PROPAGATION OF POLARIZED COSMIC MASER RADIATION IN AN ANISOTROPIC MAGNETIZED PLASMA

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The polarization plane of the cosmic maser radiation (CMR) can be rotated either in the space-time with a metric of the anisotropic Bianchi-I type or in a magnetized plasma. A unified treatment of these two phenomena is presented for cold anisotropic plasma. It is argued that the generalized expressions derived in the present study may be relevant for direct searches of a possible rotation of the plane of polarization of the cosmic maser radiation.

1. Introduction

The presence of magnetic fields within astrophysical masers is believed to be a key ingredient in determining the observed polarization characteristics of astrophysical masers. The observation of the linear and circular polarizations of maser radiation potentially provides information about the astronomical environments, in which masers occur. On the other hand, the maser polarization behaves itself differently according to whether the Zeeman shift $\nu_{\rm B}$ is larger or smaller than the linewidth $\Delta\nu_D$. The case $\nu_{\rm B} \gg \Delta\nu_D$ is well understood. Spontaneous decays occur in pure Δm transitions, producing the radiation that is fully polarized and centered at different Zeeman frequencies in different transitions. Under these circumstances, the amplification process preserves the polarization. Therefore, thermal and maser polarizations are the same. The only difference between the two cases is the disparity between the π and σ maser intensities, reflecting their different growth rates [1].

In the opposite limit, $\nu_{\rm B} \gg \Delta\nu_D$, the Zeeman components overlap, and the amplification mixes the different polarizations. The pumping processes produce radiation in three different modes of polarization with respect to the magnetic field ($\Delta m = 0, \pm1$), but only two independent combinations propagate in any direction, because the electric field must be perpendicular to the wave vector. The elimination of the longitudinal component implies specific phase relations among the three pump-generated electric fields; only waves launched with these phase relations produce superpositions that are purely transverse, so that they can be amplified by propagation in the inverted medium. The phase relations are transformed to the following ones for the polarization of maser radiation propagating at an angle θ to the magnetic axis:

$$\frac{Q}{I} = -1 + \frac{2}{3\sin^2\theta}, \qquad \frac{V}{I} = \frac{16 x x_{\rm B}}{3\cos\theta}, \tag{1}$$

where $x = (\nu - \nu_0)/\Delta\nu_D$ and $x_{\rm B} = \nu_{\rm B}/\Delta\nu_D$. This solution, which was first derived by Goldreich, Keeley & Kwan [2] in the limit $x_{\rm B} = 0$ and extended by Elitzur [1] to finite $x_{\rm B} < 1$, follows also from the requirement that the four Stokes parameters produce fractional polarizations that remain unaffected by the amplification process.

How the unpolarized radiation produced in spontaneous decays evolves into this stationary polarization solution remains an open problem. In this paper, a unified discussion of the rotation of a polarization plane in the cases of an anisotropic model of the Bianchi-I type and the Faraday rotation is presented for the cold plasma following works [3, 4].

2. Polarization Effects of CMR in a Space-Time with a Metric of the Bianchi-I Type

Let us consider the Maxwell equations for a free electromagnetic field. In the metric

$$ds^2 = dt^2 - \sum_{i=1}^{3} A_i^2(t)\left(dx^i\right)^2, \tag{2}$$

they can be written as

$$\nabla_\mu F^{\mu\nu} = 0, \quad \nabla_\mu (*F)^{\mu\nu} = 0,$$

where $F_{\mu\nu}$ is the electromagnetic-field tensor and $(*F)^{\alpha\beta} = \frac{1}{\sqrt{-g}}[\alpha\beta\gamma\eta]F_{\gamma\eta}$ is the quantity dual to it; here,

$[\alpha\beta\gamma\eta]$ is the fully antisymmetric tensor specified by the condition $[0123]=1$.

The solutions of these equations can be represented in the form of the electric- and magnetic-field vectors [5–9]

$$\mathbf{E}(t,\mathbf{x}) =$$

$$= \int d^3k \left[\mathcal{E}^\theta(t,\mathbf{k})\mathbf{e}_\theta(t,\mathbf{k}) + \mathcal{E}^\varphi(t,\mathbf{k})\mathbf{e}_\varphi(t,\mathbf{k})\right] \exp(i\mathbf{kx}),$$

$$\mathbf{H}(t,\mathbf{x}) =$$

$$= \int d^3k \left[\mathcal{H}^\theta(t,\mathbf{k})\mathbf{e}_\theta(t,\mathbf{k}) + \mathcal{H}^\varphi(t,\mathbf{k})\mathbf{e}_\varphi(t,\mathbf{k})\right] \exp(i\mathbf{kx}),$$

where

$$\mathbf{e}_\theta = \cos\theta_t\,\cos\varphi_t\frac{\mathbf{e}_1}{A_1} + \cos\theta_t\sin\varphi_t\frac{\mathbf{e}_2}{A_2} - \sin\theta_t\frac{\mathbf{e}_3}{A_3},$$

$$\mathbf{e}_\varphi = -\sin\varphi_t\frac{\mathbf{e}_1}{A_1} + \cos\varphi_t\frac{\mathbf{e}_2}{A_2}$$

are orthogonal vectors. Together with the vector

$$\mathbf{e}_k = \sin\theta_t\cos\varphi_t\frac{\mathbf{e}_1}{A_1} + \sin\theta_t\sin\varphi_t\frac{\mathbf{e}_2}{A_2} + \cos\theta_t\frac{\mathbf{e}_3}{a_3}$$

they form the vierbein unit basis in the momentum space.

The angles θ_t and φ_t can be expressed in terms of spherical coordinates defined in the momentum space through the relation

$$(k_1,k_2,k_3) = k\,(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

as follows

$$(\sin\theta_t\cos\varphi_t, \sin\theta_t\sin\varphi_t, \cos\varphi) =$$

$$= \mu^{-1}\left(\frac{\sin\theta\cos\varphi}{A_1}, \frac{\sin\theta\sin\varphi}{A_2}, \frac{\cos\varphi}{A_3}\right).$$

The coefficient μ is determined by equating the squares of both sides of this relation.

The components $\mathcal{E}^\theta$, $\mathcal{E}^\varphi$, $\mathcal{H}^\theta$, and $\mathcal{H}^\varphi$ can be written in the form

$$\mathcal{E}^\theta(t,\mathbf{k}) = \frac{1}{\sqrt{2}(2\pi)^{3/2}(-g)^{1/4}}\,\frac{\mu}{b^{1/2}}\,\left(\sigma^+ + \sigma^-\right),$$

$$\mathcal{E}^\varphi(t,\mathbf{k}) = \frac{1}{\sqrt{2}(2\pi)^{3/2}(-g)^{1/4}}\,\frac{\mu}{b^{1/2}k}\,\frac{d}{dt}\left(\sigma^- + \sigma^-\right),$$

$$\mathcal{H}^\theta(t,\mathbf{k}) = -\frac{1}{\sqrt{2}(2\pi)^{3/2}(-g)^{1/4}}\,\frac{\mu}{b^{1/2}}\,\left(\sigma^- + \sigma^-\right),$$

$$\mathcal{H}^\varphi(t,\mathbf{k}) = -\frac{1}{\sqrt{2}(2\pi)^{3/2}(-g)^{1/4}}\,\frac{\mu}{b^{1/2}k}\,\frac{d}{dt}\left(\sigma^+ + \sigma^-\right), \quad (3)$$

where

$$b = \frac{1}{\sqrt{-g}}\left(A_2^2\cos^2\varphi + A_1^2\sin^2\varphi\right),$$

and the functions $\sigma^\pm = \sigma^r$ satisfy the equation

$$\ddot{\sigma}^r - \frac{\dot{b}}{b}\dot{\sigma}^r + \left[k^2\mu^2 + rk\Delta\right]\sigma^r = 0, \quad (4)$$

$$\Delta = b\frac{d}{dt}\left(\frac{a}{b}\right), \quad a = \frac{\cos\theta\sin 2\varphi}{2\sqrt{-g}}\,(A_2^2 - A_1^2). \quad (5)$$

Following [10, 11], we describe the polarization effects of electromagnetic radiation with the aid of the polarization matrix defined in the plane orthogonal to the direction of propagation of the electromagnetic waves. This density matrix can be represented in the form

$$J^{ab} = J_+^{ab} + J_-^{ab}, \quad a,b = \theta,\varphi,$$

$$J^{ab} = \frac{1}{2}\mathcal{E}^{\{a}(t,\mathbf{k}), \quad \mathcal{E}t^{*b\}}(t,\mathbf{k}),$$

$$J_-^{ab} = \frac{1}{2}\mathcal{E}^{[a}(t,\mathbf{k}), \quad \mathcal{E}t^{*b]}(t,\mathbf{k}), \quad (6)$$

where the symbol $\{,\}$ ($[,]$) denotes the symmetrization (antisymmetrization) with respect to the corresponding superscripts.

The substitution of the electric-field components (2) into expression (5) yields

$$J_+^{\theta\theta} = \left(2(2\pi)^3(-g)^{1/2}b\right)^{-1}\mu^2|Y^+|^2,$$

$$J_+^{\varphi\varphi} = \left(2(2\pi)^3(-g)^{1/2}bk^2\right)^{-1}|\dot{Y}^-|^2,$$

$$J_+^{\theta\varphi} = -\frac{\mu}{2}\left(2(2\pi)^3(-g)^{1/2}bk\right)^{-1} 2\,\mathrm{Re}\,\dot{Y}^-\overset{*}{Y}{}^+,$$

$$J_-^{\theta\varphi} = \frac{i}{2}\frac{\mu}{2(2\pi)^3(-g)^{1/2}bk}\,2\,\mathrm{Im}\,\dot{Y}^-\overset{*}{Y}{}^+, \quad (7)$$

where

$$Y^\pm = \sigma^+ \pm \sigma^-.$$

The matrix J^{ab} is expressed in terms of the Stokes parameters which describe the polarization properties of electromagnetic waves as follows:

$$J_+^{ab} = \frac{1}{2}\begin{pmatrix} I+Q & U \\ U & I-Q\end{pmatrix}, \quad J_-^{ab} = \frac{1}{2}\begin{pmatrix} 0 & -iV \\ iV & 0\end{pmatrix}. \quad (8)$$

Here, I is the total intensity of radiation, the parameters Q and U are related to the degree of linear polarization by the equation

$$P_L = \frac{\sqrt{U^2+Q^2}}{I},$$

and V determines the degree of circular polarization via the relation

$$P_C = V/I.$$

Comparing relations (6) and (7), we find that the Stokes parameters are given by

$$I = \frac{1}{2(2\pi)^3(-g)^{1/2}bk^2}\left[|\dot{Y}^-|^2 + K_0^2|Y^+|^2\right],$$

$$Q = -\frac{1}{2(2\pi)^3(-g)^{1/2}bk^2}\left[|\dot{Y}^-|^2 - K_0^2|Y^+|^2\right],$$

$$U = -\frac{\mu}{(2\pi)^3(-g)^{1/2}bk}\,\mathrm{Re}\,\dot{Y}^-\overset{*}{Y}{}^+,$$

$$V = -\frac{\mu}{(2\pi)^3(-g)^{1/2}bk}\,\mathrm{Im}\,\dot{Y}^-\overset{*}{Y}{}^+, \quad K_0 = k\mu. \tag{9}$$

To pursue the investigation of polarization effects further, we assume that the propagation of waves in an anisotropic space-time can be described in the short-wave approximation; that is, we retain the first terms of the asymptotic expansion of the solutions of Eq. (3) for σ^r in the limit $k \to \infty (\lambda \to 0)$. Such an expansion was constructed by Sagnotti and Zwiebach [12–15]. From their results, it follows that, in the leading approximation in k^{-1} for $k \to \infty$, the required solutions can be represented as

$$Y_0^+ = \left(\frac{b\mu_0}{b_0\mu}\right)^{1/2}\left(C_0^+ e^{i\lambda} + C_0^- e^{-i\lambda}\right)e^{i\Omega},$$

$$Y_0^- = \left(\frac{b\mu_0}{b_0\mu}\right)^{1/2}\left(C_0^+ e^{i\lambda} - C_0^- e^{-i\lambda}\right)e^{i\Omega},$$

$$\dot{Y}^+ = iK_0Y^+,$$

$$\dot{Y}^- = iK_0Y^-, \tag{10}$$

where

$$\lambda = \int_{t_0}^{t}\frac{\Delta}{2\mu}\,dt', \quad \Omega = \int_{t_0}^{t}k\mu dt',$$

t_0 corresponds to an arbitrary initial instant of propagation, and $C_0^{\pm}$ are the values of the functions $\sigma^{\pm}(t)$ at the point t_0.

The substitution of the WKB solutions (9) into expressions (8) for the Stokes parameters yields

$$I = \frac{\mu\mu_0}{\sqrt{-g}(2\pi)^3b_0}\left[|C_0^+|^2 + |C_0^-|^2\right],$$

$$Q = \frac{\mu\mu_0}{(2\pi)^3\sqrt{-g}b_0}\,2\mathrm{Re}\left(C_0^+\overset{*}{\mathrm{C}}{}_0^- e^{2i\lambda}\right),$$

$$U = \frac{\mu\mu_0}{(2\pi)^3\sqrt{-g}b_0}\,2\mathrm{Im}\left(C_0^+C_0^- e^{2i\lambda}\right),$$

$$V = -\frac{\mu\mu_0}{\sqrt{-g}b_0}\left[|C_0^+|^2 - |C_0^-|^2\right].$$

Eliminating the constants from these relations, we can find the Stokes parameters as functions of time. These are given by

$$I(t) = \frac{\mu(t)}{\mu(t_0)}\sqrt{\frac{-g(t_0)}{-g(t)}}\,I(t_0),$$

$$Q(t) =$$

$$= \frac{\mu(t)}{\mu(t_0)}\sqrt{\frac{-g(t_0)}{-g(t)}}\left[Q(t_0)\cos 2\lambda(t) - U(t_0)\sin 2\lambda(t)\right],$$

$$U(t) =$$

$$= \frac{\mu(t)}{\mu(t_0)}\sqrt{\frac{-g(t_0)}{-g(t)}}\left[Q(t_0)\sin 2\lambda(t) + U(t_0)\cos 2\lambda(t)\right],$$

$$V(t) = \frac{\mu(t)}{\mu(t_0)}\sqrt{\frac{-g(t_0)}{-g(t)}}\,V(t_1).$$

It immediately follows that

$$P_L(t) = P_L(t_0), \quad P_C(t) = P_C(t_0);$$

that is, the degree of linear polarization and the degree of circular polarization do not vary with time as the electromagnetic wave propagates in an anisotropic space-time (this result is in the perfect agreement with analogous conclusions drawn in [16–19] by different methods).

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Under such conditions, the rotation of the polarization plane is the only nontrivial polarization effect. The angle τ of this rotation in the plane orthogonal to the direction of wave propagation is determined by the relation

$$\tan 2\tau = \frac{U}{Q}.$$

It follows that

$$\tan 2\tau(t) = \frac{\sin 2\lambda + \tan 2\tau(t_0)\cos 2\lambda}{\cos 2\lambda - \tan 2\tau(t_0)\sin 2\lambda}.$$

Differentiating this last relation, we find that the variable $y \equiv \tan 2\tau$ satisfies the equation

$$\frac{d}{dt}y = 2\lambda(1+y^2).$$

Solving this equation, we obtain

$$\tau(t) = \lambda(t) + \tau_0$$

or

$$\Delta\tau = \int\limits_{t_0}^{t} \frac{\Delta}{2\mu}\, dt'.$$

Taking expression (4) for Δ into account and using the linear approximation in the anisotropy parameter $\Delta\bar{A} = A_2 - A_1$ [the latter is reduced to setting $b = \mu = 1/A$, where $A = (A_1A_2A_3)^{1/3}$], we obtain

$$\Delta\tau(t) = \frac{1}{2}\cos\theta\sin 2\varphi\left(A(t)\Delta A(t) - A(t_0)\Delta A(t_0)\right).$$

We can see that the polarization plane undergoes rotation only if the wave propagates in a direction other than that specified by the coordinate values $\theta = \frac{\pi}{2}(2k+1)$ and $\varphi = \frac{\pi}{2}k\,(k = 0, 1, 2, \ldots)$ and if $\Delta A \neq 0$, that is, if the anisotropic model under study is not axially symmetric [9].

3. Cosmic Maser Radiation Polarization Effects in a Magnetized Plasma with Bianchi-I type Anisotropy

If linearly polarized CMR passes through the cold plasma containing a magnetic field, the polarization plane of a wave can be rotated since the two circular polarizations (forming the linearly polarized beam) are travelling at different speeds. This effect has been studied in a variety of different frameworks even in relativistic QED plasmas (see, for instance, [3, 20]).

The CMR has a degree of linear polarization. If CMR is linearly polarized, then its polarization plane also can be rotated provided a sufficiently strong magnetic field is present [21].

In this section, a unified discussion of rotation of the polarization plane in the case of an anisotropic model of the Bianchi-I type and the Faraday rotation will be presented for the cold plasma following works [3, 4]. A related aspect of the present analysis will be to study the range of validity of the Faraday rotation estimates.

Let us start by discussing the typical scales involved in the problem. The plasma is globally neutral, and the ion density equals the electron density, i.e. $n_\mathrm{i} \simeq n_\mathrm{e} = n_0$, where n_0 stands for the common electron–ion number density in the corresponding area of the Universe. The global neutrality of the plasma occurs for typical length scales $L \gg \lambda_\mathrm{D}$, where

$$\lambda_\mathrm{D} = \sqrt{\frac{k_\mathrm{B}T_\mathrm{ei}}{8\pi e^2 n_0}}, \tag{11}$$

is the Debye screening length, and k_B is Boltzmann constant.

If the plasma is not magnetized, the only relevant frequency scales of the problem are the plasma frequencies which can be constructed from the electron and ion densities, i.e.

$$\omega_\mathrm{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_\mathrm{e}}}, \quad \omega_\mathrm{pi} = \sqrt{\frac{4\pi e^2 n_0}{m_\mathrm{i}}}. \tag{12}$$

The frequencies given in Eq. (12) enter the dispersion relations determining the group velocity of an electromagnetic signal in the plasma. The plasma frequencies for both electrons and ions are much larger than the collision frequencies constructed from the inverse of the mean free paths. Then the plasma can be described, to a very good approximation, within a two-fluid framework [26, 27].

If the plasma is magnetized, two new frequency scales arise in the problem, namely the electron and ion gyrofrequencies, i.e.

$$\omega_\mathrm{Be} = \frac{eB_0}{m_\mathrm{e}c}, \qquad \omega_\mathrm{Bi} = \frac{eB_0}{m_\mathrm{i}c}, \tag{13}$$

where B_0 is the magnetic field strength in the corresponding area of the Universe, and c is the speed of light in vacuo.

The electron and ion gyrofrequencies, together with the plasma frequencies of Eq. (12), affect the dispersion relations in the case of a magnetized plasma.

Assuming that, at the time moment t_in on the background of the initially homogeneous and isotropic gravitational field in the Universe with the Friedman metric, there arises the homogeneous anisotropic perturbation so that, as a consequence, the metric can be represented as (2). Let us assume also that, at $t < t_{\rm in}$, the state of the EM field can be described with the density matrix with the non-zero occupation number of photons in the mode $n_0(\nu_0)$ corresponding to the black-body radiation. The latter is strictly constant at $t < t_{\rm in}$ and constant in the zeroth approximation in the anisotropy parameters at $t > t_{\rm in}$:

$$\frac{\partial}{\partial t} n_0(\nu_0) = 0.$$

The frequency ν_0 is considered to be independent of time and equal to the radiation frequency in the current epoch. With the frequency at any time moment t, it is related as follows:

$$\nu_0 A(t_0) = \nu(t) A(t), \qquad (14)$$

where $A(t)$ is the scale factor in the Friedman model at $t < t_{\rm in}$, and $A^3 = (A_1 A_2 A_3)$ at $t > t_{\rm in}$.

Defining the appropriately rescaled electron and ion densities, $n_{\rm e} = A^3 \tilde{n}_{\rm e}$ and $n_{\rm i} = A^3 \tilde{n}_{\rm i}$ in a conformally flat geometry of the Bianchi-I type background characterized by a scale factor $A(\eta)$ and by the line element (2), the continuity equations for the charge densities read

$$n'_{\rm e} + 3 w_{\rm e} {\rm H} n_{\rm e} + (w_{\rm e} + 1)\, {\rm div}\, (n_{\rm e} \mathbf{v}_{\rm e}) = 0, \qquad (15)$$

$$n'_{\rm i} + 3 w_{\rm i} {\rm H} n_{\rm i} + (w_{\rm i} + 1)\, {\rm div}\, (n_{\rm i} \mathbf{v}_{\rm i}) = 0, \qquad (16)$$

where ${\rm H} = A'/A$; the prime denotes a derivation with respect to the conformal time coordinate η; w is the barotropic index for the electron or ion fluid.

In the cold plasma, both electrons and ions are non-relativistic. Hence, the barotropic index w will be close to zero to a good approximation. For instance, the energy density of an ideal electronic gas is given by

$$\rho_{\rm e} = n_{\rm e} \left(m_{\rm e} c^2 + \frac{3}{2} k_{\rm B} T_{\rm e} \right), \qquad (17)$$

and since $w_{\rm e,i} = k_{\rm B} T_{\rm e,i}/m_{\rm e,i} c^2$, $w_{\rm e,i} \ll 1$ as far as $k_{\rm B} T_{\rm e,i} \ll m_{\rm ,ie} c^2$.

In the cold-plasma approximation, the temperature of ions and electrons vanishes. In the warm-plasma approximation, the temperature of the two charged species may be very small but non-vanishing. The warm-plasma treatment will lead, in practice, only to an effective correction of the plasma frequency. Since the cold-plasma results turn out to be, *a posteriori*, rather accurate, the discussion will be presented in terms of the cold-plasma description.

To have a self-consistent set of two-fluid equations, Eqs. (15) and (16) will be supplemented by the evolution equations of the velocity fields and of the electromagnetic field, namely,

$$\rho_{\rm e} [\mathbf{v}'_{\rm e} + {\rm H} \mathbf{v}_{\rm e} + (v^a_{\rm e} \nabla_a) \mathbf{v}_{\rm e}] = -n_{\rm e} e \left(\mathbf{E} + \frac{\mathbf{v}_{\rm e}}{c} \times \mathbf{B} \right), \qquad (18)$$

$$\rho_{\rm i} [\mathbf{v}'_{\rm i} + {\rm H} \mathbf{v}_{\rm i} + (v^b_{\rm i} \nabla_b) \mathbf{v}_{\rm i}] = n_{\rm i} e \left(\mathbf{E} + \frac{\mathbf{v}_{\rm i}}{c} \times \mathbf{B} \right), \qquad (19)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the conformally rescaled electromagnetic fields obeying the following set of generalized Maxwell equations [4]:

$${\rm div}\, \mathbf{E} = 4\pi e (n_{\rm i} - n_{\rm e}), \qquad (20)$$

$${\rm div}\, \mathbf{B} = 0, \qquad {\rm curl}\, \mathbf{E} = -\frac{1}{c} \mathbf{B}', \qquad (21)$$

$${\rm curl}\, \mathbf{B} = \frac{1}{c} \mathbf{E}' + \frac{4\pi e}{c} (n_{\rm i} \mathbf{v}_{\rm i} - n_{\rm e} \mathbf{v}_e), \qquad (22)$$

Eqs. (20)–(22) are the usual two-fluid equations [22].

Equations (15) and (16) together with Eqs. (18), (19), and (20)–(22) can then be linearized in the presence of the weak background magnetic field B_0, i.e.

$$n_{\rm e,\,i}(\eta, \mathbf{x}) = n_0 + \delta n_{\rm e,\,i}(\eta, \mathbf{x}), \ \ \mathbf{B}(\eta, \mathbf{x}) = \mathbf{B}_0 + \delta \mathbf{B}(\eta, \mathbf{x}),$$

$$\mathbf{v}_{\rm e,\,i}(\eta, \mathbf{x}) = \delta \mathbf{v}_{\rm e,\,i}(\eta, \mathbf{x}), \ \ \mathbf{E}(\eta, \mathbf{x}) = \delta \mathbf{E}(\eta, \mathbf{x}). \qquad (23)$$

Using Eq. (23), the system of equations (15)–(19) and (20)–(22) can be written as

$$\delta n'_{\rm e} + n_0\, {\rm div}\, (\delta \mathbf{v}_{\rm e}) = 0, \ \ \delta n'_{\rm i} + n_0\, {\rm div}\, (\delta \mathbf{v}_{\rm i}) = 0, \qquad (24)$$

$$\delta \mathbf{v}'_{\rm e} + {\rm H} \delta \mathbf{v}_{\rm e} = -\frac{e}{m_{\rm e}} \left[\delta \mathbf{E} + \frac{\delta \mathbf{v}_{\rm e}}{c} \times \mathbf{B}_0 \right],$$

$$\delta \mathbf{v}'_{\rm i} + {\rm H} \delta \mathbf{v}_{\rm i} = \frac{e}{m_{\rm i}} \left[\delta \mathbf{E} + \frac{\delta \mathbf{v}_{\rm i}}{c} \times \mathbf{B}_0 \right], \qquad (25)$$

$${\rm curl}\, (\delta \mathbf{E}) = -\frac{1}{c} \delta \mathbf{B}', \ {\rm div}\, (\delta \mathbf{E}) = 4\pi e (\delta n_{\rm i} - \delta n_{\rm e}), \qquad (26)$$

$${\rm curl}\, (\delta \mathbf{B}) = \frac{1}{c} \delta \mathbf{E}' + \frac{4\pi\, e\, n_0}{e} (\delta \mathbf{v}_{\rm i} - \delta \mathbf{v}_{\rm e}). \qquad (27)$$

From Eqs. (24)–(26), the relevant dispersion relations and the associated refraction indices can be obtained by treating separately the motions in parallel and perpendicularly to the magnetic field direction. Defining the current direction parallel to the magnetic field as

$$\mathbf{j}_{\parallel} = n_0 e(\delta\mathbf{v}_{\mathrm{i},\parallel} - \delta\mathbf{v}_{\mathrm{e},\parallel}),$$

Eqs. (25) yield

$$\mathbf{j}'_{\parallel} + \mathrm{H}\mathbf{j}_{\parallel} = \frac{1}{4\pi}(\omega^2_{\mathrm{p,i}} + \omega^2_{\mathrm{e,i}})\,\delta\mathbf{E}_{\parallel}. \tag{28}$$

Since a variation of the geometry is slow with respect to the typical frequencies of plasma oscillations, the following adiabatic expansions can be used:

$$\mathbf{j}_{\parallel}(\eta,\mathbf{x}) = \mathbf{j}_{\parallel,\omega}(\mathbf{x})e^{-i\int^{\eta} d\eta'\omega(\eta')},$$

$$\delta\mathbf{E}_{\parallel}(\eta,\mathbf{x}) = \delta\mathbf{E}_{\parallel,\omega}(\mathbf{x})e^{-i\int^{\eta} d\eta'\omega(\eta')}. \tag{29}$$

Thus, defining $\alpha = i\mathrm{H}/\omega \ll 1$, Eq. (28) implies that

$$\mathbf{j}_{\parallel,\omega} = \frac{i}{4\pi}\frac{\omega^2_{\mathrm{p,i}} + \omega^2_{\mathrm{p,e}}}{\omega(1+\alpha)}\delta\mathbf{E}_{\parallel,\omega}. \tag{30}$$

Inserting Eq. (30) into the parallel component of Eq. (27), the following equation can be obtained:

$$\mathrm{curl}\,(\delta\mathbf{B}_{\omega})_{\parallel}) = -i\frac{\omega}{c}\epsilon_{\parallel}(\omega,\alpha)\delta\mathbf{E}_{\parallel,\omega}\,, \tag{31}$$

where the parallel dielectric constant is

$$\epsilon_{\parallel}(\omega,\alpha) = 1 - \frac{\omega^2_{\mathrm{p,i}}}{\omega^2(1+\alpha)} - \frac{\omega^2_{\mathrm{p,e}}}{\omega^2(1+\alpha)}. \tag{32}$$

With a similar procedure, the equation of motion in the plane orthogonal to the magnetic field direction can be solved as well, and the evolution equations of the electric and magnetic fluctuations can then be written, in compact notation, as

$$\mathrm{curl}\,(\delta\mathbf{E}_{\omega}) = i\,\frac{\omega}{c}\delta\mathbf{B}, \tag{33}$$

$$\mathrm{curl}\,(\delta\mathbf{B}_{\omega}) = -i\,\frac{\omega}{c}\overline{\epsilon}(\alpha,\omega)\delta\mathbf{E}_{\omega}\,, \tag{34}$$

where $\delta\mathbf{E}_{\omega}$ and $\delta\mathbf{B}_{\omega}$ have to be understood as column matrices containing, in each row, the components of the electric and magnetic fields in each of the three spatial directions, while $\overline{\epsilon}(\omega,\alpha)$ is a 3×3 matrix given by

$$\overline{\epsilon}(\omega,\alpha) = \begin{pmatrix} \epsilon_1(\omega,\alpha) & i\epsilon_2(\omega,\alpha) & 0 \\ -i\epsilon_2(\omega,\alpha) & \epsilon_1(\omega,\alpha) & 0 \\ 0 & 0 & \epsilon_{\parallel}(\omega,\alpha) \end{pmatrix}, \tag{35}$$

where $\epsilon_{\parallel}(\omega,\alpha)$ is defined by Eq. (32); $\epsilon_{1,2}(\omega,\alpha)$ are instead

$$\epsilon_1(\omega,\alpha) =$$

$$1 - \frac{\omega^2_{\mathrm{p\,i}}(\alpha+1)}{\omega^2(\alpha+1)^2 - \omega^2_{\mathrm{B\,i}}} - \frac{\omega^2_{\mathrm{p\,e}}(\alpha+1)}{\omega^2(\alpha+1)^2 - \omega^2_{\mathrm{B\,e}}}, \tag{36}$$

$$\epsilon_2(\omega,\alpha) =$$

$$\frac{\omega_{\mathrm{B\,e}}}{\omega}\frac{\omega^2_{\mathrm{p\,e}}}{\omega^2(\alpha+1)^2 - \omega^2_{\mathrm{B\,e}}} - \frac{\omega_{\mathrm{B\,i}}}{\omega}\frac{\omega^2_{\mathrm{p\,i}}}{\omega^2(\alpha+1)^2 - \omega^2_{\mathrm{B\,i}}}. \tag{37}$$

The coordinate system can be fixed by setting $k_x = 0$ and $k_y = k\sin\theta$, $k_z = k\cos\theta$ with $\mathbf{B}_0$ oriented along the $\hat{z}$ direction. Since Eqs. (33) and (34) yield

$$\mathrm{curl}\,\mathrm{curl}\,(\delta\mathbf{B}_{\omega}) = \frac{\omega^2}{c^2}\overline{\epsilon}(\omega,\alpha)\delta\mathbf{B}_{\omega}\,, \tag{38}$$

the Fourier transformation of Eq. (38) in the coordinate system selected previously leads to the generalized Appleton–Hartree equation [4]:

$$\mathcal{A}\,\delta\mathbf{B}_{\mathbf{k},\omega} = \begin{pmatrix} \left[1-\frac{\epsilon_1}{n^2}\right] & -i\left[\frac{\epsilon_2}{n^2}\right] & 0 \\ i\left[\frac{\epsilon_2}{n^2}\right] & \left[c^2-\frac{\epsilon_1}{n^2}\right] & -sc \\ 0 & -sc & \left[s^2-\frac{\epsilon_{\parallel}(\omega,\alpha)}{n^2}\right] \end{pmatrix}\times$$

$$\times\begin{pmatrix} \delta B_{k,\omega,x} \\ \delta B_{k,\omega,y} \\ \delta B_{k,\omega,z} \end{pmatrix} = 0, \tag{39}$$

where the refraction index $n = c/v$ has been introduced so as to eliminate the comoving momentum in such a way that $k = \omega/v = n\omega/c$; we have written $c(\theta) = \cos\theta$ and $s(\theta) = \sin\theta$. From Eq. (39), we get $\mathcal{A}^{\dagger} = \mathcal{A}$, where the dagger denotes the transposition and complex conjugation of a given matrix.

The non-trivial solutions of the system of algebraic (homogeneous) equations given by formula (39) come from setting the determinant of the coefficients equal to zero, i.e. $\det\mathcal{A} = 0$. It was found that the determinant vanishes if [4]

$$s^2(\theta)\left\{\left(\frac{1}{\epsilon_{\parallel}} - \frac{1}{n^2}\right)\left[\frac{1}{n^2} - \frac{1}{2}\left(\frac{1}{\epsilon_{\mathrm{L}}} + \frac{1}{\epsilon_{\mathrm{R}}}\right)\right] -\right.$$

$$-c^2(\theta)\left[\left(\frac{1}{n^2} - \frac{1}{\epsilon_{\mathrm{L}}}\right)\left(\frac{1}{n^2} - \frac{1}{\epsilon_{\mathrm{R}}}\right)\right] = 0, \tag{40}$$

where the right-handed and left-handed dielectric constants have been defined as

$$\epsilon_{\rm R} = \epsilon_1 + \epsilon_2 =$$

$$= 1 - \frac{\omega_{\rm p\,i}^2}{\omega[\omega(\alpha+1) - \omega_{\rm B\,i}]} - \frac{\omega_{\rm p\,e}^2}{\omega[\omega(\alpha+1) + \omega_{\rm B\,e}]}, \qquad (41)$$

$$\epsilon_{\rm L} = \epsilon_1 - \epsilon_2 =$$

$$= 1 - \frac{\omega_{\rm p\,e}^2}{\omega[\omega(\alpha+1) - \omega_{\rm B\,e}]} - \frac{\omega_{\rm p\,i}^2}{\omega[\omega(\alpha+1) + \omega_{\rm B\,i}]}. \qquad (42)$$

Equation (40) reduces exactly to the Appleton–Hartree equation known from the two-fluid plasma theory [26] with a minor difference that the leading dependence upon the background geometry appears in $\epsilon_{\rm R,L}$ through the function α. The dispersion relations for a wave propagating in parallel and perpendicularly to the magnetic field direction can be obtained by setting, respectively, $\theta = 0$ and $\theta = \pi/2$ in Eq. (40). Consequently, the relevant equations determining the refraction index are, in this case,

$$(n^2 - \epsilon_{\rm R})(n^2 - \epsilon_{\rm L}) = 0, \qquad \theta = 0, \qquad (43)$$

$$(n^2 - \epsilon_{\|})[n^2(\epsilon_{\rm L} + \epsilon_{\rm R}) - 2\epsilon_{\rm L}\epsilon_{\rm R}] = 0, \quad \theta = \frac{\pi}{2}. \qquad (44)$$

Equation (43) gives the usual dispersion relations for the two circular polarizations of the electromagnetic wave, i.e. $n^2 = \epsilon_{\rm R}$ and $n^2 = \epsilon_{\rm L}$, while Eq. (44) gives those for the "ordinary" (i.e. $n^2 = \epsilon_{\|}$) and "extraordinary" (i.e. $n^2 = 2\epsilon_{\rm R}\epsilon_{\rm L}/(\epsilon_{\rm R} + \epsilon_{\rm L})$) plasma waves [26, 27].

From Eq. (43), the generalized Faraday rotation experienced by the linearly polarized CMR travelling in parallel to the magnetic field can be obtained as

$$\Delta\Phi = \frac{\omega}{2c}\left[\sqrt{\epsilon_{\rm R}} - \sqrt{\epsilon_{\rm L}}\right]\Delta L, \qquad (45)$$

where ΔL is the distance travelled by the signal in the direction parallel to the magnetic field.

According to Eqs. (12) and (13), the leading contribution to the generalized Faraday rotation arises as follows:

$$\left(\frac{\omega_{\rm Be}}{\omega_{\rm CMR}}\right)\left(\frac{\omega_{\rm pe}}{\omega_{\rm CMR}}\right)^2. \qquad (46)$$

In a complementary perspective, when analyzing the possible rotation of the CMR polarization, it seems then preferable to adopt the generalized formulae derived in the present study.

In summary, the contributions to the observed CMR polarization from the magnetized plasma and the anisotropic space-time with a metric of the Bianchi-I type are believed to be as follows:

- the degree of linear polarization and the degree of circular polarization do not vary with time as the electromagnetic wave propagates in an anisotropic space-time of Bianchi-I type;
- the polarized CMR propagates in a Bianchi-I space-time undergoes a rotation of the polarization plane without any change in the degree of polarization;
- the magnetized plasma contribution has a frequency dependence that may allow us to disentangle their relative weight, since the magnetic contribution vanishes for $\omega > \omega_{\rm CMR}$ as $1/\omega^2$.

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1. M. Elitzur, ApJ **457**, 415 (1996).
2. P. Goldreich, D.A. Keeley, and J.Y. Kwan, ApJ **179**, 111 (1973).
3. S.S. Moskaliuk, Phys. Rev. D **68**, 084023 (2003).
4. M. Giovannini, Phys. Rev. D **71**, 021301 (2005).
5. A.A. Grib and A.V. Nesteruk, Yad. Fiz. **38**, 1357 (1983).
6. S.S. Moskaliuk, A.V. Nesteruk, and S.A. Pritomanov, Preprints of Inst. for Theoretical Physics, National Acad. Sci. of Ukraine, Kiev, No. 90-40R, 90-79R (1990).
7. G.I. Kuznetsov, S.S. Moskalyuk, Yu.F. Smirnov, and V.P. Shelest, *Graphical Theory of Representations of Orthogonal and Unitary Groups and their Physical Applications* (Naukova Dumka, Kiev, 1992) (in Russian).
8. G.S. Bisnovatyi-Kogan, *Stellar Physics* (Springer, Berlin, 2001).
9. S.S. Moskaliuk and A.V. Nesteruk, Phys. At. Nucl. **58**, 1744 (1995).
10. Ya.B. Zel'dovich and I.D. Novikov, *Structure and Evolution of the Universe* (Nauka, Moscow, 1975) (in Russian).

11. A.Z. Dolginov, Yu.N. Gnedin and N.A. Silant'ev, *Propagation and Polarization of Radiation in Cosmic Media* (Nauka, Moscow, 1979) (in Russian).
12. B.W. Tolman and R.A. Matzner, Proc. R. Soc., London **392**, 391 (1984).
13. C.H. Brans, Astrophys. J. **197**, 1 (1975).
14. N. Cademi, R. Fabbri, B. Melchiorri *et al.*, Phys. Rev. D: Part. Fields **17**, 1901 (1978).
15. A. Sagnotti and B. Zwiebach, Phys. Rev. D: Part. Fields **24**, 305 (1981).
16. A.M. Cruise, Mon. Not. R. Astron. Soc. **204**, 485 (1983).
17. U. Rees, Astrophys. J. **153**, LI (1968).
18. J. Negroponte and J. Silk, Phys. Rev. Lett. **44**, 1433 (1980).
19. M.M. Basko and A.G. Polnarev, Astron. Zh. **57**, 465 (1980).
20. A.K. Ganguly, S. Konar and P.B. Pal, Phys. Rev. D **60**, 105014 (1999); J. C. D'Olivo, J.F. Nieves and S. Sahu, Phys. Rev. D **67**, 025018 (2003); A.K. Ganguly and R. Parthasarathy, Phys. Rev. D **68**, 106005 (2003).
21. A. Kosowsky and A. Loeb, Astrophys. J. **469**, 1 (1996); M. Giovannini, Phys. Rev. D **56**, 3198 (1997).
22. M. Giovannini, Int. J. Mod. Phys. D **13**, 391 (2004); G. Piccinelli and A. Ayala, Lect. Notes in Phys. **646**, 293 (2004).
23. J.A. Frieman, C.T. Hill, A. Stebbins and I. Waga, Phys. Rev. Lett. **75**, 2077 (1995).
24. S.M. Carroll, G.B. Field and R. Jackiw, Phys. Rev. D **41**, 1231 (1990); S.M. Carroll, Phys. Rev. Lett. **81**, 3067 (1998).
25. K.R.S. Balaji, R.H. Brandenberger and D.A. Easson, JCAP **0312** 008 (2003).
26. T.J.M. Boyd and J.J. Sanderson, *The Physics of Plasmas* (Cambridge Univ. Press, Cambridge, 2003).
27. H.T. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962).
28. M. Giovannini, Phys. Rev. D **61**, 063004 (2000); Phys. Rev. D **61**, 063502 (2000); M. Giovannini and M.E. Shaposhnikov, Phys. Rev. Lett. **80**, 22 (1998).
29. B. Feng, H. Li, M. z. Li, and X. m. Zhang, arXiv:hep-ph/0406269.
30. V.A. Kostelecky, R. Lehnert, and M.J. Perry, Phys. Rev. D **68**, 123511 (2003); O. Bertolami, R. Lehnert, R. Potting, and A. Ribeiro, Phys. Rev. D **69**, 083513 (2004).
31. J. Skilling, Phys. of Fluids **14**, 2523 (1971).

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ПОШИРЕННЯ ПОЛЯРИЗОВАНОГО ВИПРОМІНЮВАННЯ КОСМІЧНОГО МАЗЕРА В АНІЗОТРОПНІЙ ЗАМАГНІЧЕНІЙ ПЛАЗМІ

С.С. Москалюк

Р е з ю м е

Площина поляризації випромінювання космічного мазера може обертатися як у просторі з анізотропною метрикою типу Б'янки-I, так і в анізотропній замагніченій плазмі. У випадку холодної плазми ці явища описуються в межах єдиного підходу. Показано, що отримані в даному дослідженні узагальнені вирази можуть стати у нагоді при безпосередніх вимірюваннях величини обертання площини поляризації випромінювання космічного мазера.