho^{C_2}), \tag{63}$$

$$K = \ln(C_5\rho^{C_2^2}) + \frac{C_3}{m+1}\rho^m, \tag{64}$$

$$W = C_6\rho, \tag{65}$$

and C_4, C_5, and C_6 are three more constants of integration.

5. Solution for $k \neq 0$, $n = -1$

System (27) can be completely integrated in quite a general case where $n = -1$ (and the stress-energy tensor is $T^\mu_\nu = diag(lp, -p, -mp, p)$). As we will see below, this is the case with $W = \rho$.

It is much harder to obtain the explicit expressions for components of the metric tensor in this case. That is why, in order to completely describe the reduction of the problem to the system of only two differential equations, we have left the values $k = 0$ and $n = -1$ as a separate case.

We will use the first substitution (30)–(32) and the system that it produces (45)–(46) (the second substitution does not work in this case).

Again, we will introduce several constants of integration. All of them will be denoted according to the usual notation as C_1, C_2, and so forth.

So, we set $n = -1$ in (45)–(46) and obtain

$$F' = -k^2F^2 + H(mF - C_HH), \tag{66}$$

$$H' = -H^2. \tag{67}$$

Integrating the second equation, we have $H = \frac{1}{\rho + C_1}$. As in the previous case, we can set $C_1 = 0$, so

$$H = \frac{1}{\rho}. \tag{68}$$

Then we are going to rearrange the right-hand side of the first equation, so we have $\left(kF - \frac{m}{2k}H\right)^2$ in it and

$$\begin{aligned} F' &= -k^2F^2 + H(mF - C_HH) = \\ &= -k^2F^2 + mFH - C_HH^2 = \\ &= -k^2F^2 + 2kF\frac{m}{2k}H - \left(\frac{m}{2k}\right)^2 H^2 + \\ &+ \left(\frac{m}{2k}\right)^2 H^2 - C_HH^2 = \\ &= -\left(kF - \frac{m}{2k}H\right)^2 + \left(\left(\frac{m}{2k}\right)^2 - C_H\right) H^2. \end{aligned} \tag{69}$$

Now, we want to get $\left(kF - \frac{m}{2k}H\right)'$ on the left-hand side. Thus, we multiply the equation by k and then add $-\frac{m}{2k}H'$ to it:

$$\begin{aligned} \left(kF - \frac{m}{2k}H\right)' &= -k\left(kF - \frac{m}{2k}H\right)^2 + \\ &+ k\left(\left(\frac{m}{2k}\right)^2 - C_H\right) H^2 - \frac{m}{2k}H'. \end{aligned} \tag{70}$$

Using the second equation $H' = -H^2$, we obtain $-\frac{m}{2k}H' = \frac{m}{2k}H^2$ and

$$\begin{aligned} \left(kF - \frac{m}{2k}H\right)' &= -k\left(kF - \frac{m}{2k}H\right)^2 + \\ &+ \left[k\left(\left(\frac{m}{2k}\right)^2 - C_H\right) + \frac{m}{2k}\right] H^2. \end{aligned} \tag{71}$$

Then we denote $P = kF - \frac{m}{2k}H$ and $a = k\left(\left(\frac{m}{2k}\right)^2 - C_H\right) + \frac{m}{2k}$ and insert $H = \frac{1}{\rho}$:

$$P' = -kP^2 + \frac{a}{\rho^2}. \tag{72}$$

To solve it, we use the standard substitution $Q = \rho P$, which allows us to separate variables:

$$\begin{aligned} \rho Q' &= -kQ^2 + Q + a = -k\left(Q^2 - \frac{1}{k}Q - \frac{a}{k}\right) = \\ &= -k\left(Q^2 - 2Q\frac{1}{2k} + \frac{1}{4k^2} - \frac{1}{4k^2} - \frac{a}{k}\right) = \\ &= -k\left(\left(Q - \frac{1}{2k}\right)^2 - \frac{1+4ak}{4k^2}\right). \end{aligned} \tag{73}$$

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We denote $u = Q - \frac{1}{2k}$. Then $\frac{1+4ak}{4k^2}$ can be both positive and negative. Denoting $b = \frac{\sqrt{|1+4ak|}}{2k}$, we have two cases in view of the sign of $1 + 4ak$:

$$\frac{du}{u^2 \mp b^2} = -k\frac{d\rho}{\rho}. \tag{74}$$

Both are easy to integrate.

Hence, we use two different substitutions, depending on the sign of $1 + 4ak$. In the case of the minus sign, we consider two substitutions depending on whether $u^2 < b^2$ or $u^2 > b^2$:

$$u = b\tanh v, \quad 1 + 4ak > 0, \quad u^2 < b^2, \tag{75}$$

$$u = b\coth v, \quad 1 + 4ak > 0, \quad u^2 > b^2, \tag{76}$$

$$u = b\tan v, \quad 1 + 4ak < 0, \tag{77}$$

which gives us the following integrals (integrals in the cases where $u^2 < b^2$ or $u^2 > b^2$ are the same):

$$-\frac{1}{b}\int dv = -k\int \frac{d\rho}{\rho} - \frac{1}{b}\ln C_2,$$
$$1 + 4ak > 0, \quad u^2 < b^2, \tag{78}$$

$$-\frac{1}{b}\int dv = -k\int \frac{d\rho}{\rho} - \frac{1}{b}\ln C_2,$$
$$1 + 4ak > 0, \quad u^2 > b^2, \tag{79}$$

$$\frac{1}{b}\int dv = -k\int \frac{d\rho}{\rho} - \frac{1}{b}\ln C_2, \quad 1 + 4ak < 0, \tag{80}$$

where we have chosen the constant of integration to be $\frac{1}{b}\ln C_2$ in order to simplify our formulas.

Then

$$v = \ln\left(C_2\rho^{kb}\right), \quad 1 + 4ak > 0, \quad u^2 < b^2, \tag{81}$$

$$v = \ln\left(C_2\rho^{kb}\right), \quad 1 + 4ak > 0, \quad u^2 > b^2, \tag{82}$$

$$v = \ln\left(C_2\rho^{-kb}\right), \quad 1 + 4ak < 0. \tag{83}$$

Thus,

$$u = b\tanh\ln\left(C_2\rho^{kb}\right), \quad 1 + 4ak > 0, \quad u^2 < b^2, \tag{84}$$

$$u = b\coth\ln\left(C_2\rho^{kb}\right), \quad 1 + 4ak > 0, \quad u^2 > b^2, \tag{85}$$

$$u = b\tan\ln\left(C_2\rho^{-kb}\right), \quad 1 + 4ak < 0. \tag{86}$$

Using $\tanh(\ln x) = \frac{x^2-1}{x^2+1}$ and $\coth(\ln x) = \frac{x^2+1}{x^2-1}$, we have

$$u = b\,\frac{C_2^2\rho^{2kb} - 1}{C_2^2\rho^{2kb} + 1}, \quad 1 + 4ak > 0, \quad u^2 < b^2, \tag{87}$$

$$u = b\,\frac{C_2^2\rho^{2kb} + 1}{C_2^2\rho^{2kb} - 1}, \quad 1 + 4ak > 0, \quad u^2 > b^2, \tag{88}$$

$$u = b\tan\ln\left(C_2\rho^{-kb}\right), \quad 1 + 4ak < 0. \tag{89}$$

We now collect all the substitutions and roll back to the original function F: $F = \frac{1}{k}\left(\frac{m}{2k}\frac{1}{\rho} + \frac{1}{\rho}\left(u + \frac{1}{2k}\right)\right) = \frac{1}{\rho}\left(\frac{m+1}{2k^2} + \frac{1}{k}u\right)$. The condition $u^2 < b^2$ transforms into $F < \frac{1}{k\rho}\left(\frac{m+1}{2k} + b\right)$. Thus, together with $H = \frac{1}{\rho}$, we have

$$F = \frac{1}{\rho}\left(\frac{m+1}{2k^2} + \frac{b}{k}\frac{C_2^2\rho^{2kb} - 1}{C_2^2\rho^{2kb} + 1}\right),$$
$$1 + 4ak > 0, \quad F < \frac{1}{k\rho}\left(\frac{m+1}{2k} + b\right), \tag{90}$$

$$F = \frac{1}{\rho}\left(\frac{m+1}{2k^2} + \frac{b}{k}\frac{C_2^2\rho^{2kb} + 1}{C_2^2\rho^{2kb} - 1}\right),$$
$$1 + 4ak > 0, \quad F > \frac{1}{k\rho}\left(\frac{m+1}{2k} + b\right), \tag{91}$$

$$F = \frac{1}{\rho}\left(\frac{m+1}{2k^2} + \frac{b}{k}\tan\ln\left(C_2\rho^{-kb}\right)\right),$$
$$1 + 4ak < 0, \tag{92}$$

$$H = \frac{1}{\rho}. \tag{93}$$

Now we will tidy up our notation. Recall that $a = k\left(\left(\frac{m}{2k}\right)^2 - C_H\right) + \frac{m}{2k}$. Then

$$1 + 4ak = 1 + m^2 - 4k^2C_H + 2m =$$
$$= (m+1)^2 - 4k^2C_H, \tag{94}$$

so the condition $1 + 4ak > 0$ reads $(m+1)^2 > 4k^2C_H$.

We denote

$$\beta = \sqrt{|1 + 4ak|} = \sqrt{|(m+1)^2 - 4k^2C_H|},$$

so $b = \frac{\beta}{2k}$. Then we have

$$F = \frac{1}{2k^2\rho}\left(m + 1 + \beta\frac{C_2^2\rho^{\beta} - 1}{C_2^2\rho^{\beta} + 1}\right),$$
$$(m+1)^2 > 4k^2C_H, \quad F < \frac{m + 1 + \beta}{2k^2\rho}, \tag{95}$$

$$F = \frac{1}{2k^2\rho}\left(m+1+\beta\frac{C_2^2\rho^\beta+1}{C_2^2\rho^\beta-1}\right),$$
$$(m+1)^2 > 4k^2C_H, \quad F > \frac{m+1+\beta}{2k^2\rho}, \tag{96}$$
$$F = \frac{1}{2k^2\rho}\left(m+1+\beta\tan\ln\left(C_2\rho^{-\frac{\beta}{2}}\right)\right),$$
$$(m+1)^2 < 4k^2C_H, \tag{97}$$
$$H = \frac{1}{\rho}, \tag{98}$$
$$\beta = \sqrt{|(m+1)^2 - 4k^2C_H|}. \tag{99}$$

The choice between the first two solutions depends on the initial condition (in ρ) for F, which can be expressed through the constant of integration C_2 (basically, take the initial condition $F(\rho_0)$ at some point ρ_0 and compare it with $\frac{m+1+\beta}{2k^2\rho_0}$). These two solutions can be combined in a single formula. Denote $C_2^2 = \frac{a_1}{a_2}$. Then, in both these solutions, a_1 and a_2 should have the same sign. However, these solutions differ only by the sign in front of C_2^2. So, if we allow a_1 and a_2 to have different signs, then we embrace both solutions simultaneously. We have

$$F = \frac{1}{2k^2\rho}\left(m+1+\beta\frac{a_1\rho^\beta-a_2}{a_1\rho^\beta+a_2}\right),$$
$$(m+1)^2 > 4k^2C_H, \tag{100}$$
$$F = \frac{1}{2k^2\rho}\left(m+1+\beta\tan\ln\left(C_2\rho^{-\frac{\beta}{2}}\right)\right),$$
$$(m+1)^2 < 4k^2C_H, \tag{101}$$
$$H = \frac{1}{\rho}, \tag{102}$$
$$\beta = \sqrt{|(m+1)^2 - 4k^2C_H|}. \tag{103}$$

Using the original substitution (30)–(32) and (44), we obtain

$$\Omega = kF, \tag{104}$$
$$\Phi = (m+1)F - C_H H, \tag{105}$$
$$\Psi = H. \tag{106}$$

Then, remembering that $\Omega = U'$, $\Phi = K'$, and $\Psi = W'/W$ and gathering all together, we obtain

$$U = \int \frac{1}{2k\rho}\left(m+1+\beta\frac{a_1\rho^\beta-a_2}{a_1\rho^\beta+a_2}\right)d\rho + C_3, \tag{107}$$
$$K = \int \frac{1}{2k^2\rho}\Big((m+1)^2 - 2k^2C_H +$$
$$+(m+1)\beta\frac{a_1\rho^\beta-a_2}{a_1\rho^\beta+a_2}\Big)d\rho + C_4, \tag{108}$$
$$W = C_5\rho, \tag{109}$$
$$(m+1)^2 > 4k^2C_H, \tag{110}$$
$$\beta = \sqrt{|(m+1)^2 - 4k^2C_H|}, \tag{111}$$

and C_3, C_4, and C_5 are constants of integration.

For $(m+1)^2 > 4k^2C_H$, we have

$$U = \int\frac{1}{2k\rho}\left(m+1+\beta\tan\ln\left(C_2\rho^{-\frac{\beta}{2}}\right)\right)d\rho + C_3, \tag{112}$$
$$K = \int \frac{1}{2k^2\rho}\Big((m+1)^2 - 2k^2C_H +$$
$$+(m+1)\beta\tan\ln\left(C_2\rho^{-\frac{\beta}{2}}\right)\Big)d\rho + C_4, \tag{113}$$
$$W = C_5\rho, \tag{114}$$
$$(m+1)^2 < 4k^2C_H, \tag{115}$$
$$\beta = \sqrt{|(m+1)^2 - 4k^2C_H|}. \tag{116}$$

The case $(m+1)^2 > 4k^2C_H$ can be integrated to the end. We have to calculate the integral

$$\int \frac{\beta}{2k\rho}\frac{a_1\rho^\beta - a_2}{a_1\rho^\beta + a_2}d\rho. \tag{117}$$

We rewrite it as

$$\int \frac{\beta}{2k\rho}\frac{a_1\rho^{\beta/2} - a_2\rho^{-\beta/2}}{a_1\rho^{\beta/2} + a_2\rho^{-\beta/2}}d\rho =$$
$$= \int \frac{\beta}{2k}\frac{a_1\rho^{\beta/2-1} - a_2\rho^{-\beta/2-1}}{a_1\rho^{\beta/2} + a_2\rho^{-\beta/2}}d\rho \tag{118}$$

and note that the function in the numerator is the derivative of the function in the denominator:

$$\frac{d}{d\rho}\left(a_1\rho^{\beta/2} + a_2\rho^{-\beta/2}\right) = \frac{\beta}{2}\left(a_1\rho^{\beta/2-1} - a_2\rho^{-\beta/2-1}\right). \tag{119}$$

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Eventually, we have

$$U = \frac{m+1}{2k} \ln\rho + \frac{1}{k} \ln\left(a_1\rho^{\beta/2} + a_2\rho^{-\beta/2}\right) + C_3, \quad (120)$$

$$K = \frac{(m+1)^2 - 2k^2 C_H}{2k^2} \ln\rho + $$
$$+ \frac{m+1}{k^2} \ln\left(a_1\rho^{\beta/2} + a_2\rho^{-\beta/2}\right) + C_4, \quad (121)$$

$$W = C_5\rho, \quad (122)$$

$$(m+1)^2 > 4k^2 C_H, \quad (123)$$

$$\beta = \sqrt{|(m+1)^2 - 4k^2 C_H|}. \quad (124)$$

The constants of integration C_3 and C_5 can be set to 0 and 1, respectively, by the choice of the coordinates t and φ.

6. Equilibrium Points of the System of Equations and the Energy Conditions

Here, we are going to find the equilibrium points of the systems of equations (45), (46) and (53), (54) and to study those of them, which are hyperbolic.

We start with system (45), (46). The equilibrium points of a system of ordinary first-order differential equations are points, in which all first derivatives of the unknown functions equal zero. Setting F' and H' equal to zero in (45), (46), we have

$$-k^2F^2 + (mF - C_H H)(H + (n+1)F) = 0, \quad (125)$$

$$-H(H + (n+1)F) = 0. \quad (126)$$

In this section, F and H will denote temporarily the equilibrium points of the system.

Solving this system, we find the following points: First and foremost, the equilibrium point at $F = 0$, $H = 0$. If k, m, and n satisfy the condition $k^2 = m(n+1)$, then there is a whole additional set of equilibrium points parametrized as $F = \mu$, $H = 0$, or, in other words, it is the axis F in the phase plane of the system.

It turns out that, about the point $F = 0$, $H = 0$, all eigenvalues of the linearization of the system (Jacobian matrix) equal zero, so this point is not hyperbolic, and the behavior of the solutions near this point can be complicated.

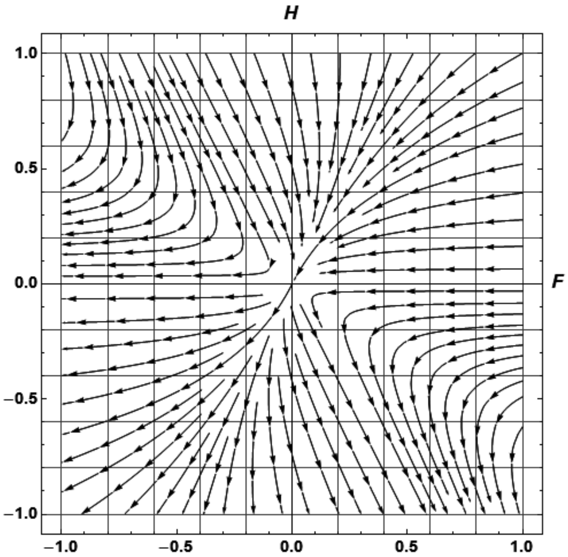

Fig. 1. Phase portraits of system (45), (46) in the case of a static electric field ($l = -1,\ m = -1,\ n = -1,\ k = -1$) for $C_H = -1/4$

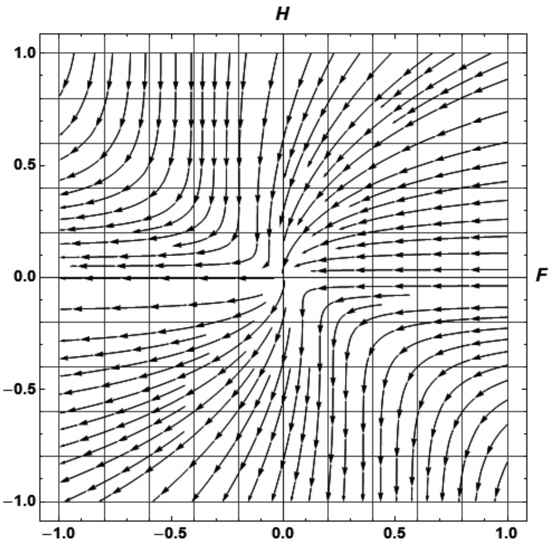

Fig. 2. Phase portraits of system (45), (46) in the case of a static electric field ($l = -1,\ m = -1,\ n = -1,\ k = -1$) for $C_H = 1/4$

In the case of the points $F = \mu$, $H = 0$, eigenvalues $\lambda(F, H)$ (eigenvalue λ at an equilibrium point (F, H)) are $\lambda_1(F, H) = 0$, $\lambda_2(F, H) = -\mu(n+ +1)$. The first zero eigenvalue indicates that we deal with a line of

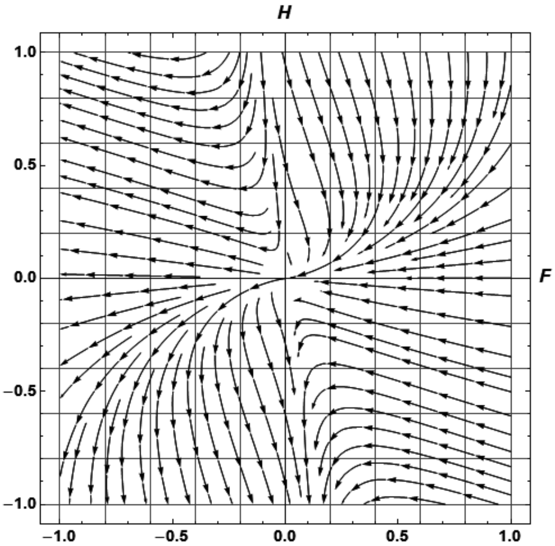

Fig. 3. Perfect fluid: $l = 1$, $m = 1$, $n = 1$, $k = 2$, $C_H = -1/4$

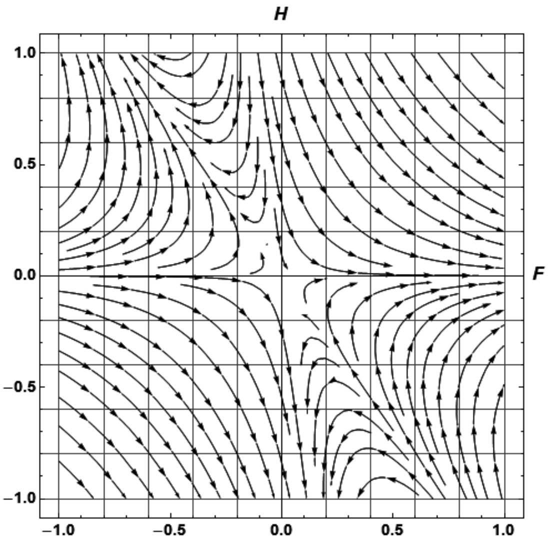

Fig. 4. Cosmological constant: $l = -1$, $m = 1$, $n = 1$, $k = 1$, $C_H = -1/4$

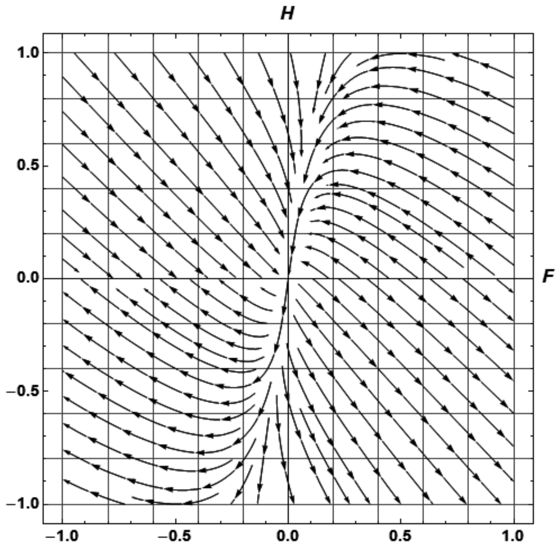

Fig. 5. Phase portrait of the system for $k = -2$, $m = -2$, $n = -3$, $C_H = -1/4$. Note the equilibrium points along the axis F

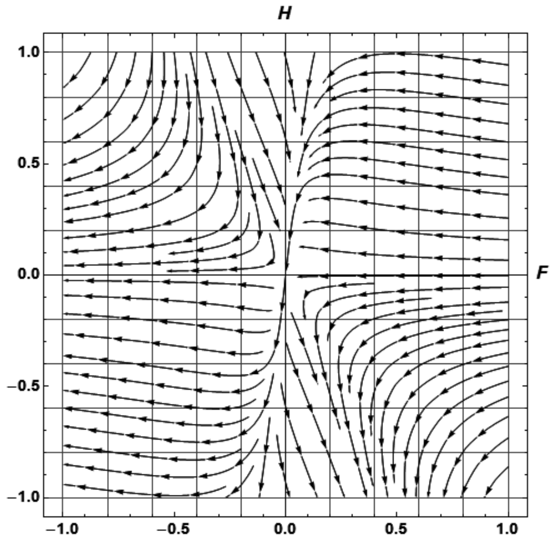

Fig. 6. Phase portrait of the system that differs only in k from the case in Fig. 5: $k = -3$, $m = -2$, $n = -3$, $C_H = -1/4$

equilibrium points. The values of the second eigenvalue indicate that a solution either approaches the corresponding point (negative values) or moves away from it (positive values), at least far away from the point $F = 0$, $H = 0$.

The set of equilibrium points of system (53), (54) is similar to that of the first system. The equilibrium points are $F = 0$, $H = 0$, and, if $k^2 = m(n + 1)$, also a set $F = 0$, $H = \mu$ (note that, in this case, it is the axis H).

The point $F = 0$, $H = 0$ has the zero eigenvalues of the corresponding Jacobian matrix. The eigenvalues on the axis H are the same $\lambda_1(F, H) = 0$, $\lambda_2(F, H) = -\mu(n + 1)$.

As we can see, if $k^2 = m(n + 1)$, then the system of equations acquires a new set of equilibrium points, so it is of interest to get at least a superficial idea of the behavior of the system in that case. In order to do this, we will consider several phase portraits of the system, which are computed numerically.

At first, we are going to look at phase portraits in the well-known cases: static electric field, perfect fluid, and solution with cosmological constant.

In order to do this, we also have to specify the constant of integration C_H. If we write down system (45), (46) in the case of a static electric field ($l = -1$, $m = -1$, $n = -1$, $k = -1$), then C_H must be negative. For example, $C_H = -1/4$ corresponds to the Mukherjee solution [29]. In Figs. 1 and 2, we present two cases for $C_H = -1/4$ and $C_H = 1/4$.

The next pair of phase portraits (Figs. 3 and 4) is the perfect fluid ($l = 1$, $m = 1$, $n = 1$, $k = 2$) and the system with the cosmological constant ($l = -1$, $m = 1$, $n = 1$, $k = 1$); $C_H = -1/4$ in both cases.

Figures 5 and 6 show the case where $k^2 = m(n+1)$ ($k = -2$, $m = -2$, $n = -3$, $C_H = -1/4$) in comparison with a case that differs from the former only in k ($k = -3$, $m = -2$, $n = -3$, $C_H = -1/4$).

The energy conditions [31] pose some restrictions on the physically sensible values of parameters l, m, and n. In our case, they lead to the following set of inequalities:

$$lp \geq 0, \tag{127}$$

$$|lp| \geq |p|, \tag{128}$$

$$|lp| \geq |mp|, \tag{129}$$

$$|lp| \geq |np|. \tag{130}$$

Dividing by $|p|$, we have

$$lp \geq 0, \tag{131}$$

$$|l| \geq |1|, \tag{132}$$

$$|l| \geq |m|, \tag{133}$$

$$|l| \geq |n|. \tag{134}$$

The first inequality simply sets the sign of l depending on the sign of p.

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С.Б. Григорьев, А.Б. Леонов

РІВНЯННЯ ЕЙНШТЕЙНА У ВИПАДКУ СТАТИЧНОЇ ЦИЛІНДРИЧНОЇ СИМЕТРІЇ ТА ДІАГОНАЛЬНИЙ ТЕНЗОР ЕНЕРГІЇ-ІМПУЛЬСУ ІЗ ВЗАЄМНО ПРОПОРЦІЙНИМИ КОМПОНЕНТАМИ

Р е з ю м е

Розглядаються рівняння Ейнштейна у випадку статичної циліндричної симетрії. Вибраний тензор енергії-імпульсу має вигляд діагональної матриці з взаємно пропорційними компонентами. Декілька відомих точних розв'язків задовольняють такі умови (розв'язок зі статичним електричним полем, частина розв'язків з ідеальною рідиною, розв'язок з космологічною сталою). Коефіцієнти пропорційності між компонентами тензора енергії-імпульсу виступають параметрами, що дозволяють вивчати більш загальний випадок (а також знаходити нові точні розв'язки для окремих значень коефіцієнтів). Розділення змінних дозволяє привести систему рівнянь до спрощеної системи з двох звичайних диференціальних рівнянь першого порядку. Знайдено точний розв'язок системи для широкого діапазону значень коефіцієнтів. Вивчено точки рівноваги системи рівнянь, на основі аналізу яких з'ясована якісна поведінка розв'язків для випадків простих станів рівноваги.