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MICROSCOPIC DESCRIPTION OF NONEXTENSIVE SYSTEMS IN THE FRAMEWORK OF THE ISING MODEL

To describe the behavior of nonextensive systems, the deformed Ising Hamiltonian is introduced by substituting the spin variable s_i by the deformed one s_i^q. In the framework of mean-field theory, the phase transition paramagnet–ferromagnet is investigated for the deformed partition function. The influence of the non-extensive parameter q on the free-energy density and the steady-state value of order parameter is studied in the Landau approximation.

Keywords: Ising model, Hamiltonian, order parameter.

1. Introduction

As long ago as in 1865, R. Clausius developed the concept and introduced the term "entropy" in the context of classical thermodynamics and taking no microscopic interaction into account. The properties of a system that naturally arose in the context of the Clausius concept include the extensivity (i.e. additivity) of system's entropy, which is connected with the number of system's elements at the microscopic level. L. Boltzmann and, later, J.W. Gibbs proposed the relation $S_{\mathrm{BG}} = -k\sum_{i=1}^{W} p_i \ln p_i$ that couples the Clausius entropy with system's microstates (here, W is the number of relevant microstates, p_i is the probability of the i-th state realization, and k is the Boltzmann constant). However, it turned out that the Boltzmann–Gibbs theory is not universal, but has a restricted scope of applications.

On the one hand, this theory is based on the assumption that all elements in the system are independent, which is a cornerstone of entropy additivity (extensivity). In addition, a hypothesis of molecular chaos was used, according to which the particles in the system do not correlate at all with one another before they collide. Of course, for the majority of macroscopic physical systems, the interaction forces between particles are short-range, extending only on a restricted number of the nearest neighbors. However, what about those complicated systems, in which long-range interaction reveals itself? Moreover, we have to take into account that a weaker chaos is usually realized at the microscopic level, and the sensitivity to external conditions grows according to the power law rather than the exponential one.

On the other hand, this theory describes a specific steady state, which is called the *thermodynamic equilibrium*. However, as is known, the complicated physical, biological, social, and other systems, which dominantly dwell in *nonequilibrium* stationary states, have attracted more and more attention recently. As a result, there arises a natural question: Can a more general theory be developed, a specific case of which–provided the thermodynamic equilibrium and the independence of system's elements–is the Boltzmann–Gibbs one. There is no unambiguous answer to this question till now. However, in 1988, C. Tsallis made an attempt to expand the scope of applicability for statistical mechanics and thermodynamics. As a result, a new direction of researches emerged, *nonextensive statistical mechanics and thermodynamics*. The basis of this approach is formed by the generalized expression for the entropy,

$$S_q = -k\sum_{i=1}^{W} p_i^q \ln_q p_i = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1},$$

which is reduced to the ordinary Boltzmann's entropy in the limit $q \to 1$, where q is the deformation parameter.

For today, the scope of nonextensive statistical mechanics applications is rather wide, which is evidenced by plenty of examples from various domains of science [1]. Every such a case deserves a special analysis, because some of them are proved experimentally and/or substantiated theoretically, whereas the others remain only phenomenological observations, when the deformation parameter q is determined by a direct substitution (mainly owing to the uncertainty that takes place in the microscopic world). At last, some cases can be described only at the probabilistic level because of the lack of relevant data. Moreover, one has to take into account that, in some examples, the parameter q governs the distribution degree; in others, it may be connected with the sensitivity to external conditions, the multifractal character, and so on.

In astrophysics and cosmology, the relations to the q-deformed theory were established for self-gravitating systems [2], the velocity distribution for spiral galaxies [3], the solar neutrino problem [4–7], cosmic microwave background radiation [8–10], and distribution of the cosmic ray energy [11]. In solid-state physics, the similar relations were revealed, while studying the high-temperature superconductivity [12], Bose–Einstein condensation [13], and electron strong coupling [14]. In nonlinear dynamics, a special attention is focused on the application of the Tsallis theory to three-dimensional turbulence [15]. In addition, in the framework of this theory, the Arrhenius law is obeyed for the abnormal diffusion as well [16]. The phenomenon of the self-organized criticality–in particular, for the model of biological evolution [17]–seems to be close to the concepts that arose in the framework of the existing nonextensive formalism.

One more example should be noted, where the statistical theory of nonextensive systems is applied. It is a description of objects with finite dimensions. The importance of their researches grew with the development of nanotechnologies. For instance, for the problem concerning the separation of a macrosystem into several parts, the accuracy of additivity persistence, making allowance for the difference between the surface energies for the whole system and their separated parts, was found to approximately equal to the size ratio between the atom and the system. Thus, the smaller the system dimension, the larger are the nonadditivity effects. Really, for a finite number of particles, N, the deformation parameter equals [18]

$$q = \left(1 - \frac{\alpha}{d} N^{-1}\right)^{-1}, \tag{1}$$

where α is the similarity index for the coordinate dependence of the Hamiltonian (e.g., for a harmonic oscillator, $\alpha = 2$), and d is the dimensionality of the system. The short-range potentials ($\alpha > 0$) are characterized by the values $q \geq 1$, whereas the long-range ones ($-d \leq \alpha \leq 0$) by $q \leq 1$. If $\alpha < -d$, the Boltzmann–Gibbs statistics is applicable [19]. In the thermodynamic limit, $N \to \infty$, we obtain the value $q = 1$ for the conventional statistics. As the number of particles, N, diminishes, the difference $|q - 1|$ grows and reaches the maximum value $\alpha/(d - \alpha)$ if $\alpha > 0$ or $|\alpha|/(d + |\alpha|)$ if $\alpha < 0$.

This work is devoted to the consideration of a microscopic theory of nonextensive systems in the framework of the Ising model. The article structure is as follows. In Section 2, the fundamentals of a q-deformed algebra, in the framework of which the formalism of nonextensive systems is built, are described. In Section 3, a q-deformed Ising Hamiltonian for the description of nonextensive systems is proposed, a relation for possible values of parameter q is obtained, and the average fractional value of spin is calculated. In Section 4, the partition function over all microstates of a nonextensive system and the free energy density are determined. The latter bring about the classical Landau expansion in the limit $q \to 1$. Section 5 is devoted to the analysis of the equilibrium order parameter.

2. Formalism of a Nonextensive Statistical System

Unlike the conventional statistical ensemble, the nonextensive system obeys Tsallis' statistics [1], in which the states are not distributed according to the Gibbs probability, but the escort one [20],

$$P_q(x) = \frac{p^q(x)}{\int p^q(x)\mathrm{d}x}, \quad p(x) = Z_q^{-1} \exp_q(x), \tag{2}$$

where the partition function Z_q is defined by the normalization condition for the initial probability $p(x)$. The latter is given, in turn, by the deformed Tsallis' exponent

$$\exp_q(x) := [1 + (1-q)x]_+^{\frac{1}{1-q}},$$

$$[y]_+ \equiv \max(y, 0), \tag{3}$$

which is reduced to the ordinary exponential function $e^x = \exp_1(x)$ in the limit $q \to 1$. Accordingly, Tsallis' logarithm, which plays the role of a function inverse to the exponential one, Eq. (3), is defined by the equality

$$\ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q}. \tag{4}$$

In addition, the sum, difference, product, and quotient operations of two positive quantities look like [21]

$$x \oplus_q y := x + y + (1-q)xy,$$
$$x \ominus_q y := \frac{x - y}{1 + (1-q)y},$$
$$x \otimes_q y := \left[x^{1-q} + y^{1-q} - 1\right]_+^{\frac{1}{1-q}},$$
$$x \oslash_q y := \left[x^{1-q} - y^{1-q} + 1\right]_+^{\frac{1}{1-q}}. \tag{5}$$

In particular, functions (3) and (4) satisfy the rules

$$\ln_q(x \otimes_q y) = \ln_q x + \ln_q y,$$
$$\ln_q(x \oslash_q y) = \ln_q x - \ln_q y,$$
$$\ln_q(xy) = \ln_q x \oplus_q \ln_q y,$$
$$\ln_q(x/y) = \ln_q x \ominus_q \ln_q y,$$
$$\exp_q(x) \otimes_q \exp_q(y) = \exp_q(x + y),$$
$$\exp_q(x) \oslash_q \exp_q(y) = \exp_q(x - y),$$
$$\exp_q(x) \exp_q(y) = \exp_q(x \oplus_q y),$$
$$\exp_q(x) / \exp_q(y) = \exp_q(x \ominus_q y). \tag{6}$$

The q-factorial

$$n!_q := 1 \otimes_q 2 \otimes_q \cdots \otimes_q n \tag{7}$$

of a natural integer $n \gg 1$ is determined by the Stirling formula [22]

$$\ln_q(n!_q) \simeq$$
$$\simeq \begin{cases} \left(\dfrac{n}{2-q} + \dfrac{1}{2}\right) \ln_q n - \dfrac{n-1}{2-q}, & 0 < q \neq 2; \\ \left[n - \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right)\right] - \ln n, & q = 2. \end{cases} \tag{8}$$

3. Master Equations

The Ising model forms a basis for the microscopic theory of phase transitions [23]. In contrast to the Heisenberg model, where the spin at every site of a regular lattice can acquire an arbitrary value (see Fig. 1, *b*), the advantage of the Ising model consists in its simplicity, i.e. only two values are supposed to be allowable for the spin at every site, $s_i = \pm 1$ (see Fig. 1, *a*). Nevertheless till now, there is no exact analytical description of the phase transition paramagnet–ferromagnet (PM–FM) in the framework of the Ising model in the three-dimensional case. However, this model allows the basic properties of the PM–FM phase transition to be described qualitatively in the framework of the mean-field approximation. But, as was already mentioned, the Gibbs–Boltzmann statistics is not suitable for the description of objects with finite dimensions, and the more general approach by Tsallis [1] has to be engaged in this case. The following question arises: How does such a "deformation" affect the phase transition scenario? From the formal point of view, the generalization of the Gibbs–Boltzmann statistics occurs owing to the substitution of the initial probability, p_i, by the deformed one, p_i^q. Let us extend this procedure at the microscopic level, i.e. let us use the deformed Ising Hamiltonian, which is obtained by substituting the spin s_i at the i-th site by its deformed variant, s_i^q.

In accordance with the formulated problem, the Hamiltonian acquires the form

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j}^{N} J_{i,j} s_i^q s_j^q - h \sum_i s_i^q. \tag{9}$$

Here, the summation is carried out over all N lattice sites with the indices $i \neq j$, J_{ij} is the potential of the effective interaction, h is the external field, $s_i = \pm 1$ is the spin value at every site, q is the parameter of

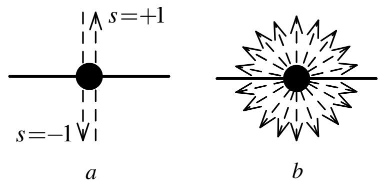

Fig. 1. Schematic diagram of possible spin directions at the sites of a regular lattice in the Ising (a) and Heisenberg models (b)

extensivity, and the factor $\frac{1}{2}$ takes into account that every site is summed up twice. Let us take into account that, in the framework of the mean-field approximation,

– the multiplier s_i^q is substituted by the averaged value $\langle s^q \rangle$,

– the main contribution to the Hamiltonian is made by only the interaction with the nearest neighbors, the number of which equals z;

– the potential of the effective interaction is reduced to a positive constant, $J > 0$, the sign of which determines the material type (FM).

As a result, the effective Hamiltonian looks like

$$\mathcal{H}_{\text{ef}} = \sum_i \varepsilon_i, \tag{10}$$

where

$$\varepsilon_i = -h_q s_i^q; \quad h_q = h + T_c \langle s^q \rangle. \tag{11}$$

In the last expression, the critical temperature $T_c = zJ$ is introduced. Then, the quantity ε_i can be regarded as the energy of the i-th site.

Since the Ising model has a discrete symmetry, being invariant under the transformations of spin, $s_i \to -s_i$, and external field, $h \to -h$, our theory makes sense provided that the condition $(-1)^q = -1$ is satisfied. Then, using the complex representation for -1, it is possible to obtain an additional condition for possible q-values,

$$q = \frac{2m+1}{2n+1}, \tag{12}$$

Some possible values of parameter q (see Eq. (12)) in the forms of ordinary and decimal fractions for various m and n

m	n	q	q
0	5	1/11	0.091
0	2	1/5	0.2
1	2	3/5	0.6
3	4	7/9	0.778
1	1	1	1
10	9	21/19	1.105
3	2	7/5	1.4
4	2	9/5	1.8
5	2	11/5	2.2
6	2	13/5	2.6

where m and n are integers $(m, n = 0, \pm 1, \pm 2, \ldots)$ (see Table).

For the further consideration, it is necessary to determine the average of $\langle s^q \rangle$ with fractional power exponent q. An analogous problem was examined in work [24]. For instance, if the variable s has an initial distribution $P(s)$, there can exist another distribution for the stochastic variable $x = s^q$, $P_q(x)$. In addition, we may assume that those two distributions are related to each other by the relation $s^q P(s) ds = x P_q(x) dx$. If we designate a quantity averaged over the distribution $P_q(x)$ as $\langle \ldots \rangle_q$ and use the ordinary angular brackets $\langle \ldots \rangle$ for the notation of the value calculated by averaging over the initial distribution $P(s)$, the relation $\langle s^q \rangle = \langle x \rangle_q$ can be obtained. As was proved in work [24], this problem makes sense only in the case of self-similar systems, where the distribution function has a power-law dependence, i.e. $P(s) = N_p^{-1} s^{-\mu}$, where μ is the power exponent, and N_p is the normalization constant,

$$N_p = \frac{1}{|1-\mu|} a^{\mu - 1}, \quad a \to 0. \tag{13}$$

Let us analyze the power exponent μ. From work [24], it is known that, if $1 < \mu < 2$, so that the fractal dimensionality of the phase space $D = 2 - \mu$ is less than 1, the system is always disordered. But, in the case $0 < \mu < 1$, the fractal dimensionality $D > 1$, and the system can be in the ordered state. Therefore, the further consideration of all possible dependences for the equilibrium value of order parameter will be carried out only in the interval $\mu \in (0, 1)$. As a result, the fractional average can be presented in the form

$$\langle s^q \rangle = \beta^{-1} \left(N_p (2-\mu) \right)^{\beta - 1} \langle s \rangle^{\beta}, \tag{14}$$

where the notation

$$\beta \equiv \frac{q + 1 - \mu}{2 - \mu}. \tag{15}$$

is introduced.

In the initial microscopic theory developed on the basis of the Ising model, the order parameter distinguishing between the disordered (PM) and ordered (FM) states is the averaged spin, $\langle s_i \rangle = \langle s \rangle = \eta$. In the case of nonextensive systems, it is reasonable to use this relation again. As a result, we obtain the effective Hamiltonian of a nonextensive system in the

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form

$$\mathcal{H}_{\rm ef} = -\sum_i \left(h + CT_c\eta^{\beta}\right) s_i, \tag{16}$$

where

$$C \equiv \beta^{-1} \left(N_p(2-\mu)\right)^{\beta-1}. \tag{17}$$

Since Hamiltonian (16) is negative, zero cannot be selected as the reference point; otherwise, the system would have had an infinitely larger negative energy in the ordered state. To avoid it, the self-action has to be taken into account, which reflects the Le Chatelier principle. For the first time, it was done by Academician M.M. Bogolyubov in the 1950s, while explaining the phenomenon of superfluidity at the microscopic level. In his theory, the Hamiltonian component that reflects the action is given by the square-law term

$$\mathcal{H}_0 = \frac{N}{2} T_c\,\eta^2. \tag{18}$$

The final form of the effective Hamiltonian of the system looks like

$$\mathcal{H}_{\rm ef} = \frac{N}{2} T_c\,\eta^2 - h\sum_i s_i - CT_c\,\eta^{\beta}\sum_i s_i. \tag{19}$$

4. Construction of Phenomenological Theory on the Basis of a Microscopic Scenario

The phenomenological theory of phase transitions developed by L.D. Landau is known to allow a relation for the equilibrium order parameter $\eta \equiv \langle s\rangle$ to be obtained. In the problems of this kind, the average value is determined using the Gibbs distribution

$$P\{s_i\} = Z^{-1} \exp\left(-\frac{\mathcal{H}_{\rm ef}}{T}\right), \tag{20}$$

where Z is the partition function, $\mathcal{H}_{\rm ef}$ is Hamiltonian (19), and T is the temperature expressed in terms of energy units.

First, let us determine the partition function for all sets of possible spin orientations at all sites,

$$Z = Z_0 \sum_{\{s_i\}} \exp\left(\alpha \sum_i s_i\right). \tag{21}$$

Here,

$$Z_0 \equiv \exp\left(-\frac{N}{2}\frac{T_c}{T}\eta^2\right), \tag{22}$$

$$\alpha \equiv \frac{h}{T} + C\frac{T_c}{T}\eta^{\beta}. \tag{23}$$

The exponential function in formula (21) includes the sum over all sites. Therefore, it is possible to express it as a product of site exponents and, afterward, rearrange it with the sum over all spin sets. As a result, we obtain the expression

$$Z = Z_0 \prod_{i=1}^{N} \sum_{s_i} \exp(\alpha s_i), \tag{24}$$

in which only those terms were left that are relevant for the given i-th site, i.e. $\{s_i\} \to s_i$. Since $s_i = \pm 1$, the sum in Eq. (24) can be easily calculated,

$$Z = Z_0 (2\cosh\alpha)^N. \tag{25}$$

Substituting relations (22) and (23) into Eq. (25), we obtain

$$Z = \exp\left(-\frac{N}{2}\frac{T_c}{T}\eta^2\right)\left[2\,{\rm ch}\left(\frac{h}{T} + C\frac{T_c}{T}\eta^{\beta}\right)\right]^N. \tag{26}$$

From thermodynamics, it is known that the free energy is connected with the partition function by the relation

$$F = -T\ln Z. \tag{27}$$

Substituting the partition function of the nonextensive system (26) into Eq. (27), we obtain the ultimate expression for the free energy of the nonextensive system,

$$F = \frac{N}{2}T_c\eta^2 - TN\ln 2 - TN\ln\cosh\left(\frac{h}{T} + C\frac{T_c}{T}\eta^{\beta}\right). \tag{28}$$

Here, the first term corresponds to the energy of the action itself. The second term does not depend on the order parameter, but plays an important role, because it determines the energy decrease associated with the disorder growth. The last term plays the main role in Eq. (28). It cannot be reduced to a standard form, e.g., the Landau one. However, it should not be so, because energy (28) was obtained from the microscopic theory and remains valid at any temperature T; whereas, first, the Landau phenomenological theory is valid in a vicinity of the critical temperature

T_c, and, second, it does not take the nonadditivity property into account.

To build a phenomenological theory, it is necessary to expand the hyperbolic cosine and the logarithm in Eq. (28) in series. As a result, we obtain the following general formula for the free energy:

$$F = \frac{N}{2} T_c \eta^2 - TN \ln 2 -$$
$$-\frac{TN}{2}\left[\left(\frac{h}{T} + C\frac{T_c}{T}\eta^\beta\right)^2 - \frac{1}{6}\left(\frac{h}{T} + C\frac{T_c}{T}\eta^\beta\right)^4\right]. \quad (29)$$

Now, let us introduce a notation for the free energy density per one site,

$$f = \frac{F}{N}. \quad (30)$$

In the approximation $h = 0$, Eq. (29) can be rewritten in the form

$$f = -T\ln 2 + \frac{T_c}{2}\eta^2 - \frac{C^2T_c^2}{2T}\eta^{2\beta} + \frac{C^4T_c^4}{12T^3}\eta^{4\beta}. \quad (31)$$

The first term in Eq. (31) arose due to a reduction of the free energy owing to the transition of the system from the disordered state into the ordered one (as a change of the free energy, only the difference in the reference points for the ordered and disordered phases is meant). This term does not depend on the order parameter, which is not typical of the phenomenological theory, because it contains excess information that cannot be obtained with the use of approximate methods. Hence, the term $-T\ln 2$ can be neglected in the phenomenological case. As a result, we obtain

$$f = \frac{T_c}{2}\eta^2 - \frac{C^2T_c^2}{2T}\eta^{2\beta} + \frac{C^4T_c^4}{12T^3}\eta^{4\beta}. \quad (32)$$

Bearing in mind that every relation used in the framework of the nonextensive mechanics acquires the corresponding "classical" form in the limit $q \to 1$, let us analyze how the energy density (32) looks in this case,

$$f_{q\to 1} = \frac{T - T_c}{2}\eta^2 + \frac{1}{4}\frac{T_c}{3}\eta^4. \quad (33)$$

Here, we took into account that, according to Eq. (17), the coefficient $C \equiv 1$ as $q \to 1$. Comparing Eq. (33) with the Landau series expansion

$$f_L = \frac{A}{2}\eta^2 + \frac{B}{4}\eta^4,$$

we may write down that, in our case, $B = T_c/3$ and $A = \alpha(T - T_c)$, where $\alpha \equiv 1$.

Omitting the term proportional to $\ln 2$, let us consider the free energy density obtained from Eq. (29) in the limit as $q \to 1$,

$$f_{q\to 1} = \frac{T - T_c}{2}\eta^2 + \frac{1}{4}\frac{T_c}{3}\eta^4 - \eta h + O(h, \eta). \quad (34)$$

Since the term designated as $O(h, \eta)$ contains products of h and η with power exponents larger than 1, those terms can be neglected in the weak-field approximation. As a result, Eq. (34) is reduced to the standard Landau form for the free energy that makes allowance for the external field,

$$f_L = \frac{A}{2}\eta^2 + \frac{B}{4}\eta^4 - \eta h. \quad (35)$$

Hence, in the limit as $q \to 1$, relations (33) and (35) bring about the conventional Landau phenomenological theory.

5. Equilibrium Order Parameter Value

From Eq. (32), according to the equilibrium condition $\frac{\partial f}{\partial \eta} = 0$ and for the case of the zero external field, we obtain

$$\eta\frac{T_c}{T}\left[T - \beta C^2 T_c \eta^{2(\beta-1)} + \frac{\beta C^4 T_c^3}{3T^2}\eta^{2(2\beta-1)}\right] = 0. \quad (36)$$

The first stationary solution, $\eta_0 = 0$, evidently corresponds to a disordered state. By putting the expression in the brackets equal to zero and by dividing it by the critical temperature T_c, we obtain the equation for the stationary order parameter in the ordered state,

$$\frac{T}{T_c} - \beta C^2 \eta_0^{2(\beta-1)} + \frac{\beta C^4}{3}\frac{T_c^2}{T^2}\eta_0^{2(2\beta-1)} = 0. \quad (37)$$

By analogy with the Landau theory, let us consider our system near the critical temperature, i.e. in the case where $|T - T_c| \ll T_c$. In addition, let us introduce the dimensionless temperature

$$\theta = \frac{T}{T_c}. \quad (38)$$

As a result, one can obtain the dependence $\theta(\eta_0)$,

$$\theta = \beta C^2 \eta_0^{2(\beta-1)}\left(1 - \frac{C^2}{3}\eta_0^{2\beta}\right). \quad (39)$$

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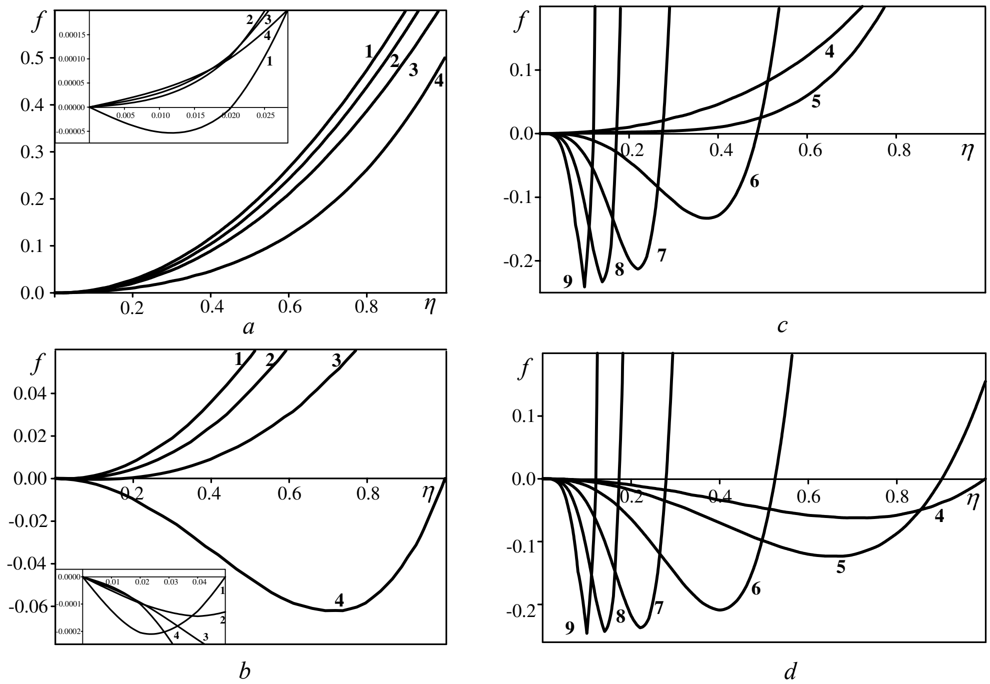

Fig. 2. Dependences of the dimensionless free energy (41) on the order parameter η for $a = 0.001$ and $\mu = 0.5$. The dimensionless temperature $\theta = 0.5$ (panels *a* and *c*) and 1.5 (panels *b* and *d*). The parameter $q = 1/5$ (*1*), 3/5 (*2*), 7/9 (*3*), 1 (*4*), 21/19 (*5*), 7/5 (*6*), 9/5 (*7*), 11/5 (*8*), and 13/5 (*9*)

Taking into account that the critical temperature T_c was selected as a temperature measure unit, it would be natural to select the corresponding units for other variables as well. For the order parameter, we may use the well-known relation of the Landau theory, $\eta_{0L} = (-\varepsilon)^{1/2}$, which reflects the dependence of the equilibrium order parameter on the relative temperature $\varepsilon \equiv \frac{T-T_c}{T_c}$. Comparing η_{0L} with η_0 from Eq. (39) as $q \to 1$, we obtain the measure unit for the order parameter, $\eta_s \equiv \sqrt{3}$. We choose the unit $f_s \equiv \eta_s^2 T_c$ for the free energy density and $h_s \equiv T_c$ for the field. Then, the reduced (dimensionless) variants of Eqs. (32) and (39) read

$$f = \frac{\theta}{2}\eta^2 - \frac{3^{\beta-1}C^2}{2}\eta^{2\beta} + \frac{3^{2(\beta-1)}C^4}{4}\eta^{4\beta}, \tag{40}$$

$$\theta = \beta C^2 \eta_0^{2(\beta-1)}\left(1 - \frac{C^2}{3}\eta_0^{2\beta}\right). \tag{41}$$

The corresponding dependences are depicted in Figs. 2 and 3, respectively.

As one can see from Fig. 2, the more the parameter of nonextensivity differs from the critical value $q = 1$, the smaller value of the order parameter is needed for the ordered state to be realized. Moreover, as curve *1* (see the inset in Fig. 2, *a*) and curves *6* to *9* (see Fig. 2, *c*) testify, the ordered state can be realized even at temperatures lower than the critical one.

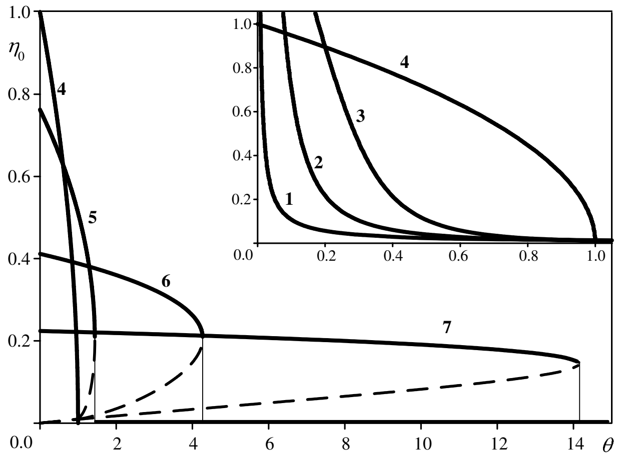

Fig. 3. Dependences of the equilibrium order parameter η_0 on the dimensionless temperature θ for $a = 0.001$ and $\mu = 0.5$. The parameter $q = 1/5$ (*1*), 3/5 (*2*), 7/9 (*3*), 1 (*4*), 21/19 (*5*), 7/5 (*6*), 9/5 (*7*)

In Fig. 3, the temperature dependences of the equilibrium order parameter are depicted for various q-values. One can see that, at $q < 1$, the character of the dependence changes (cf. curves *1* to *3* in Fig. 3 with curve *4* in the same figure). It should also be noted that the largest values of order parameter for curves *2* and *3* are attained at temperatures in the interval $(0.1 \div 0.2)T_c$, but this result can be explained by the fact that dependence (32) was obtained in the approximation $|T - T_c| \ll T_c$. While analyzing curves *5* to *7* in Fig. 3, a conclusion can be drawn that the equilibrium value of order parameter undergoes a jump. This means that the phase transition in nonextensive systems can be realized by the mechanism of a phase transition of the first kind (this conclusion can be verified only experimentally). An alternative consists in that the values $q > 1$ of this parameter are not realized in this case.

6. Conclusions

Recently, a lot of theorists, while studying complicated systems, have been attempting, more and more often, to generalize the Boltzmann–Gibbs statistics with the help of Tsallis' approach. This generalization made it possible to describe many phenomena and effects, which were observed earlier, but have not been interpreted theoretically. In the present work, Tsallis' approach is applied to the Ising Hamiltonian and to the partition function describing a microscopic system consisting of a set of spin variables. As a result, it was demonstrated that, in the framework of the mean-field approximation, the microscopic consideration can give rise to a phenomenological approach that characterizes the transition paramagnet–ferromagnet in nonextensive systems.

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МІКРОСКОПІЧНИЙ ОПИС
НЕЕКСТЕНСИВНИХ СИСТЕМ
У РАМКАХ МОДЕЛІ ІЗІНГА

Р е з ю м е

Деформований гамільтоніан Ізінга для опису поведінки неекстенсивних систем було представлено шляхом заміни спінової змінної s_i на деформовану s_i^q. У рамках теорії середнього поля було досліджено фазовий перехід парамагнетик–феромагнетик для деформованої статсуми. У наближенні Ландау було проаналізовано вплив параметра неекстенсивності q на густину вільної енергії та на стаціонарне значення параметра порядку.