SPIN FLUIDS IN HOMOGENEOUS AND ISOTROPIC SPACE–TIMES*

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We consider a Weyssenhoff fluid assuming that the space-time is homogeneous and isotropic, therefore being relevant for cosmological considerations of gravity theories with torsion. It is explicitly shown that the Weyssenhoff fluids obeying the Frenkel condition or the Papapetrou–Corinaldesi condition are incompatible with the cosmological principle which restricts the torsion tensor to have only the vector and axial vector components. Moreover, it turns out that the Weyssenhoff fluid obeying the Tulczyjew condition is also incompatible with the cosmological principle. This condition has not been analyzed so far in this context. Based on this result, we propose to reconsider a number of previous works that analyzed cosmological solutions of the Einstein–Cartan theory, since their spin fluids did not obey usually the cosmological principle.

1. Introduction

Over the last years, cosmology has become a very active field of research containing many open questions that require the further investigation. Hawking, Penrose, and others have shown that, under fairly general assumptions, solutions of Einstein's field equations evolve singularities. This, from a conceptual point of view, is rather unsatisfactory, we refer the reader to [1].

When cosmological models with torsion were first studied, it was hoped for that the inclusion of torsion would help to avoid these singularities. Unfortunately, this could only be achieved assuming quite unrealistic matter models (see, e.g., [2]). It turns out, however, that most of these cosmological models with torsion did not satisfy the cosmological principle, sometimes also known as the Copernican principle, that strongly restricts the metric and the torsion tensor.

The Copernican principle states that the Universe is spatially homogeneous and isotropic on very large scales. This principle takes the following mathematically precise form. The four-dimensional $4d$ space-time manifold $(\mathcal{M}, g)$ is foliated by $3d$ constant time space-like hypersurfaces which are the orbits of a Lie group G acting on $\mathcal{M}$, with the isometry group $SO(3)$. Following the Copernican principle [3], we assume all fields to be invariant under the action of G

$$\mathcal{L}_\xi g_{\mu\nu} = 0\,, \qquad \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0\,, \tag{1}$$

where ξ are the (six) Killing vectors generating the space-time isometries. The metric tensor is denoted by $g_{\mu\nu}$, $T^\lambda{}_{\mu\nu}$ denotes the torsion tensor, and Greek indices label the holonomic components. For the rest of the paper, only anholonomic components of tensors labelled by Latin indices are used.

Kopczyński initiated the investigation of cosmological models with torsion in [4] and [5], who assumed a Weyssenhoff fluid to be the source of both curvature and torsion. In [4], a non-singular universe with torsion was constructed and an anisotropic model of the universe with torsion was analyzed in [5]. The cosmological principle in the above strict sense (1) was first developed in the Einstein–Cartan theory by Tsamparlis in [3], where it

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was also suggested to reconsider the results in [4,5], since the Weyssenhoff fluid turns out to be incompatible with the cosmological principle (see also [6]). The spin tensor used by [5] in a cosmological context had just one non-vanishing component $S_{23} = K$, where K was assumed to be a function of the time variable $K = K(t)$. Such a spin tensor, as we will show below, is not compatible with the cosmological principle, a fact that was noted by Kopczyński. It should also be pointed out that if we require only the metric of Friedman–Robertson–Walker (FRW) type and put no restrictions on the torsion tensor, then the Weyssenhoff fluid can consistently be used as a source for curvature and torsion (see the energy-momentum tensor Eq. (5.2) and (5.3) in [6]). However, by doing so, one must drop the second condition of (1), $\mathcal{L}_\xi T^\lambda{}_{\mu\nu} = 0$, and use a weaker notion of the cosmological principle. For a recent example where the so-called cosmological model with macroscopic spin fluid was analyzed, see [7].

Applying the restrictions (1) to $g_{\mu\nu}$ yields the FRW-type metric

$$ds^2 = dt^2 - \Big(\frac{a(t)}{1+\frac{k}{4}r^2}\Big)^2(dx^2+dy^2+dz^2), \qquad (2)$$

where $r^2 = x^2+y^2+z^2$, and the 3-space is spherical for $k=1$, flat for $k=0$, and hyperbolic for $k=-1$. If we impose the restrictions (1) on the torsion tensor [3], its allowed components are

$$\begin{aligned} &T_{xxt} = T_{yyt} = T_{zzt} = h(t)\,, \\ &T_{xyz} = T_{zxy} = T_{yzx} = f(t)\,, \end{aligned} \qquad (3)$$

where we follow the notation of [8]. Hence, the cosmological principle allows a vector torsion component $T_t = T^a{}_{at} = 3h$, along the world lines, and an axial vector component $T_{xyz} = f\epsilon_{xyz}$, within the hypersurfaces of constant time. Such a totally skew-symmetric torsion tensor in cosmology was considered earlier in [9], where $h = 0$ was assumed. The geometry parameter k was redefined to include the remaining torsion by $\bar{k} = k - f^2a^2/2$. Also with $h=0$, the cosmological inflation could be explained by torsion in [10], using a rough model.

2. Cosmological Field Equations

The spin-connection 1-form $\tilde{\omega}^i{}_j$ in theories with torsion can be split into a torsion-free part (the usual spin-connection 1-form $\omega^i{}_j$ related to the Christoffel symbol Γ^k_{ij}) and a contortion 1-form part $K^i{}_j$ that takes the torsion of space-time into account:

$$\tilde{\omega}^i{}_j = \omega^i{}_j + K^i{}_j\,. \qquad (4)$$

Here, the torsion tensor (Cartan's torsion) and the contortion tensor are related by the following algebraic relation

$$T^i = \frac{1}{2}T^i{}_{jk}e^i\wedge e^j = De^i = de^i + \tilde{\omega}^i{}_je^j = K^i{}_j\wedge e^j\,, \qquad (5)$$

where we used that $\omega^i{}_j$ is a torsion-free connection. The latter relation between torsion and contortion also implies that their vector and axial vector components are simply related by

$$T_{[ijk]} = K_{[ijk]}\,, \qquad T_{ij}{}^j = \frac{1}{2}K_{ji}{}^j\,. \qquad (6)$$

Since the metric and the contortion (or torsion) components that are compatible with the cosmological constant are fixed, one can compute any geometrical quantity of interest.

Metric (2) gives rise to the basis 1-forms

$$e^t = dt\,, e^{x,y,z} = \frac{a(t)}{1+\frac{k}{4}r^2}dx,y,z\,, \qquad (7)$$

which together with the non-vanishing torsion components (4) yield the non-vanishing connection 1-forms

$$\begin{aligned} \omega^t{}_x &= \frac{\dot{a}}{a}e^x + he^x\,, \\ \omega^t{}_y &= \frac{\dot{a}}{a}e^y + he^y\,, \\ \omega^t{}_z &= \frac{\dot{a}}{a}e^z + he^z\,, \\ \omega^x{}_y &= \frac{ky}{2a}e^x - \frac{kx}{2a}e^y - \frac{f}{2}e^z\,, \\ \omega^x{}_z &= \frac{kz}{2a}e^x - \frac{kx}{2a}e^z + \frac{f}{2}e^y\,, \\ \omega^y{}_z &= \frac{kz}{2a}e^y - \frac{ky}{2a}e^z - \frac{f}{2}e^x \end{aligned} \qquad (8)$$

that are computed from (4); $T^i = de^i + \tilde{\omega}^i{}_je^j$, where the torsion two form can be obtained from (4) via $T^i = (1/2)T^i{}_{jk}e^j\wedge e^k$. The field equations of the Einstein–Cartan theory [2] are obtained by varying the usual Einstein–Hilbert action with respect to the vielbein and the spin-connection as independent variables

$$R^i{}_j - \frac{1}{2}R\delta^i_j = 8\pi\,\Sigma^i{}_j\,,$$

$$T^i{}_{jk} - \delta^i_jT^l{}_{lk} - \delta^i_kT^l{}_{jl} = 8\pi\,s^i{}_{jk}\,, \qquad (9)$$

$\Sigma^i{}_j$ is the canonical energy-momentum tensor, and $s^i{}_{jk}$ is the spin tensor.

3. Weyssenhoff Fluid

The ideal Weyssenhoff fluid [11] is a generalization of the ideal fluid to take into account the properties of spin and torsion in space-time. Its canonical energy-momentum tensor is given by

$$\Sigma_{ij} = p_i u_j + P(u_i u_j + g_{ij})\,,$$

$$p_i = \rho u_i - u^l \nabla_k (u^k S_{li})\,, \tag{10}$$

$$s^i{}_{jk} = u^i S_{jk}\,, \tag{11}$$

where p_i is the momentum density of the fluid, and u_i is the fluid's velocity. By ρ and P, we denoted the energy density and the pressure of the fluid, respectively. The intrinsic angular momentum tensor S_{ij} satisfies the relation

$$S_{ij} = -S_{ji}\,. \tag{12}$$

The spin tensor S^{ij} can be decomposed into two 3-vectors

$$\boldsymbol{\mu} := (S^{01}, S^{02}, S^{03})\,, \tag{13}$$

that, in case we assume the Frenkel condition [12], vanishes in the rest-frame. The second vector

$$\boldsymbol{\sigma} := (S^{23}, S^{31}, S^{12}) \tag{14}$$

in the rest-frame can be regarded as the spin density.

Integrability of the particles' equations of motion requires one more condition that the spin tensor has to satisfy,

$$\zeta^i S_{ji} = 0\,, \tag{15}$$

where the vector ζ^i is usually taken to be the velocity vector of the fluid u^i, following Frenkel [12]. It is also possible to choose the momentum density according to Tulczyjew [13]. Another frequently used condition was put forward by Papapetrou and Corinaldesi [14] who assumed the condition

$$S^{ti} = 0\,, \tag{16}$$

where t stands for the time component of the spin tensor. In the following sections, we investigate whether a Weyssenhoff fluid obeying one of the three presented integrability conditions is compatible with the cosmological principle.

4. Frenkel Condition

If we assume the Frenkel condition [12] $\zeta^i = u^i$, then the spin contribution of the energy-momentum tensor can be rewritten as

$$u^l \nabla_k (u^k S_{li}) = u^l S_{li} \nabla_k u^k + u^l u^k \nabla_k S_{li} =$$

$$= u^l u^k \nabla_k S_{li} = -a^l S_{li}\,. \tag{17}$$

In the third and fourth steps, the Frenkel condition was necessary for the modifications, and we introduced the acceleration of the fluid a^j defined by $a^j = (u^k \nabla_k) u^j$. Hence, Eqs. (10) and (11) with regard for the Frenkel condition yield

$$\Sigma_{ij} = \rho u_i u_j - P(u_i u_j + g_{ij}) + a^l S_{li} u_j\,,$$

$$s^i{}_{jk} = u^i S_{jk}\,, \qquad u^i S_{ik} = 0\,. \tag{18}$$

This implies that, at the zero acceleration a^j of a fluid, one is back at the Einstein gravity [15]. The interpretation of the contribution of the spin angular momentum tensor in (10) in terms of the acceleration strongly depends on the Frenkel condition.

The totally skew-symmetric part of the torsion tensor (3) is allowed by the cosmological principle. Since the four velocity u^i enters the definition of the spin tensor (11), a Weyssenhoff-like fluid cannot be the source of the totally skew-symmetric torsion component (3). On the other hand, it is the Frenkel condition (15) with $\zeta^i = u^i$ that does not allow the Weyssenhoff fluid to be a source of the trace components (3) of the torsion tensor. More explicitly, multiplying Eq. (9b) for the torsion field by δ^j_i leads to

$$-2T^l{}_{lk} = 8\pi s^l{}_{lk} = 8\pi u^l S_{lk} = 0\,, \tag{19}$$

where (11) and the Frenkel condition were taken into account for the last steps. Therefore, we have explicitly shown that the Weyssenhoff fluid obeying the Frenkel condition is incompatible with the cosmological principle put forward in [3].

5. Papapetrou–Corinaldesi Condition

Assuming the Papapetrou–Corinaldesi [14] condition S^{tj} has the following consequences for the torsion tensor implied by the spin fluid. As before, since the fluid's four velocity enters the definition of a spin tensor, the totally skew-symmetric torsion has to vanish. Second, the traced torsion field equation (9b) yields

$$-2T^l{}_{lk} = 8\pi s^l{}_{lk} = 8\pi u^l S_{lk} = 8\pi\delta^l_t S_{lk} = 8\pi S_{tk} = 0\,, \tag{20}$$

where the vanishing of the trace part of the torsion tensor is identically the condition of Papapetrou–Corinaldesi (16).

Therefore, we again conclude that also the Weyssenhoff fluid obeying the Papapetrou–Corinaldesi condition is incompatible with the cosmological principle (since we have $u^i = \delta^i_0$, the Papapetrou–Corinaldesi condition and the Frenkel one in fact take the same form).

6. Tulczyjew Condition

According to our information, it has not been analyzed so far if the Weyssenhoff fluid obeying the Tulczyjew condition [13] is compatible with the cosmological principle. As in the previous section, the fluid cannot be a source of the totally skew-symmetric component of the torsion tensor, because the four velocity u^i of the fluid is present in Eq. (11). However, in this case, (15) does not vanish identically on general ground, and we arrive at

$$-2T^l{}_{lk} = 8\pi s^l{}_{lk} = 8\pi u^l S_{lk}\,, \tag{21}$$

where the last term on the right-hand side need not to vanish. The Tulczyjew condition, Eq. (15) with $\zeta^i = p^i$ explicitely written out, leads to

$$S_{ij}p^j = S_{ij}\big(\rho u^j - u_l\nabla_k(u^k S^{lj})\big) = 0\,, \tag{22}$$

which can be used to express the last term on (21) by

$$\rho S_{ij}u^j = S_{ij}u_l\nabla_k(u^k S^{lj})\,. \tag{23}$$

The fluid's four velocity simply reads $u^j = \delta^j_0$, and Eq. (23) yields, by taking (8) into account, the following form of the Tulczyjew condition that the spin fluid has to satisfy:

$$\rho S_{i0} = S_{ij}\left(\dot S_{0j} + \Gamma_0 S_{0j}\right), \tag{24}$$

where the dot means the differentiation with respect to time t. In contrast to the two previous cases, we find that this condition does not imply the vanishing of the resulting trace of the torsion tensor. For the trace of the Christoffel symbol, we find $\Gamma_0 = \Gamma^k_{k0} = 3(H+h)$, where, by H, we denoted the Hubble parameter defined by $H = \dot a/a$. Equations (24) provide us with for conditions ($i = 0,1,2,3$) which we will analyze in more details. For $i = 0$, the left-hand side of (24) vanishes, and one is left with

$$0 = S_{0j}\left(\dot S_{0j} + \Gamma_0 S_{0j}\right) = -S^{0j}\left(\dot S_{0j} + \Gamma_0 S_{0j}\right), \tag{25}$$

which can be written in an equivalent form by using the introduced vectors $\boldsymbol\mu$ and $\boldsymbol\sigma$ in (13) and (14) and leads to

$$0 = (\dot{\boldsymbol\mu} + \Gamma_0\,\boldsymbol\mu)\cdot\boldsymbol\mu\,, \tag{26}$$

where $\cdot$ means the usual inner product of vectors.

From this, we conclude that the vectors $\boldsymbol\mu$ and $(\dot{\boldsymbol\mu} + \Gamma_0\boldsymbol\mu)$ are orthogonal to each other. For the remaining values $i = 1,2,3$, condition (24) takes the following form in terms of the three-vector:

$$\boldsymbol\mu = \frac{1}{\rho}(\dot{\boldsymbol\mu} + \Gamma_0\,\boldsymbol\mu)\times\boldsymbol\sigma\,. \tag{27}$$

Here, we now see that (26) is not an independent equation since we derive from the latter that $\boldsymbol\mu \perp (\dot{\boldsymbol\mu} + \Gamma_0\boldsymbol\mu)$ and, moreover, $\boldsymbol\mu \perp \boldsymbol\sigma$. Therefore, we find the following non-vanishing components of the induced torsion tensor via the field equations (21):

$$T^l{}_{l0} = 0\,,$$

$$T^l{}_{li} = 4\pi\boldsymbol\mu\,, \tag{28}$$

where $\boldsymbol\mu$ is given by Eq. (27). Note that the index i only takes the values $1,2,3$, and we furthermore suppressed the explicit index for the vector $\boldsymbol\mu$. We can now try to continue the construction of a spin fluid that is compatible with cosmological principle, namely the Weyssenhoff fluid obeying the Tulczyjew condition. In principle, we have two possibilities: (**a**) we choose the vector $\boldsymbol\mu$ of the spin tensor so that it satisfies condition (26), and we choose the spin density vector $\boldsymbol\sigma$ so that Eq. (27) is satisfied. On the other hand, (**b**) allows us to prescribe the spin density vector $\boldsymbol\sigma$. Then, in order to get an allowed $\boldsymbol\mu$, one has to solve the vector differential equation (27), so that each solution satisfies (26). However, one must be careful with the above result. The cosmological principle allows a vector torsion component, but only along the world lines, $T^l{}_{lt} \neq 0$. The other components are excluded. Therefore, also the Weyssenhoff fluid obeying the Tulczyjew condition is incompatible with the cosmological principle, since (28a) vanishes identically.

7. Conclusions and Outlook

The restrictions that follow from assuming the homogeneity and the isotropy on the very large scales of the Universe (the cosmological principle) allow one metric component and two torsion components; the vector and axial vector components of the torsion tensor. We showed that the Weyssenhoff fluids obeying either the Frenkel or the Papapetrou–Corinaldesi condition are incompatible with this principle. Furthermore, we analyzed the Tulczyjew condition which, in principle, allows one to construct a non-trace-free torsion tensor. However, its time component, allowed by the cosmological principle, vanishes identically. Therefore, it has been shown that no spin fluid obeying the common integrability conditions is compatible with the cosmological principle. This rather surprising result shows the necessity to reconsider the previous works on cosmology with torsion, since none of these results can be regarded as a truly cosmological model with torsion.

Furthermore, it raises the question, whether an integrability condition exists that allows a spin fluid to have homogeneous and isotropic torsion components. The construction of such a spin fluid, if possible, could be a subject of the further research. If it will turn out that such a spin fluid does not exist, this would have quite significant consequences for the physical applicability of such models. The possible non-existence would indicate that the Weyssenhoff fluid is not a very good model for a macroscopic spin fluid. If, on the other hand, such a cosmological spin fluid can be constructed, it would be very interesting to study its properties. For example, the consequences of a truly cosmological spin fluid on the singularities, mentioned in the introduction, were worth a thorough investigation. Moreover, it would then be possible to reconsider some of the previously suggested models in a real cosmological fashion. Finally, we would like to mention the possibility of applying the cosmological principle to the more general hyperfluid [16, 17]. However, a axial vector component for the torsion tensor cannot be obtained from the hyperfluid, since a generalized form of Eq. (11) essentially enters the spin tensor.

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СПІНОВІ РІДИНИ В ОДНОРІДНОМУ ТА ІЗОТРОПНОМУ ПРОСТОРІ-ЧАСІ

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Р е з ю м е

Розглянуто рідину Віссенхофа у припущенні, що простір-час є однорідним та ізотропним і тому придатний для космологічного розгляду теорій гравітації з крученням. Показано, що рідини Віссенхофа, для яких виконується умова Френкеля або умова Папапетроу–Коріналдезі, є несумісними з космологічним принципом, згідно з яким тензор кручення має тільки векторну і аксіальну векторну компоненти. Більше того, виявилося, що рідина Віссенхофа, що задовольняє умову Тульчієва, також несумісна з космологічним принципом. Але цю умову не було проаналізовано повністю в цьому контексті. Ґрунтуючись на цих результатах, запропоновано переглянути деякі попередні роботи, в яких проаналізовано космологічні розв'язки теорії Ейнштейна–Картана, оскільки їхні спінові рідини звичайно не задовольняють космологічний принцип.