THE ELECTROSTATIC POTENTIAL IN NONLOCAL POLARIZABLE MEDIA

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We consider the interaction of charged grains with a medium that is capable to a nonlocal polarization. The potential distribution problem has been solved with regard for the polarization effects that arise in the spatial charge field of a smoky plasma in the presence of charged condensed grains. It is shown that there could be the local minima that testify to the existence of a stable equilibrium of equally charged condensed grains due to such an environment at particular values of the polarizability and plasma parameters.

Under certain conditions, the spatial arranged structures of condensed charged grains, e.g., a plasma crystal, are formed in heterogeneous systems like a dusty [1] or smoky [2] plasma. The cause for such a structurization is the interaction between condensed charged grains in the presence of a plasma, but the origins of the interaction in dusty and smoky plasmas essentially differ [2–5], although it has the electrostatic nature in both cases.

The interaction of condensed charged grains with a plasma has been considered from different positions [1–7]. In the majority of papers, the problem of estimation of the interaction potential energy for two charged grains in a plasma was reduced to solving the Poisson–Boltzmann equation for the electrostatic self-consistent potential. The solution of an appropriate boundary-value problem in a linear approximation revealed the Debye electrostatic screening concept. The further research of more exact solutions of this equation in the case of great values of the potential has shown that it is impossible to neglect the interaction between electrons and ions. It is necessary to refer to methods of statistical physics for a more precise effective potential description of charged grains in such media. While studying the interaction between moving charged grains with a plasma, it is necessary to use the linear response theory [8–11]. In this case, the essential role is played by the relaxation of the dielectric permeability of the medium after a perturbation.

Another way to consider the interaction between charged grains and the medium is to involve the bulk plasma potential [5] which is defined locally and makes sense of an initial level of the Debye potential. Then the interaction between charged condensed grains is realized by means of their influence on properties of the medium, namely on the bulk plasma potential. As a result, the interaction potential of charged grains in a plasma is represented in the form of a sum of the Debye and bulk plasma potential which can have a minimum under certain conditions.

The above-mentioned papers have analyzed the structure of a spatial charge and the potential distribution. Nevertheless, the polarization effects in a spatial charge volume which influence the potential distribution character were not considered in an explicit form. Therefore, the present paper is devoted to the statement and the solution of the problem on potential distribution in a plasma, by directly taking the polarization effects into account.

Let us consider a low-temperature plasma where the condensed charged grain is located. The grain temperature is equal to the plasma temperature T, and the average concentration of electrons is equal to the average concentration of ions n. For definiteness, we will assume that the grain size can be neglected. There appears some charge Q on the grain surface, and the induced spatial charge is formed in a vicinity of the grain as a result of the interphase interaction. It is natural that its distribution is defined by the value of a grain surface charge. If the grain is charged positively, it attracts charges of the opposite sign. Therefore, a negatively charged layer arises near the grain surface.

Thus, the interaction between the grain and the plasma causes a nonzero volume charge that can be interpreted as the plasma polarization. It is obvious that, in terms of the linear Debye theory, the Debye radius should be taken as the polarization characteristic scale.

On the microscale, the displacement of electrical charges in the plasma under the influence of the effective field $\mathbf{E}$ is equivalent to the appearance of a displacement charge density ρ_{ind} that depends on the effective field $\mathbf{E}$ or the corresponding potential φ. Using the Gauss theorem in the differential form, we obtain the self-consistent equation for the effective field in the medium:

$$\text{div}\mathbf{E} = \frac{1}{\varepsilon_0}\left(\tilde{\rho}_{\text{ind}}(\mathbf{E}) + \rho\right). \qquad (1)$$

For the effective potential, we have an equation of the Poisson form

$$\triangle\varphi = -\frac{1}{\varepsilon_0}\left(\rho_{\text{ind}}(\varphi) + \rho\right), \qquad (2)$$

where ε_0 is the dielectric constant, and ρ is the free charge distribution function.

Instead of expressing the induced charge density in the plasma by terms of the Boltzmann distribution, we will present it through a polarization vector $\mathbf{P}$ which linearly depends on the electric field strength:

$$\tilde{\rho}_{\text{ind}} = -\text{div}\mathbf{P}. \qquad (3)$$

As the polarization vector magnitude is influenced not only by the external field of a free charge, but also by the field of the displacement charge density which was formed due to polarization, the polarization vector at any point should be expressed through the electrostatic field of all points of the medium. Then the resulted polarization vector can be expressed in the convolution form as

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0\chi\int\limits_{\mathbf{r}'}\mathbf{E}(\mathbf{r}')M(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}', \qquad (4)$$

where the integration is performed in the total $\mathbf{r}'$ space, $M(\mathbf{r}'-\mathbf{r})$ is the averaging kernel which is responsible for the contribution of the electrostatic field of all induced volume charges to the polarization vector. Thus, the polarization vector becomes nonlocal in the given model. If the kernel quickly decreases to zero with increase in the distance between points, then the polarization possess the local character, and $M(\mathbf{r}'-\mathbf{r}) \to \delta(\mathbf{r}'-\mathbf{r})$. The explicit type of the kernel requires the additional research, but it can be modeled with regard for the above-described properties.

According to the present model, we can express the averaging kernel in such a form:

$$M(\mathbf{r}'-\mathbf{r}) = \frac{\exp\left[-\frac{(\mathbf{r}'-\mathbf{r})^2}{R^2}\right]}{\pi^{3/2}R^3}. \qquad (5)$$

Here, R is the averaging length which depends on parameters of the medium. Substituting expressions (3)–(5) in (1), we obtain the equation for the effective field strength in the medium:

$$\text{div}\left(\mathbf{E}(\mathbf{r}) + \chi\int\limits_{\mathbf{r}'}\mathbf{E}(\mathbf{r}')M(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}'\right) = \frac{\rho(\mathbf{r})}{\varepsilon_0}. \qquad (6)$$

Expressing the field strength through its potential in this equation, we obtain

$$\triangle\varphi(\mathbf{r}) + \chi\text{div}_{\mathbf{r}}\int\limits_{\mathbf{r}'}\nabla\varphi(\mathbf{r}')M(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}' = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \qquad (7)$$

where the vector index below the divergence operator denotes the operation applied on the corresponding set of variables. Let us bring the divergence operator under the integral sign (integration is carried out over the other set of variables) and use the identity $\text{div}(\mathbf{A}f(\mathbf{r})) = \mathbf{A}\nabla f(\mathbf{r})$, where $\mathbf{A}$ is any vector, and $f(\mathbf{r})$ is any function. Then we get the equation for the effective electrostatic potential which should be searched in a class of functions which quickly fall down to zero at the infinite distance from sources. Thus, it is necessary to solve the boundary-value problem

$$\triangle\varphi(\mathbf{r}) + \chi\int\limits_{\mathbf{r}'}\nabla\varphi(\mathbf{r}')\cdot\nabla_{\mathbf{r}}M(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}' = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \qquad (8)$$

$$\varphi(\mathbf{r})\big|_{\mathbf{r}\to\infty} = 0. \qquad (9)$$

The boundary condition (9) allows us to use the Fourier integral transformation in order to solve the problem. It is obvious that such a transformation takes place in the Fourier space: $\rho(\mathbf{r}) \to \hat{\rho}(\mathbf{k})$, $\triangle\varphi(\mathbf{r}) \to -k^2\hat{\varphi}(\mathbf{k})$, $\int\limits_{\mathbf{r}'}\nabla\varphi(\mathbf{r}')\nabla_{\mathbf{r}}M(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}' \to$ $\to -(2\pi)^{3/2}k^2\hat{\varphi}(\mathbf{k})\hat{M}(\mathbf{k})$.

Thus, with the specified boundary condition (9), the equation for the effective potential in the Fourier space can be presented as

$$-k^2\hat{\varphi}(\mathbf{k})-(2\pi)^{3/2}\chi k^2\hat{\varphi}(\mathbf{k})\hat{M}(\mathbf{k})=-\frac{\hat{\rho}(\mathbf{k})}{\varepsilon_0}. \quad (10)$$

Its solution looks as

$$\hat{\varphi}(\mathbf{k})=\frac{\hat{\rho}(\mathbf{k})}{\varepsilon_0 k^2\left(1+(2\pi)^{3/2}\chi\hat{M}(\mathbf{k})\right)}. \quad (11)$$

To get a more exact expression, we will find the Fourier transform of the averaging kernel:

$$\hat{M}(\mathbf{k})=\frac{1}{(2\pi)^{3/2}}\frac{1}{(\pi)^{3/2}R^3}\int\limits_{\mathbf{r}'}\exp\left[-\frac{r^2}{R^2}-i\mathbf{k}\,\mathbf{r}\right]d\mathbf{r}'=$$

$$=\frac{1}{(2\pi)^{3/2}}\frac{1}{(\pi)^{3/2}R^3}I(k_x)I(k_y)I(k_z),$$

$$I(p)=\int\limits_{-\infty}^{\infty}\exp\left[-\frac{q^2}{R^2}-iqp\right]dq=\sqrt{\pi}R\exp\left[-\frac{p^2R^2}{4}\right].$$

It can be reduced to

$$\hat{M}(\mathbf{k})=\frac{1}{(2\pi)^{3/2}}\exp\left[-\frac{k^2R^2}{4}\right]. \quad (12)$$

The Fourier transform of the effective potential is as follows:

$$\hat{\varphi}(\mathbf{k})=\frac{\hat{\rho}(\mathbf{k})}{\varepsilon_0 k^2\left(1+\chi\exp\left[-\frac{k^2R^2}{4}\right]\right)}. \quad (13)$$

We now perform the inverse Fourier transformation of expression (13):

$$\varphi(\mathbf{r})=\frac{1}{(2\pi)^{3/2}\varepsilon_0}\int\limits_{\mathbf{k}}\frac{\hat{\rho}(\mathbf{k})\exp[i\mathbf{k}\,\mathbf{r}]}{k^2\left(1+\chi\exp\left[-\frac{k^2R^2}{4}\right]\right)}d\mathbf{k}=$$

$$=\frac{1}{(2\pi)^3\varepsilon_0}\int\limits_{\mathbf{r}'}\rho(\mathbf{r}')\int\limits_{\mathbf{k}}\frac{\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')]}{k^2\left(1+\chi\exp\left[-\frac{k^2R^2}{4}\right]\right)}d\mathbf{r}'d\mathbf{k}.$$

It is possible to represent this expression through the convolution with an appropriate Green's function:

$$\varphi(\mathbf{r})=\int\limits_{\mathbf{r}'}\rho(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}', \quad (14)$$

$$G(\mathbf{r}-\mathbf{r}')=\frac{1}{(2\pi)^3\varepsilon_0}\int\limits_{\mathbf{k}}\frac{\exp[i\mathbf{k}\,(\mathbf{r}-\mathbf{r}')]}{k^2\left(1+\chi\exp\left[-\frac{k^2R^2}{4}\right]\right)}d\mathbf{k}. \quad (15)$$

As the Fourier transform of the obtained Green's function depends just on the radius-vector $\mathbf{k}$, the Fourier original should also depend on $|\mathbf{r}-\mathbf{r}'|$. Therefore, we choose a system of coordinates so that $\mathbf{r}-\mathbf{r}'$ be parallel to the direction of k_z. Then we can integrate (15) in the spherical system of coordinates:

$$G(\mathbf{r}-\mathbf{r}')=\frac{1}{(2\pi)^3\varepsilon_0}\times$$

$$\times\int\limits_0^{\infty}\int\limits_0^{\pi}\int\limits_0^{2\pi}\frac{\exp[ik|\mathbf{r}-\mathbf{r}'|\cos(\theta)]}{k^2\left(1+\chi\exp\left[-\frac{k^2R^2}{4}\right]\right)}k^2\,dk\,d\theta\,d\varphi,$$

$$G(\mathbf{r}-\mathbf{r}')=\frac{1}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|}K(|\mathbf{r}-\mathbf{r}'|), \quad (16)$$

$$K(|\mathbf{r}-\mathbf{r}'|)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin(k|\mathbf{r}-\mathbf{r}'|)}{k}\frac{dk}{1+\chi\exp\left[-\frac{k^2R^2}{4}\right]}. \quad (17)$$

Expressions (16) and (17) define the Green's function of the boundary-value problem (8), (9). For convenience, we make substitution $k|\mathbf{r}-\mathbf{r}'|=t$ in (17). Using (14), we can write the potential of a point-like grain with charge Q located at the origin of coordinates as

$$\varphi(r)=\frac{Q}{4\pi\varepsilon_0 r}K\left(\frac{r}{R}\right), \quad (18)$$

$$K(p)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin(t)}{t}\frac{dt}{1+\chi\exp\left[-\frac{t^2}{4p^2}\right]}, \quad (19)$$

where the variable $p=\frac{r}{R}$.

Like the Debye theory, we use the shielding radius $\tilde{D}$ as a distance where the effective grain potential in the medium is smaller than the corresponding Coulomb potential by e times. By virtue of equalities (18) and (19), we obtain the equation, from which the shielding radius can be determined:

$$K\left(\frac{\tilde{D}}{R}\right)=e. \quad (20)$$

Let us find the asymptotics of the obtained potential for various parameters of the system.

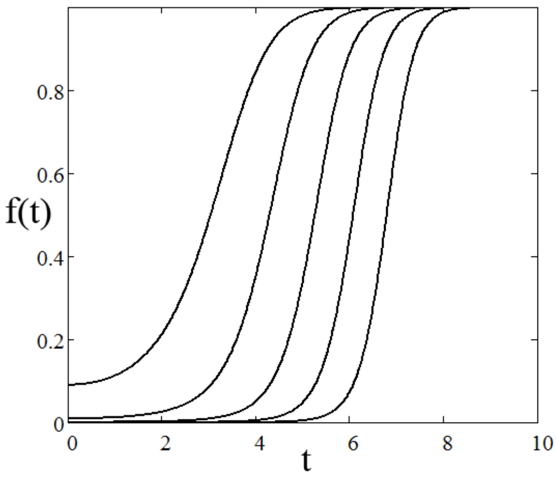

Fig. 1. Form of the function $f(t)$ for great values of the polarizability

Polarizability of the medium tends to zero: $\chi \ll 1$. Thus, one can decompose the denominator in expression (19) up to the first order in the infinitesimal value

$$K_{\chi\to 0}(p) = \frac{2}{\pi}\int\limits_0^\infty \frac{\sin(t)}{t}\left(1-\chi\exp\left[-\frac{t^2}{4p^2}\right]\right)dt. \quad (21)$$

Assuming that

$$\int\limits_0^\infty \frac{\sin(t)}{t}dt = \frac{\pi}{2}, \quad \int\limits_0^\infty \frac{\sin(t)}{t}\exp\left[-\frac{t^2}{4p^2}\right]dt = \frac{\pi}{2}\,\mathrm{erf}(p), \quad (22)$$

we obtain the effective field potential in the form

$$\varphi_{\chi\to 0}(r) = \frac{Q}{4\pi\varepsilon_0 r}\left(1-\chi\,\mathrm{erf}\left(\frac{r}{R}\right)\right). \quad (23)$$

The observation point lies much more further than the averaging length: $r \gg R$.
In this case, $p^{-1}\to 0$ in formula (19). In order to obtain such an asymptotics, we need to make some transformations:

$$K(p)=\frac{2}{\pi}\int\limits_0^\infty \frac{\sin(t)}{t}\,\frac{1+\chi\exp\left[-\frac{t^2}{4p^2}\right]-\chi\exp\left[-\frac{t^2}{4p^2}\right]}{1+\chi\exp\left[-\frac{t^2}{4p^2}\right]}dt=$$

$$=1-\frac{2}{\pi}\int\limits_0^\infty \frac{\sin(t)}{t}\,\frac{dt}{1+\chi^{-1}\exp\left[\frac{t^2}{4p^2}\right]}. \quad (24)$$

After the decomposition of the exponent index up to a first infinitesimal term, we obtain an approximate form of (19):

$$K_{R\to 0}(p) = 1-\frac{2}{\pi}4p^2\chi\int\limits_0^\infty \frac{\sin(t)}{t}\,\frac{dt}{4p^2(1+\chi)+t^2}. \quad (25)$$

Taking into account that

$$\int\limits_0^\infty \frac{\sin(t)}{t}\,\frac{dt}{a^2+t^2} = \frac{\pi}{2}\left(1-e^{-a}\right)$$

and $1+\chi=\varepsilon$, where ε is the dielectric permeability, we finally obtain

$$K_{R\to 0}(p) = \frac{1}{\varepsilon}+\frac{\chi}{\varepsilon}e^{-2p\sqrt{\varepsilon}}, \quad (26)$$

$$\varphi_{R\to 0}(r) = \frac{Q}{4\pi\varepsilon_0 r}\left(\frac{1}{\varepsilon}+\frac{\chi}{\varepsilon}\exp\left[-\frac{2r\sqrt{\varepsilon}}{R}\right]\right). \quad (27)$$

We see that, in this case, the resulting potential is represented by a sum of the Coulomb and Debye-like potentials, where the Debye radius is $D = \frac{2\sqrt{\varepsilon}}{R}$.

Polarizability of the medium tends to infinity: $\chi \gg 1$. It is considered that, for such media as plasmas, the polarizability and the dielectric permeability are sufficiently great. It is obvious because the electric charges are free and capable to neutralize an external electric field easily.

In Fig. 1, we shown the function from (19):

$$f(t) = \frac{1}{1+\chi\exp\left[-\frac{t^2}{4p^2}\right]}. \quad (28)$$

It is convenient to approximate $f(t)$ by a step-function when $\chi \gg 1$:

$$f(t) = \begin{cases} 0, & t<t^*; \\ 1, & t>t^*, \end{cases} \quad (29)$$

where t^* is the inflection point of $f(t)$ which can be determined from the equation

$$e^{x^*} = \chi\frac{2x^*+1}{2x^*-1}, \quad x^* = \frac{{t^*}^2}{4p^2}. \quad (30)$$

For $\chi \gg 1$, we get $x^* = \ln(\chi)$, $t^* = 2p\sqrt{\ln\chi}$, and $\chi=\varepsilon$. In view of these relations, we can write an asymptotic expression for (18) and (19):

$$K(p)_{\chi\to\infty} = \frac{2}{\pi}\int\limits_{t^*}^\infty \frac{\sin(t)}{t}dt = 1-\mathrm{Si}(2p\sqrt{\ln\varepsilon}), \quad (31)$$

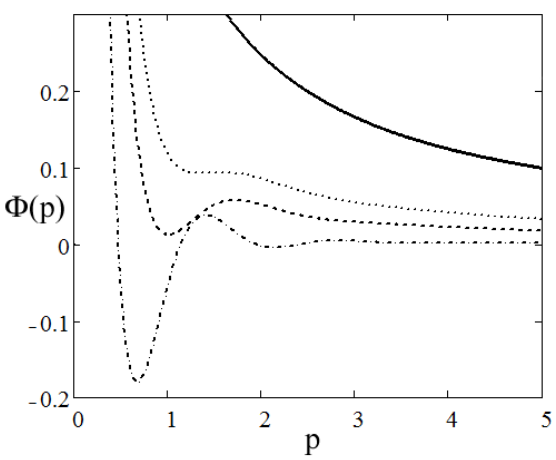

Fig. 2. Dependence $\Phi(p)$ for $\chi = 1,\ 5,\ 10,\ 100$

$$\mathrm{Si}(x) = \frac{2}{\pi}\int\limits_0^x \frac{\sin(t)}{t}\,dt;$$

$$\varphi(r)_{\chi\to\infty} = \frac{Q}{4\pi\varepsilon_0 r}\left(1 - \mathrm{Si}\left(2\frac{r}{R}\sqrt{\ln\varepsilon}\right)\right). \quad (32)$$

The solution obtained is represented by a zero-based oscillating function with degraded extrema by modulus. Therefore, we can get shielding radius, by using expression (20):

$$\tilde{D} = \frac{R}{2\sqrt{\ln\varepsilon}}. \quad (33)$$

It is convenient to represent the exact solution (18) and (19) in the dimensionless form. Changing of the variable $\varphi(p) = \Phi(p)\frac{Q}{4\pi\varepsilon_0 R}$, we will get the expression

$$\Phi(p) = \frac{2}{\pi}\frac{1}{p}\int\limits_0^\infty \frac{\sin(t)}{t}\frac{1}{1+\chi\exp\left[-\frac{t^2}{4p^2}\right]}\,dt. \quad (34)$$

Figure 2 presents the effective potential of the dimensionless charge of grains for various values of the polarizability χ. A plot is shifted down by increasing the polarizability. Consequently, the extrema appear.

Figure 3 shows the dependence of $\Phi(p)$ extrema on the dimensionless distance for various values of the polarizability χ. It is seen that, at $\chi \approx 4$, the first pair of extrema appears and then diverges. At $\chi \approx 12$, the second pair arises. With increase in the polarizability, new pairs of extrema appear, but their values tend to zero rapidly.

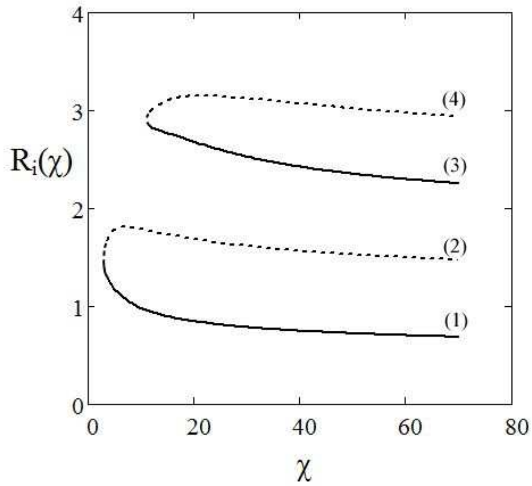

Fig. 3. First four extrema of the potential $\Phi(p)$ versus the polarizability. Solid lines are minima, and dashed lines are maxima

Thus, the consideration of the plasma medium polarization near charged grains allows us to make the following conclusions:

1) For finite polarizability values at far distances, the potential is represented by a sum of the Coulomb and Debye potentials. If the polarizability magnitude is considerable, the exponential damping makes its significant contribution, and it can be interpreted as the Debye shieling.

2) As the polarizability of the medium near charged grains increases, the set of minima and maxima appears, which testifies to the presence of a complicated interaction of equally charged grains with the possible establishment of a stable equilibrium.

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ЕЛЕКТРОСТАТИЧНИЙ ПОТЕНЦІАЛ У СЕРЕДОВИЩІ З НЕЛОКАЛЬНОЮ ПОЛЯРИЗАЦІЄЮ

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Р е з ю м е

У роботі розглянуто взаємодію зарядженої частинки з необмеженим плазмовим середовищем, здатним до нелокальної поляризації. Розв'язано відповідну самоузгоджену задачу про розподіл потенціалу з урахуванням поляризаційних ефектів, які з'являються в шарі просторового заряду димової плазми за наявності заряджених конденсованих частинок. Показано, що при певних значеннях поляризації та параметрів плазми з'являються локальні мінімуми, що свідчать про наявність існування в такому середовищі локальної стійкої рівноваги однаково заряджених конденсованих частинок.