CORRELATION FUNCTIONS OF A CHARGED SCALAR FIELD IN THE BACKGROUND OF NONCOMMUTATIVE $U(1)$ GAUGE FIELD

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We consider a complex charged scalar field coupled to a constant background non-commutative $U(1)$ gauge field and calculate the correlation function of two gauge-invariant composite operators. This calculation illustrates an interplay between the gauge transformations in gauge theories on noncommutative spaces and a space-time geometry. We show that the noncommutative gauge invariance is restored for higher-order correlators, though the Green's function itself is not invariant. The correlation functions reveal a singular behavior in the case where the Seiberg–Witten map becomes singular; i.e., there is no equivalent commutative description.

1. Introduction

Recently, noncommutative gauge theories drew attention owing to a certain progress achieved in the string theory [1]. In their work [2], Seiberg and Witten proved that the theory of open strings in a strong background B-field gives an effective noncommutative field theory on the D-brane worldvolume. It is one of the stimulating reasons to study the interaction between the commutative and ordinary field theories, because such a research can make our understanding of interaction between open and closed strings deeper.

In this work, we study the theory of a noncommutative scalar field that interacts with a background gauge one. The latter has a constant strength, being not dynamic. Despite a seeming simplicity, this system allows some effects connected with gauge transformations in noncommutative theories and the Seiberg–Witten map (the correspondence between the ordinary and noncommutative descriptions of the same system) to be illustrated.

We use the operators of scalar fields to construct gauge invariant operators and calculate the correlation functions for the latter. The result is radially symmetric, though the intermediate calculation stages contain an arbitrary coordinate scaling depending on the choice of background field gauge. This symmetry "restoration" takes place owing to the redefinition of gauge invariant operators in accordance with gauge transformations in the noncommutative theory. It is worth noting that these correlation functions become singular just when the Seiberg–Witten map diverges. This occurs at such a strength of the background gauge field, when there is no equivalent ordinary theory (i.e. a theory on the ordinary space).

In the next section, we make a review of notions for the noncommutative space and the deformation quantization. In particular, their application is considered in the context of quantum mechanics, where they are taken from. Really, the simplest way for the noncommutativity of the phase space in quantum mechanics to reveal itself is the context of deformation quantization (the Moyal $\star$-product). In this work, we study a noncommutativity of the Moyal type in the coordinate space. In Section 3, we expound the emergence of spatial noncommutativity from a definite string theory, as it was done in work [2]. In Section 4, we formulate the classical gauge theory with matter fields on the noncommutative space and introduce the Seiberg–Witten map [2], which represents an equivalence between noncommutative and ordinary gauge theories. In Section 5, we construct gauge invariant operators ("observables") and calculate their correlation functions. The background field in our calculations has a constant strength. Such a free (quadratic in the scalar field ϕ) field theory is equivalent, in some respect, to the effective conventional theory. This equivalence becomes especially transparent from the physical viewpoint, when the vector potential of a background gauge field is a linear function [4]. In this case, the construction of an effective theory on the ordinary space demands for a certain gauge-dependent coordinate scaling. That is why the definition of gauge invariant correlation functions includes the $\star$-product rather than the usual one. We hope that just the $\star$-product will provide the symmetry restoration for correlation functions. The gauge invariance of redefined correlation functions is verified by direct calculations using a formal spectral definition of Green's functions. The two-point gauge invariant function is calculated in the form of a series, in which every term is explicitly gauge invariant and radially symmetric.

2. Noncommutative Spaces: a Brief Review

In this section, we examine the coordinate noncommutativity; mainly, in connection with quantum mechanics. Nevertheless, the idea that the coordinate noncommutativity manifests itself as a certain deformation of the algebra of functions is universal. This concept expands far beyond the scope of quantum-mechanical applications, which are dealt with in this section. Historically, the ideas of deformation quantization were formulated for the first time in the quantum-mechanical context, and that is why we use it in this section.

Fields in the noncommutative theory acquire values in the deformed algebra of functions, where the ordinary pointwise product is replaced by the associative noncommutative Moyal $\star$-product. There is a Weyl–Moyal correspondence (the "WM-correspondence"), which is a convenient tool for making calculations in the noncommutative field theories. The WM-correspondence is an isomorphism between the deformed algebra of functions on a noncommutative manifold with a constant noncommutativity matrix θ^{ij} (i.e. an algebra, in which fields in the noncommutative theory get values) and the algebra of operators in an auxiliary Hilbert space. In this work, we widely apply the WM-correspondence, because we study physical systems that arise in the string theory after taking the Seiberg–Witten limit (it is a coordinate noncommutativity of the Moyal type that is inherent to these systems).

As was said above, noncommutative spaces are met in the physical context for the first time in quantum mechanics. The canonical quantization requires that the coordinates (x, p) in the phase space be substituted by differential operators $(\hat{x}, \hat{p})$, which satisfy the commutation relation

$$\hat{x}\hat{p} - \hat{p}\hat{x} = \mathrm{i}\mathbb{1}. \tag{1}$$

A natural question arises: Can the quantum mechanics be so formulated that the transition to the quasi-classical regime would be the most transparent? The answer to this question is given by deformation quantization [1].

In order to construct a $\star$-product, it is necessary to formulate a "rule of ordering", which allows the operator $\hat{\mathcal{O}}_f(\hat{x}, \hat{p})$ to be unambiguously derived from a function $f(x, p)$ defined on the phase space. We use a symmetric ordering ("Weyl's ordering"), which corresponds to

$$xp \to \frac{1}{2}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right). \tag{2}$$

For arbitrary functions, this technique can be formulated as follows. Let a classical observable $f(x, p)$ be presented in the form of its Fourier transform,

$$f(x,p) = \int \frac{d^2k}{2\pi}\tilde{f}(k_x, k_p)\mathrm{e}^{\mathrm{i}(k_x x + k_p p)}. \tag{3}$$

Then, the corresponding quantum-mechanical operator is calculated by the formula

$$\hat{\mathcal{O}}_f = \int \frac{d^2k}{2\pi}\tilde{f}(k_x, k_p)\mathrm{e}^{\mathrm{i}(k_x \hat{x} + k_p \hat{p})}. \tag{4}$$

The application of another ordering changes the form of the exponential function on the right-hand side of this equation. For instance, the xp-ordering gives rise to the substitution $\mathrm{e}^{\mathrm{i}(k_x x + k_p p)} \to \mathrm{e}^{\mathrm{i}k_x\hat{x}}\mathrm{e}^{\mathrm{i}k_p\hat{p}}$. We confine the consideration to the symmetric ordering. The fixed rule of ordering establishes a mutually unambiguous correspondence between classical observables (functions on the phase space) and quantum-mechanical operators.

Quantum-mechanical operators can be presented as pseudo-differential operators defined in a Hilbert space of wave functions $\mathcal{H}$. In the coordinate representation, the kernel of such a Hilbert–Schmidt operator is

$$K_f(x,y) \equiv \langle x|\hat{\mathcal{O}}_f|y\rangle = \int\int \frac{dz\,dp}{2\pi} f(z,p)\times$$

$$\times e^{\mathrm{i}p(x-y)}\delta\left(z - \frac{x+y}{2}\right). \tag{5}$$

There is also an inverse formula, which allows a classical observable to be restored knowing the kernel of the corresponding quantum operator:

$$f(z,p) = \int\int dx\,dy\,K(x,y)\mathrm{e}^{-\mathrm{i}p(x-y)}\delta\left(z - \frac{x+y}{2}\right). \tag{6}$$

In this case, the real-valued functions $f(x, p)$ correspond to Hermitian operators $\hat{\mathcal{O}}_f$ and *vice versa*. Generally speaking, the complex conjugation of a function $f(x, p)$ is associated with the Hermitian conjugation of the corresponding quantum-mechanical operator:

$$\hat{\mathcal{O}}_{\bar{f}} = \hat{\mathcal{O}}_f^{\dagger}. \tag{7}$$

Another important property of the Weyl–Moyal correspondence is a capability to calculate the traces of operators in terms of integrals over the phase space:

$$\mathrm{tr}_{\mathcal{H}}\hat{\mathcal{O}}_f = \int\int \frac{dx\,dp}{2\pi} f(x,p). \tag{8}$$

After establishing the correspondence between the classical and quantum-mechanical observables, quantum mechanics can be formulated in terms of functions on the phase space. Really, let the $\star$-product of two functions f and g on the phase space be defined as follows:

$$\hat{\mathcal{O}}_{f\star g} = \hat{\mathcal{O}}_f \circ \hat{\mathcal{O}}_g \,. \tag{9}$$

Such a $\star$-product looks like (we explicitly write Planck's constant):

$$(f \star g)(x,p) = f \mathrm{e}^{\mathrm{i}\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)/2} g \,. \tag{10}$$

The arrows over the derivative operators define whether they act on the function from the left or from the right (i.e., on f or g, respectively). $\star$-product (10) is known as the Weyl–Moyal product, since it corresponds to the symmetric ordering routine (4). To an accuracy of the first order of $\hbar$, this $\star$-product can be expressed in terms of the ordinary product and the Poisson bracket $\{\cdot,\cdot\}$:

$$f \star g = fg + \frac{\mathrm{i}\hbar}{2}\{f,g\} + O(\hbar^2) \,. \tag{11}$$

In the quasi-classical regime ($\hbar \ll 1$), the $\star$-product becomes an ordinary one, and the commutator is reduced to the Poisson bracket:

$$[f,g]_\star \equiv f \star g - g \star f \;=\; \mathrm{i}\hbar\{f,g\} + O(\hbar^3) \,. \tag{12}$$

The Weyl–Moyal correspondence is a partial case of the deformation quantization problem. The general problem of deformation quantization consists in finding a one-parametric (with the parameter $\hbar$) associative deformation of the function algebra, which reproduces the Poisson bracket in the quasi-classical regime ($\hbar \to 0$) [7]. A remarkable property of the Moyal case is that the Poisson bracket of coordinate functions is constant.

The formulation of quantum mechanics in terms of the deformation quantization should be appended by a calculation rule for observables. Although the concept of a wave function cannot be reformulated in terms of the phase space only and, as a consequence, does not exist in the deformation quantization context, the average values of observables can be calculated with the help of the density matrix operator $\hat{\rho}$. The inverse transform of this operator at the symmetric ordering is widely known as the Wigner function $W(x,p)$. Equation (6) allows one to calculate the Wigner function for the known density matrix. For a pure state with a (normalized) wave function $\psi(x)$, the corresponding Wigner function is

$$W(x,p) = \int d\xi \, \mathrm{e}^{\mathrm{i}p\xi} \bar{\psi}\left(x + \frac{\xi}{2}\right) \psi\left(x + \frac{\xi}{2}\right) . \tag{13}$$

The average value of an observable f can be calculated as follows:

$$\langle f \rangle \equiv \mathrm{tr}\,_{\mathcal{H}} \hat{\mathcal{O}}_W \hat{\mathcal{O}}_f \;=\; \int\!\!\int \frac{dx\,dp}{2\pi} W(x,p) f(x,p) \,. \tag{14}$$

In general, the following formula takes place:

$$\int\!\!\int \frac{dx\,dp}{2\pi} f \star g = \int\!\!\int \frac{dx\,dp}{2\pi} fg \,. \tag{15}$$

It is this property that allows the $\star$-product 0n the right-hand side of expression (14) to be substituted by the ordinary product. Note that the interpretation of the Wigner function as a probability density on the phase space in the context of Eq. (14) is impossible, because it can acquire negative values. Only after integrating over x (or p), the Wigner function becomes non-negative. Generally speaking, the capability to calculate average values is an exclusive property of Moyal quantization. Such a calculation cannot be carried out in the general context of deformation quantization [7].

3. Emergence of a Noncommutative Space in the String Theory

In this section, we consider a connection between certain regimes in the string theory and the emergence of a noncommutative space in the field theory. In general, the emergence of noncommutative field theories in the string theory was considered in works [2, 9] in detail.

The most important component that is responsible for the emergence of noncommutativity is the Neveu–Schwarz 2-form B. At its propagation, a string traces out a two-dimensional worldsheet Σ (unlike a one-dimensional world line in the case of a particle propagation). The interaction of a string with the background B-field stems from the term

$$\int_\Sigma B \,. \tag{16}$$

Geometrically, the structure of this interaction reminds the interaction of a gauge field with a point-like particle. A unique difference is that, for a particle, according to the dimensionality of its world line, the interaction with a 1-form A (the electromagnetic field gauge potential) is natural.

In terms of coordinate fields, the action functional for a string that propagates in a flat space with the metrics g_{ij} is

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{ij}\partial_a x^i \partial^a x^j - \frac{\mathrm{i}}{2}\int_\Sigma B_{ij}\epsilon^{ab}\partial_a x^i \partial_b x^j \,. \tag{17}$$

The second term represents the interaction between the string and the B-field. Similarly to what was done in work [2], we consider the case of a constant field B (i.e. it does not depend on the coordinates x^i). This condition demands that the form of B should be given accurately:

$$B = \frac{1}{2}B_{ij}\, dx^i \wedge dx^j \ = \ d\left(\frac{1}{2}B_{ij}x^i dx^j\right) \ \equiv \ dA\,. \qquad (18)$$

Hence, for an open string, the ends of which are attached to a D-brane, the interaction with the B-field is reduced to a linear integral along the (oriented) boundary of a worldsheet. Really, we may take advantage of the Stokes theorem,

$$\int_\Sigma B = \int_{\partial\Sigma} A\,. \qquad (19)$$

Then, the string ends behave as two charges that form a dipole.

The equations of motion determine boundary conditions for the coordinates with indices i along the brane,

$$g_{ij}\partial_n x^j + 2\pi\mathrm{i}\alpha' B_{ij}\partial_t x^j|_{\partial\Sigma} = 0\,. \qquad (20)$$

In this equation, ∂_n is a normal derivative, and ∂_t a derivative in the direction tangent to the Σ-boundary. Note that, in the case of a strong B-field, mixed boundary conditions become of the Dirichlet type.

In work [2], a worldsheet of a string with the topology of a disk was studied. The propagator calculated at two points located at the disk boundary is

$$\langle x^i(\tau)x^j(\tau')\rangle = -\alpha' G^{ij}\log(\tau-\tau')^2 +$$

$$+\frac{\mathrm{i}}{2}\theta^{ij}\,\mathrm{sign}(\tau-\tau')\,. \qquad (21)$$

In new notations, the effective metrics is

$$G^{ij} = -(2\pi\alpha')^2\left[\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right]^{ij}; \qquad (22)$$

and the matrix of noncommutativity looks like

$$\theta^{ij} = -(2\pi\alpha')^2\left[\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right]^{ij}. \qquad (23)$$

The explicit form of the propagator demonstrates that the commutator of two coordinate fields is nonzero,

$$[x^i(\tau), x^j(\tau)] = \mathrm{i}\theta^{ij}\,. \qquad (24)$$

Important is the Seiberg–Witten (SW) regime, when α' is small (or the momenta are small in the string propagation space).[1] In this regime, the operator product is

$$\mathrm{e}^{\mathrm{i}px}(\tau)\,\mathrm{e}^{\mathrm{i}qx}(\tau') = \mathrm{e}^{\mathrm{i}px}\star\mathrm{e}^{\mathrm{i}qx}(\tau'), \qquad (25)$$

where the $\star$-product is the Moyal $\star$-product

$$f_1\star f_2(x) = f_1(x)\mathrm{e}^{\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}f_2(x) \qquad (26)$$

with the matrix of noncommutativity

$$\theta^{ij} = (B^{-1})^{ij}\,. \qquad (27)$$

The low-energy dynamics of open strings in the SW-regime corresponds to the noncommutative field theory.

As was explained above, a constant B-field in the string propagation space is equivalent to a magnetic field A on the brane worldvolume. From this point of view, the coordinate noncommutativity is not a feature that is inherent to the string theory only. Really, let us consider the action functional for a charged particle in a magnetic field,

$$S = m\int\frac{\dot{x}^2}{2}\,dt - \int A_i dx^i\,. \qquad (28)$$

In the regime of a strong magnetic field, the second term dominates, and the action functional becomes simpler,

$$S = -\int A_i\dot{x}^i dt\,. \qquad (29)$$

The canonical momenta are

$$p_i \equiv \frac{\partial\mathcal{L}}{\partial\dot{x}^i} \ = \ -A_i(x)\,. \qquad (30)$$

For the constant magnetic field B, we can select the gauge

$$A_i = -\frac{1}{2}B_{ij}x^j\,, \qquad (31)$$

and then the Poisson bracket for two coordinates becomes nonzero,

$$\{x^i, x^j\} = \left(\frac{1}{B}\right)^{ij}. \qquad (32)$$

[1] More exactly, in the SW-regime, it is necessary to pass to the limits $\alpha' \sim \epsilon^{1/2} \to 0$ and $g_{ij} \sim \epsilon \to 0$ along the directions with a nonzero B-field. Just these directions become noncommutative. All other fields are finite.

This expression precisely corresponds to the matrix of noncommutativity (27).

In quantum mechanics, the wave function of a particle in a magnetic field can be parametrized using quantum numbers: the discrete energy levels (Landau levels) and one of the coordinates of the orbit center. In the regime of a strong magnetic field, the orbit radius decreases, and the difference between neighbor energy levels grows. A projection onto the lower Landau level takes place, and only a single quantum number—one of the particle coordinates – remains. The impossibility of determining two coordinates simultaneously is a direct consequence of the uncertainty relation.

We use the parametrization of a $\star$-product

$$f_1 \star f_2(x) = \int d^d x'\, d^d x''\, K(x, x', x'') f_1(x') f_2(x'') \quad (33)$$

with the integral kernel

$$K(x, x', x'') = \frac{1}{\pi^d \det(\theta^{\mu\nu})}\, \mathrm{e}^{-2\mathrm{i}(x'-x)^\mu (\theta^{-1})_{\mu\nu} (x''-x)^\nu}\,. \quad (34)$$

In this work, we consider a space with two noncommutating spatial coordinates,

$$\left[x^1, x^2\right] = \mathrm{i}\theta^{12} \equiv \mathrm{i}\theta. \quad (35)$$

In this case [5],

$$K(x, y, z) = \frac{1}{\pi^2 \theta^2} \mathrm{e}^{-\frac{2\mathrm{i}}{\theta}\left(x^2(y^1 - z^1) + y^2(z^1 - x^1) + z^2(x^1 - y^1)\right)}. \quad (36)$$

We use this expression while calculating various $\star$-products. If additional commutating coordinates are available, they are usual parameters from the $\star$-product viewpoint.

4. Classical gauge theory on a noncommutative space

The action functional in the classical field theory on a noncommutative space differs from that in the ordinary theory by a substitution of the $\star$-product for the ordinary pointwise product between fields in the corresponding Lagrangian. In the case of gauge theory, such a replacement is accompanied by the change of the gauge transformation rule:

$$A_i \to U \star A_i \star \bar{U} - \mathrm{i}\partial_i U \star \bar{U}. \quad (37)$$

In this expression, $\bar{U}(x)$ is a function pointwise complex conjugate to $U(x)$, as it is in the ordinary theory. Note, however, that the unitary conditions for $U(x)$ differ from those in the ordinary theory. This results in a redefinition of covariant derivatives that acts on a charged scalar field:

$$D_i \phi = \partial_i \phi - \mathrm{i} A_i \star \phi\,. \quad (38)$$

Like the ordinary gauge theory, the gauge field strength in the noncommutative gauge theory is defined as a commutator of two covariant derivatives,

$$F_{ij} = \mathrm{i}\,[D_i, D_j] = \partial_i A_j - \partial_j A_i - \mathrm{i}[A_i, A_j]_\star\,. \quad (39)$$

Note that the term with the commutator of two gauge fields $[A_i, A_j]_\star$ cannot be rejected even when the gauge group in the ordinary gauge theory is $U(1)$.[2] The gauge field strength is transformed as follows:

$$F_{ij} \to U \star F_{ij} \star \bar{U}\,. \quad (40)$$

In work [2], it was emphasized that, from the viewpoint of the string theory, the commutative and noncommutative descriptions of the same system are equivalent. Accordingly, there exists a map between the fields in the ordinary and noncommutative gauge theories, known as the Seiberg–Witten map (SW-map). If two fields $A_{\rm ord}$ and $A'_{\rm ord}$ of the ordinary theory are connected with the generating function $U_{\rm ord}$ by a gauge transformation, there exists such a generating function U in the noncommutative theory that identifies the field images under the action of an SW-map $A(A_{\rm ord})$ and $A'(A'_{\rm ord})$. Therefore, the following diagram is commutative:

$$\begin{array}{ccc} A_{\rm ord} & \xrightarrow{U_{\rm ord}} & A'_{\rm ord} \\ \downarrow{\scriptstyle \rm SW} & & \downarrow{\scriptstyle \rm SW} \\ A & \xrightarrow{U(U_{\rm ord}, A)} & A' \end{array}$$

An important fact is the dependence of U on $U_{\rm ord}$ and A. In the absence of A-dependence, such a construction of the SW-map would provide the isomorphism between the ordinary and noncommutative gauge groups. Naturally, it is impossible. Therefore, the SW-map identifies only gauge equivalence classes rather than gauge transformations.

The field system we focus our attention on consists of a complex scalar field against the background of a

[2] By analogy with work [2], we use the term "ordinary" instead of "commutative" for the theory on a space with commutating coordinates.

noncommutative $U(1)$ gauge field. The action functional is constructed following the usual way:

$$S = -\int \bar{\phi}(-D_i D^i + m^2)\phi. \quad (41)$$

The metric tensor has the Euclidean (++) signature. The covariant derivatives operate according rule (38). The gauge transformations are generated by $\star$-unitary U's,

$$\phi \to U \star \phi\,, \quad (42)$$

$$\bar{\phi} \to \bar{\phi} \star \bar{U}\,; \quad (43)$$

with the following "$\star$-unitarity" condition:

$$\bar{U} \star U = 1 = U \star \bar{U}. \quad (44)$$

Both indicated conditions are necessary owing to the infinite dimensionality of the function space: there can exist such U's that $\bar{U} \star U = 1$ and $U \star \bar{U} \neq 1$. (The vector space with basic vectors $\{|1\rangle, |2\rangle, \ldots\}$ and the operator U, for which

$$U|n\rangle = |n+1\rangle, \quad (45)$$

compose the well-known example [6]. In this case, $U^\dagger U = \mathbb{1}$, but $UU^\dagger = \mathbb{1} - |1\rangle\langle 1|$.)

Note that the invariance of action (41) is preserved even if[3]

$$\bar{U} \star U = 1, \quad U \star \bar{U} = 1 - P\,; \quad (46)$$

where P is a certain projection operator, i.e. $P \star P = P$. Such a generating function is "topologically nontrivial", namely, $U = \mathrm{e}_\star^{\mathrm{i}f}$ for no real-valued f. Under the action of such a transformation, the field strength transforms as follows:

$$F_{ij} \to U \star F_{ij} \star \bar{U} + U \star (A_j \star \partial_i \bar{U} - A_i \star \partial_j \bar{U}) \star P +$$

$$+\mathrm{i}(\partial_i U \star \partial_j \bar{U} - \partial_j U \star \partial_i \bar{U}) \star P\,. \quad (47)$$

This transformation breaks the gauge invariance of the total action in a gauge theory, which contains a term with $F_{ij}F^{ij}$ (this term is absent in our case where the gauge field is not dynamic). After a U-transformation of type (46), the gauge field A_i is not real-valued anymore.

[3] The author expresses his gratitude to O. Morozov, who drew his attention to this fact.

Below, we will deal with a field with constant strength $F_{12} = F$. We select the corresponding linear potential

$$A_1 = -\alpha_1 x^2, \quad A_2 = \alpha_2 x^1\,. \quad (48)$$

In terms of the parameters $\alpha_{1,2}$, the field strength reads

$$F = \alpha_1 + \alpha_2 + \theta\alpha_1\alpha_2\,. \quad (49)$$

Generally speaking, the field strength F is not gauge invariant. However, in our case, it is constant and is not changed under the influence of gauge transformations (the constant F $\star$-commutes with all generating functions and is not changed at the conjugation). Without loss of generality, we adopt that $F > 0$.

A detailed analysis of various gauges was made in work [4]. Without repeating this analysis, we construct a one-parametric family of generating functions U_t. The gauge transformations generated with the help of these generating functions leave a potential of type (48) in the same class,

$$U_t = \frac{1}{\cosh t}\mathrm{e}^{\frac{2\mathrm{i}}{\theta}x^1 x^2 \tanh t}\,; \quad (50)$$

$$U_0 = 1, \quad \bar{U}_t = U_{-t}, \quad U_{t_1} \star U_{t_2} = U_{t_1+t_2}\,. \quad (51)$$

As a result of the transformation generated by U_t, the parameters α_i are changed as follows:

$$\alpha_1 \to \mathrm{e}^{-2t}\alpha_1 - \frac{2}{\theta}\mathrm{e}^{-t}\sinh t\,, \quad (52)$$

$$\alpha_2 \to \mathrm{e}^{2t}\alpha_2 + \frac{2}{\theta}\mathrm{e}^{t}\sinh t\,. \quad (53)$$

Another property of the constant-strength field is a possibility to calculate the SW-map [2] in the explicit form,

$$F = \left(\mathbb{1} + F_{\mathrm{ord}}\theta\right)^{-1} F_{\mathrm{ord}}\,; \quad (54)$$

$$F_{\mathrm{ord}} = F\left(\mathbb{1} - \theta F\right)^{-1}. \quad (55)$$

This relation is very important for the physical interpretation of our result.

5. Quantum-mechanical Theory

As was already emphasized, the part of the action in the noncommutative theory that contains a product of no more than two fields coincides with that in the ordinary theory (as a consequence of Eq. (15)). In our case, the theory contains only terms that are quadratic in ϕ and $\bar{\phi}$. Therefore, it can be analyzed with the help of standard techniques. Really, knowing the eigenfunctions f_n and eigenvalues λ_n of the operator $(-D_i D^i + m^2)$, it is possible to calculate the Green's function (the two-point function) using the formal spectral definition[4]

$$G(x_{(1)}, x_{(2)}) = -\sum_n \frac{1}{\lambda_n} f_n(x_{(1)}) \bar{f}_n(x_{(2)}) . \tag{56}$$

The subscript (quantum number) n can be discrete or continuous. Just like the ordinary case, the two-point function is

$$\left\langle \phi(x_{(1)}) \bar{\phi}(x_{(2)}) \right\rangle = \int \mathcal{D}\phi\, \mathcal{D}\bar{\phi}\, \mathrm{e}^{\mathrm{i}S[\phi,\bar{\phi}]} \phi(x_{(1)}) \bar{\phi}(x_{(2)}) =$$

$$= \mathrm{i} G(x_{(1)}, x_{(2)}) . \tag{57}$$

Such a "free" propagator not is gauge invariant, and it is transformed according to the formula

$$G(x_{(1)}, x_{(2)}) \to U(x_{(1)}) \star G(x_{(1)}, x_{(2)}) \star \bar{U}(x_{(2)}). \tag{58}$$

To verify this fact, it is necessary to make sure that the integration measure $\mathcal{D}\phi \mathcal{D}\bar{\phi}$ is invariant with respect to gauge transformations. At this point, it turns out again that both conditions are necessary in Eq. (44). At the verification, it is convenient to use the Weyl–Moyal correspondence. Let an operator, which corresponds to the field ϕ and acts in an auxiliary Hilbert space $\mathcal{H}$, be designated as $\hat{\mathcal{O}}_\phi$. Then the integration measure looks like

$$\mathcal{D}\phi\, \mathcal{D}\bar{\phi} = N \prod_{m,n \geq 0} d\phi_{mn} d\bar{\phi}_{mn}, . \tag{59}$$

The matrix elements are calculated as follows:

$$\phi_{mn} = \langle m | \hat{\mathcal{O}}_\phi | n \rangle , \tag{60}$$

$$\bar{\phi}_{mn} = \langle m | \hat{\mathcal{O}}_\phi^\dagger | n \rangle ; \tag{61}$$

where $\{|m\rangle\}$ is a certain convenient choice of basis vectors in the Hilbert space. For a generating function U of type (45), the following relation takes place:

$$\langle m | \hat{\mathcal{O}}_U \hat{\mathcal{O}}_\phi | n \rangle = \begin{cases} \phi_{m-1,n} , & m \geq 1 , \\ 0 , & m = 0 . \end{cases} \tag{62}$$

[4] The subscripts at coordinates x^i are coordinate indices, the parenthesized subscripts are the numbers of positions in two-point and higher-order functions.

A similar formula takes place for $\bar{\phi} \leftrightarrow \hat{\mathcal{O}}_\phi^\dagger$, so even the integration range is not invariant in this case. Note that the transformation law (58) can also be derived from expansion (56). In the case where the transformation generated by U has an inverse one, i.e. U has a left inverse function, which also generates a gauge transformation – in case (46), $\bar{U}$ does not satisfy this condition – all eigenfunctions are in the bijective correspondence before and after the transformation, so that the transformation rule (58) is fulfilled.

Noncommutative field theories are not local, because the $\star$-product contains an infinite number of derivatives. However, if one of the factors is a polynomial, the series in formula (26) includes a finite number of terms. Just this situation happens, when a scalar field interacts with the linear potential (48). If we select $A_1 = -Fx^2$ and $A_2 = 0$, then $D_1 = (1 + \frac{F\theta}{2})\partial_1 + \mathrm{i}Fx^2$ and $D_2 = \partial_2$. Hence, in this case, the effect of noncommutativity is a simple stretching of the coordinate x^1. On the other hand, at the gauge $A_1 = 0$ and $A_2 = Fx^1$, the other coordinate is stretched out. If a symmetric gauge is applied, the both coordinates are equally stretched out. From this example, it becomes clear that gauge invariant correlation functions are changed in comparison with those obtained in the theory on a commutative space. Really, the naive correlator $\langle :\bar{\phi}\phi(x_{(1)}): :\bar{\phi}\phi(x_{(2)}): \rangle$ is non-invariant with respect to the gauge group of noncommutative theory. It should be replaced by [5]

$$\left\langle :\bar{\phi} \star \phi(x_{(1)}): :\bar{\phi} \star \phi(x_{(2)}): \right\rangle . \tag{63}$$

If we denote $\beta_i = 1 + \frac{\alpha_i \theta}{2}$, the covariant derivatives look like

$$D_1 = \beta_1 \partial_1 + \mathrm{i}\alpha_1 x^2 , \tag{64}$$

$$D_2 = \beta_2 \partial_2 - \mathrm{i}\alpha_2 x^1 . \tag{65}$$

The problem of finding the eigenfunctions f_n can be solved taking advantage of the ansatz

$$f_n(x) = \exp\left(\mathrm{i} \frac{\alpha_2}{\beta_2} x^1 x^2 \right) g_n(x) . \tag{66}$$

In this case, we obtain the following equation:

$$\left\{ -\beta_1^2 \partial_1^2 - \beta_2^2 \partial_2^2 - 2 \left(\frac{\beta_1^2 \alpha_2}{\beta_2} + \alpha_1 \beta_1 \right) x^2 \partial_1 \right.$$

$$\left. + \left(\frac{\beta_1^2 \alpha_2}{\beta_2} + \alpha_1 \beta_1 \right)^2 (x^2)^2 + m^2 \right\} g_n = \lambda_n g_n . \tag{67}$$

[5] We refer to this correlator as a two-point one.

It can be used to find the sought eigenfunctions and eigenvalues,

$$f_{n,k} = \frac{\sqrt[4]{F}}{\sqrt{2\pi|\beta_2|}} \exp\left\{ \mathrm{i}\left(\frac{\alpha_2}{\beta_2} x^1 x^2 + k x^1 \right) \right\} \times$$

$$\times \psi_n \left(\frac{x^2\sqrt{F}}{\beta_2} + \frac{\beta_1 k}{\sqrt{F}} \right), \tag{68}$$

$$\lambda_n = (2n+1)F + m^2. \tag{69}$$

In our notations, ψ_n means the n-th normalized wave function of a one-dimensional harmonic oscillator with a frequency of unity (the Hermite polynomial times $\mathrm{e}^{-\frac{x^2}{2}}$). Correlator (63) can be calculated using the Wick theorem, as it is in the ordinary theory:

$$\left\langle :\bar\phi \star \phi(x_{(1)}) : : \bar\phi \star \phi(x_{(2)}) : \right\rangle = -G(x_{(1)}, x_{(2)}) \times$$

$$\times \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{ij}(\overrightarrow{\partial}_{(1)i}\overleftarrow{\partial}_{(1)j} + \overleftarrow{\partial}_{(2)i}\overrightarrow{\partial}_{(2)j})} G(x_{(2)}, x_{(1)}). \tag{70}$$

It can be expressed in terms of eigenfunctions f_n as follows:

$$\left\langle :\bar\phi \star \phi(x_{(1)}) : : \bar\phi \star \phi(x_{(2)}) : \right\rangle = -\sum_{n_1,n_2=0}^{\infty} \frac{1}{\lambda_{n_1}\lambda_{n_2}} \times$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dk_1\, dk_2\, \left(\bar f_{n_1,k_1} \star f_{n_2,k_2}(x_{(1)}) \right) \times$$

$$\times \left(\bar f_{n_2,k_2} \star f_{n_1,k_1}(x_{(2)}) \right). \tag{71}$$

The right-hand side of Eq. (71) can be calculated with the use of an integral representation of the $\star$-product with the help of kernel (36),

$$\bar f_{n_1,k_1} \star f_{n_2,k_2}(x) = -\frac{|2+\alpha_2\theta|\sqrt{F}}{4\pi|1+\alpha_2\theta|} \times$$

$$\times \psi_{n_1}\Big(\frac{(2+\alpha_2\theta)(2+F\theta)}{4(1+\alpha_2\theta)\sqrt{F}} k_1 + \frac{(2+\alpha_2\theta)F\theta}{4(1+\alpha_2\theta)\sqrt{F})} k_2 +$$

$$+ x^2\sqrt{F}\Big)\, \psi_{n_2}\Big(\frac{(2+\alpha_2\theta)F\theta}{4(1+\alpha_2\theta)\sqrt{F})} k_1 +$$

$$+ \frac{(2+\alpha_2\theta)(2+F\theta)}{4(1+\alpha_2\theta)\sqrt{F}} k_2 + x^2\sqrt{F}\Big)\, \mathrm{e}^{\mathrm{i}x^1 \frac{(k_2-k_1)(2+\alpha_2\theta)}{2(1+\alpha_2\theta)}}. \tag{72}$$

In the course of calculations, it is useful to fulfill the following change of variables:

$$k_1 = \frac{\sqrt{F}}{2+F\theta+\alpha_1\theta}\big((2+F\theta)\xi_1 - F\theta\xi_2 \big), \tag{73}$$

$$k_2 = \frac{\sqrt{F}}{2+F\theta+\alpha_1\theta}\big(-F\theta\xi_1 + (2+F\theta)\xi_2 \big), \tag{74}$$

$$\Big| \det\Big(\frac{\partial(k_1,k_2)}{\partial(\xi_1,\xi_2)} \Big) \Big| = \frac{4F(1+\alpha_2\theta)}{(1+\alpha_1\theta)(2+\alpha_2\theta)^2} \times$$

$$\times\, \mathrm{sign}(1+F\theta)\,; \tag{75}$$

Then

$$\bar f_{n_1,k_1} \star f_{n_2,k_2}(x) = -\frac{|2+\alpha_2\theta|\sqrt{F}}{4\pi|1+\alpha_2\theta|} \times$$

$$\times \psi_{n_1}\left(x^2\sqrt{F} + \xi_1 \right) \psi_{n_2}\left(x^2\sqrt{F} + \xi_2 \right) \mathrm{e}^{\mathrm{i}x^1\sqrt{F}(\xi_2-\xi_1)}. \tag{76}$$

Since the eigenvalues λ_n are independent of k, the integration over k_1 and k_2 (or ξ_1 and ξ_2) in sum (71) can be carried out explicitly. After shifting the integration variable, $\xi_i \to \xi_i - (x^2_{(1)} + x^2_{(2)})\sqrt{F}/2$, the integral on the right-hand side of Eq. (71) reads

$$\int d\xi_1\, d\xi_2\, \psi_{n_1}\left(\frac{x^2\sqrt{F}}{2} + \xi_1 \right) \psi_{n_1}\left(-\frac{x^2\sqrt{F}}{2} + \xi_1 \right) \times$$

$$\times \psi_{n_2}\left(\frac{x^2\sqrt{F}}{2} + \xi_2 \right) \psi_{n_2}\left(-\frac{x^2\sqrt{F}}{2} + \xi_2 \right) \times$$

$$\times \exp\left\{ 2\mathrm{i}\frac{x^1\sqrt{F}}{2}(\xi_1 - \xi_2) \right\};$$

$$x \equiv x_1 - x_2. \tag{77}$$

An important property is the definite parity of the functions ϕ_n. According to it, the last expression becomes simpler,

$$\int d\xi\, \psi_n\left(-\frac{x^2\sqrt{F}}{2} + \xi \right) \psi_n\left(\frac{x^2\sqrt{F}}{2} + \xi \right) \times$$

$$\times \exp\left\{2\mathrm{i}\frac{x^1\sqrt{F}}{2}\xi\right\} = (-1)^n \int d\xi\, \psi_n\left(\frac{x^2\sqrt{F}}{2} - \xi\right) \times$$

$$\times\psi_n\left(\frac{x^2\sqrt{F}}{2} + \xi\right) \exp\left\{2\mathrm{i}\frac{x^1\sqrt{F}}{2}\xi\right\} =$$

$$= \frac{(-1)^n}{2}\phi_n\left(\frac{x^2\sqrt{F}}{2}, \frac{x^1\sqrt{F}}{2}\right). \tag{78}$$

Here, ϕ_n is the Wigner function defined on the phase space (x^1, x^2) and corresponding to the pure quantum-mechanical state $|\psi_n\rangle$. This function is a counterpart of the operator $|\psi_n\rangle\langle\psi_n|$ with respect to the WM-correspondences ($\hbar = 1$). In our case,

$$\phi_n(x) = 2(-1)^n \mathrm{e}^{-|x|^2} L_n(2|x|^2), \tag{79}$$

where L_n is the n-th Laguerre polynomial. These functions compose a complete set of one-dimensional radially symmetric $\star$-projectors that satisfy the equation $\phi \star \phi = \phi$ [3]. The final result is

$$\left\langle :\bar{\phi}\star\phi(x_{(1)})::\bar{\phi}\star\phi(x_{(2)}):\right\rangle = -\frac{1}{|1+F\theta|\pi^2}\times$$

$$\times\left(\sum_{n=0}^{\infty}\frac{(-1)^n F\phi_n(\frac{x\sqrt{F}}{2})}{4((2n+1)F+m^2)}\right)^2; x \equiv x_{(1)} - x_{(2)}. \tag{80}$$

Here, the two-point correlation function can be factorized like the ordinary theory,

$$\langle :\bar{\phi}\phi(x_{(1)})::\bar{\phi}\phi(x_{(2)}):\rangle = -|\langle\bar{\phi}(x_{(1)})\phi(x_{(2)})\rangle|^2. \tag{81}$$

Hence, in the noncommutative theory, expression (80) is also the full square (not only a $\star$-square) of a certain gauge invariant function. In the limiting case $F \to 0$, we obtain

$$\sum_n \frac{(-1)^n F\phi_n(\frac{x\sqrt{F}}{2})}{4((2n+1)F+m^2)} \sim \sum_n \frac{(-1)^n F\phi_n(\frac{x\sqrt{F}}{2})}{4m^2} =$$

$$= \frac{\delta^{(2)}(x)}{m^2}, \tag{82}$$

In this regime, the correlator becomes singular. It is also worth noting that the factor $1/(1+F\theta)$ in formula (80) is singular, when $F\theta = -1$. Then the correlation function becomes singular irrespective of the spatial distance between two points. This singularity has a physical meaning. We recall that the noncommutativity matrix $\theta = 1/B$ in the Seiberg–Witten regime. With this identification, SW-map (55) looks like

$$F_{\mathrm{ord}} = F\frac{1}{B-F}B. \tag{83}$$

Hence, if $B = F$ or $\theta F = \mathbb{1}$ (just this situation is realized in our case: $F_{12}\theta^{21} \equiv -\theta F = 1$), the theory has no equivalent commutative description. This phenomenon is responsible for the appearance of a singularity at $F = -1/\theta$.

Higher-order correlators are calculated identically as those in the ordinary theory. A single difference is that the Green's functions are multiplied with the use of the $\star$-product, as in Eq. (70). It is the $\star$-product that ensures the gauge invariance of calculated correlators. In the course of calculation of the n-point function, the following change of the integration variables can be done:

$$\frac{2+\alpha_2\theta}{1+\alpha_2\theta}k_i \to k_i. \tag{84}$$

Then, the Jacobian eliminates the non-invariant factor on the right-hand side of Eq. (72), so that each term in the sum is explicitly gauge invariant (i.e. it depends only on F). Correlators with $n > 2$ points are not reduced anymore to projector solitons [3]. For instance, at $n = 3$, there emerge terms of the type

$$\int dk_1\, dk_2\, dk_3\, \mathrm{e}^{\frac{\mathrm{i}}{2}\left(x^1_{(1)}(k_3-k_1)+x^1_{(2)}(k_1-k_2)+x^1_{(3)}(k_2-k_3)\right)}\times$$

$$\times\psi_{n_1}\left(x^2_{(1)}\sqrt{F} + \frac{(2+F\theta)k_1 + F\theta k_3}{4\sqrt{F}}\right)\times$$

$$\times\psi_{n_3}\left(x^2_{(1)}\sqrt{F} + \frac{(2+F\theta)k_3 + F\theta k_1}{4\sqrt{F}}\right)\times$$

$$\times\psi_{n_2}\left(x^2_{(2)}\sqrt{F} + \frac{(2+F\theta)k_2 + F\theta k_1}{4\sqrt{F}}\right)\times$$

$$\times\psi_{n_1}\left(x^2_{(2)}\sqrt{F} + \frac{(2+F\theta)k_1 + F\theta k_2}{4\sqrt{F}}\right)\times$$

$$\times\psi_{n_3}\left(x^2_{(3)}\sqrt{F} + \frac{(2+F\theta)k_3 + F\theta k_2}{4\sqrt{F}}\right)\times$$

$$\times\psi_{n_2}\left(x_{(3)}^2\sqrt{F}+\frac{(2+F\theta)k_2+F\theta k_3}{4\sqrt{F}}\right), \qquad (85)$$

and the change similar to (75) does not work. It is easy to see that expression (85) does not vary, if all coordinates $x_{(i)}$ are shifted identically. Therefore, the result depends only on the relative position of points.

It is also of interest to construct the generating functional. For this purpose, the action is to be summed up with a current that corresponds to the composite operator $\bar{\phi}\star\phi$:,

$$S\to S+\int J(x)\,\bar{\phi}\star\phi(x)=S+$$

$$+\int A(x',x'')\bar{\phi}(x')\phi(x'')\,; \qquad (86)$$

$$A(x',x'')=\int dx\,J(x)\,K(x,x',x'')\,. \qquad (87)$$

Then

$$Z[J]=N\det(\mathrm{i}G^{-1}+\mathrm{i}A)=\det(1+GA), \qquad (88)$$

and the generating functional for connected diagrams is

$$W[J]=\log Z[J]\;=\;\mathrm{tr}\log(1+GA)=$$

$$=\mathrm{tr}(GA)-\frac{1}{2}\,\mathrm{tr}(GA)^2+\ldots. \qquad (89)$$

The normal ordering $:\bar{\phi}\star\phi:$ results in a removal of the first term from the right-hand side. It is clear that the previous results are reproduced in this approach; namely, the variation $\frac{\delta A(x',x'')}{\delta J(x)}$ generates the kernel $K(x,x',x'')$, and, after integrating over x' and x'', the $\star$-product is reproduced.

6. Final Remarks

The results obtained can be generalized to the case of the 2+1-dimensional field theory in the presence of a constant magnetic field. Then, the Green's functions look like

$$G(x_{(1)},x_{(2)})=-\frac{\mathrm{i}}{2}\sum_n\frac{\mathrm{e}^{-\mathrm{i}\sqrt{\lambda_n}|x^0_{(1)}-x^0_{(2)}|}}{\sqrt{\lambda_n}}\times$$

$$\times f_n(\mathbf{x}_{(1)})\bar{f}_n(\mathbf{x}_{(2)})\,, \qquad (90)$$

with the eigenfunctions f_n not changing. Hence, the most interesting properties of the theory survive. A two-point gauge invariant function can be calculated in terms of Wigner functions (noncommutative projector solitons). This result holds true for a wide class of potentials.

The substitution of the ordinary product of Green's functions by the $\star$-one restores the gauge invariance. As a result, an arbitrary, gauge-dependent scaling of coordinates disappears from correlation functions, and the seeming paradox disappears as well. This work is aimed at the explicit verification of the statements made above concerning the gauge invariance in a noncommutative field theory.

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КОРЕЛЯЦІЙНІ ФУНКЦІЇ ЗАРЯДЖЕНОГО СКАЛЯРНОГО ПОЛЯ В ЗОВНІШНЬОМУ НЕКОМУТАТИВНОМУ КАЛІБРОВНОМУ ПОЛІ З ГРУПОЮ $U(1)$

О. Соловйов

Р е з ю м е

Розглянуто заряджене скалярне поле в некомутативному просторі на тлі зовнішнього калібровного поля сталої напруженості з групою $U(1)$. Обчислено кореляційні функції двох калібровно інваріантних композитних операторів. Проілюстровано зв'язок між калібровними перетвореннями в некомутативній теорії поля та геометрією простору. Доведено відновлення калібровної інваріантності вищих кореляторів, незважаючи на те, що функція Гріна не є інваріантною. Результат як функція зовнішнього поля демонструє сингулярну поведінку саме тоді, коли відображення Зайберга–Вітена стає невизначеним. В цьому випадку не існує еквівалентної комутативної картини.