BBGKY HIERARCHY AND DYNAMICS OF CORRELATIONS

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We derive the BBGKY hierarchy for the Fermi and Bose many-particle systems, using the von Neumann hierarchy for the correlation operators. The solution of the Cauchy problem of the formulated hierarchy in the case of an n-body interaction potential is constructed in the space of sequences of trace-class operators.

1. Introduction

In recent years, a large progress in the mathematical theory of the BBGKY hierarchy for quantum many-particle systems is observed. A good example of such progress is the rigorous derivation of quantum kinetic equations that describe the Bose condensate [1–3]. In the original works of Bogolyubov [4–6], the solution of the initial-value problem of the BBGKY hierarchy was constructed in the form of an iteration series. The same representation of the solution is also used in modern works [7–9].

In the case of the Maxwell–Boltzmann statistics, the solution was also constructed in the form of the expansion over particle clusters, whose evolution is governed by the cumulants of the groups of operators for the von Neumann equations [10] (or by reduced cumulants [11]). These expansions of the solution were constructed on the base of the non-equilibrium grand canonical ensemble [11, 12].

In this paper, we propose an alternative method of description of the evolution of quantum many-particle systems. States of such systems are described in terms of correlation operators, whose evolution is governed by the von Neumann hierarchy. Based on the solution of such hierarchy, we define the s-particle (marginal) density operators and derive the BBGKY hierarchy [5] that can describe the evolution of infinite-particle systems.

Using the solution of the von Neumann hierarchy for the correlation operators, we derive a formula for the solution of the Cauchy problem of the BBGKY hierarchy in the form of an expansion over particle clusters, whose evolution is governed by the cumulants of the group of operators of finitely many Fermi or Bose particles.

The usual iteration series representation of the solution can be obtained from the constructed solution with the use of analogs of the Duhamel formulas for some classes of interaction potentials.

Let us outline the structure of the present work. In Section 2, we define the basic notions of the Fermi and Bose many-particle systems and introduce the evolution equations for the correlation operators. In Section 3, we introduce the s-particle density operators based on the solution of the von Neumann hierarchy for correlation operators and derive the BBGKY hierarchy. In Section 4, we construct the solution of the initial-value problem of the BBGKY hierarchy in the case of the initial data satisfying the chaos property.

2. Evolution of Correlations of Fermi and Bose Many-particle Systems

We consider a quantum system of a non-fixed (i.e., arbitrary but finite) number of identical (spinless) particles with unit mass $m = 1$ in the space $\mathbb{R}^\nu$, $\nu \geq 1$ (in the terminology of statistical mechanics, it is known as a non-equilibrium grand canonical ensemble [12]) that obey the Fermi–Dirac or Bose–Einstein statistics.

States of the system belong to the space $\mathfrak{L}^1_\pm(\mathcal{F}^\pm_\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{L}^1_\pm(\mathcal{H}^\pm_n)$ of sequences $f = \big(I, f_1, \dots, f_n, \dots\big)$ of trace-class operators $f_n \equiv f_n(1,\dots,n) \in \mathfrak{L}^1_\pm(\mathcal{H}^\pm_n)$, satisfying the symmetry condition $f_n(1,\dots,n) = f_n(i_1,\dots,i_n)$, if $\{i_1,\dots,i_n\} \in \{1,\dots,n\}$, where $\mathcal{H}$ is a Hilbert space associated with a single particle, $\mathcal{H}^\pm_n = \mathcal{H}^{\hat{\otimes} n}$ is the symmetric (or antisymmetric) tensor product of n Hilbert spaces $\mathcal{H}$; $\mathcal{F}^\pm_\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^\pm_n$ is the Fock space over the Hilbert space $\mathcal{H}$. The spaces $\mathfrak{L}^1_\pm(\mathcal{F}^\pm_\mathcal{H})$ are equipped with the trace norm

$$\|f\|_{\mathfrak{L}^1_\pm(\mathcal{F}^\pm_\mathcal{H})} = \sum_{n=0}^{\infty} \|f_n\|_{\mathfrak{L}^1_\pm(\mathcal{H}^\pm_n)} = \sum_{n=0}^{\infty} \mathrm{Tr}_{1,\dots,n}|f_n(1,\dots,n)|.$$

By $\mathfrak{L}^1_{0,\pm}$, we denote the everywhere dense set in $\mathfrak{L}^1_{\pm}(\mathcal{F}^{\pm}_{\mathcal{H}})$ of finite sequences of degenerate operators [13] with infinitely differentiable kernels with compact supports. Note that the space $\mathfrak{L}^1_{\pm}(\mathcal{F}^{\pm}_{\mathcal{H}})$ contains the sequences of operators more general than those determining the states of systems of particles. Hereafter, we assume that $\mathcal{H} = L^2(\mathbb{R}^\nu)$.

The Bose–Einstein and Fermi–Dirac statistics endow the state operators with additional symmetry properties. We illustrate them on the kernels of operators[11]. Let $f_n(q_1,\ldots,q_n;q'_1,\ldots,q'_n)$ be the kernel of the operator $f_n \in \mathfrak{L}^1_{\pm}(L^{2,\pm}_n)$. In the case of the Bose–Einstein statistics, the kernel is a function that is symmetric with respect to permutations in each group of arguments:

$$f_n(q_1,q_2,...,q_n;q'_1,q'_2,...,q'_n) =$$

$$= f_n(q_{\pi(1)},q_{\pi(2)},...,q_{\pi(n)};q'_{\pi'(1)},q'_{\pi'(2)},...,q'_{\pi'(n)}),$$

and, in the case of the Fermi–Dirac statistics, the corresponding kernel is antisymmetric:

$$f_n(q_1,q_2,...,q_n;q'_1,q'_2,...,q'_n) = (-1)^{(|\pi|+|\pi'|)}\times$$

$$\times f_n(q_{\pi(1)},q_{\pi(2)},...,q_{\pi(n)};q'_{\pi'(1)},q'_{\pi'(2)},...,q'_{\pi'(n)}),$$

where $\pi \in \mathfrak{S}_n$ and $\pi' \in \mathfrak{S}_n$ are the permutation functions, $\mathfrak{S}_n$ is a symmetric group, i.e. the group of all the permutations of the set {1,2,...,n}, $|\pi| = 0, |\pi'| = 0$ if the permutation is even, and $|\pi| = 1, |\pi'| = 1$ if it is odd.

We define the permutation operator $p_\pi : \mathfrak{L}^1(\mathcal{H}^{\otimes n}) \to \mathfrak{L}^1(\mathcal{H}^{\otimes n})$ in terms of kernels of operators by the following formula:

$$(p_\pi f_n)(q_1,q_2,...,q_n;q'_1,q'_2,...,q'_n) =$$

$$= f_n(t,q_{\pi(1)},q_{\pi(2)},...,q_{\pi(n)};q'_1,q'_2,...,q'_n).$$

Let us introduce the symmetrization operator $\mathcal{S}^+_n : \mathfrak{L}^1(\mathcal{H}^{\otimes n}) \to \mathfrak{L}^1_+(\mathcal{H}^+_n)$ and the antisymmetrization operator $\mathcal{S}^-_n : \mathfrak{L}^1(\mathcal{H}^{\otimes n}) \to \mathfrak{L}^1_-(\mathcal{H}^-_n)$ by the formulas

$$\mathcal{S}^+_n = \frac{1}{n!}\sum_{\pi\epsilon\mathfrak{S}_n} p_\pi,$$

$$\mathcal{S}^-_n = \frac{1}{n!}\sum_{\pi\epsilon\mathfrak{S}_n} (-1)^{|\pi|} p_\pi. \qquad (1)$$

The Hamiltonian of the system, $H = \bigoplus_{n=0}^{\infty} H_n$, is a self-adjoint operator with the domain $\mathcal{D}(H) = \{\psi = \oplus\psi_n \in \mathcal{F}^{\pm}_{\mathcal{H}} \mid \psi_n \in \mathcal{D}(H_n) \in \mathcal{H}^{\pm}_n, \sum_n \|H_n\psi_n\|^2 < \infty\} \subset \mathcal{F}^{\pm}_{\mathcal{H}}$. On the subspace of infinitely differentiable symmetric (or antisymmetric) functions with compact supports $\psi_n \in L^{2,\pm}_0(\mathbb{R}^{\nu n}) \subset L^{2,\pm}(\mathbb{R}^{\nu n})$, the n-particle Hamiltonian H_n acts according to the formula ($H_0 = 0$)

$$H_n\psi_n = -\frac{\hbar^2}{2}\sum_{i=1}^{n}\Delta_{q_i}\psi_n +$$

$$+\sum_{k=1}^{n}\sum_{i_1<\ldots<i_k=1}^{n}\Phi^{(k)}(q_{i_1},\ldots,q_{i_k})\psi_n, \qquad (2)$$

where $\Phi^{(k)}$ is a k-body interaction potential satisfying the Kato conditions [13], and $h = 2\pi\hbar$ is the Planck constant.

We describe states of the system with Hamiltonian (2) by the sequence $g(t) = (I, g_1(t,1),\ldots,g_n(t,1,\ldots,n),\ldots) \in \mathfrak{L}^1_{\pm}(\mathcal{F}^{\pm}_{\mathcal{H}})$ of correlation operators, whose evolution is determined by the initial-value problem of the von Neumann hierarchy:

$$\frac{d}{dt}g_n(t,Y) = -\mathcal{N}_n(Y)g_n(t,Y) +$$

$$+\sum_{\substack{\mathrm{P}:\,Y=\bigcup_i X_i,\\ |\mathrm{P}|\neq 1}}\ \sum_{\substack{Z_1\subset X_1,\\ Z_1\neq\emptyset}}\cdots\sum_{\substack{Z_{|\mathrm{P}|}\subset X_{|\mathrm{P}|},\\ Z_{|\mathrm{P}|}\neq\emptyset}}(-\mathcal{N}_{\mathrm{int}}^{(\sum_{r=1}^{|\mathrm{P}|}|Z_r|)}(Z_1,\ldots,Z_{|\mathrm{P}|}))\times$$

$$\times\mathcal{S}^{\pm}_n\prod_{X_i\subset\mathrm{P}} g_{|X_i|}(t,X_i), \qquad (3)$$

$$g_n(t,Y)\big|_{t=0} = g_n(0,Y), \qquad n \geq 1, \qquad (4)$$

where $\sum_{\mathrm{P}}$ is the sum over all possible partitions of the set $Y = \{1,\ldots,n\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets, $X_i \subset Y$, $\sum_{Z_j\subset X_j}$ is a sum over all subsets $Z_j \subset X_j$, for $f_n \in \mathfrak{L}^1_{0,\pm}(\mathcal{H}^{\pm}_n) \subset \mathcal{D}(\mathcal{N}_n) \subset \mathfrak{L}^1_{\pm}(\mathcal{H}^{\pm}_n)$, the von Neumann operator $\mathcal{N}_n$ is defined by

$$\mathcal{N}_n f_n = -\frac{i}{\hbar}\big(f_n H_n - H_n f_n\big),$$

and

$$\mathcal{N}^{(k)}_{\mathrm{int}} f_n = -\frac{i}{\hbar}(f_n\Phi^{(k)} - \Phi^{(k)} f_n).$$

In the case of the Maxwell–Boltzmann statistics, this hierarchy was studied in work [14].

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We note that the relation between correlation operators defined by (3)-(4) and the density operators $D(t) \in \mathfrak{L}^1_{\pm}(\mathcal{F}^{\pm}_{\mathcal{H}})$ has the form

$$g_n(t,Y) = D_n(t,Y) + \sum_{\substack{\mathrm{P}:\ Y=\bigcup_i X_i,\\ |\mathrm{P}|\neq 1}} (-1)^{|\mathrm{P}|-1}\times$$

$$\times(|\mathrm{P}|-1)!\, \mathcal{S}^{\pm}_n \prod_{X_i\subset \mathrm{P}} D_{|X_i|}(t,X_i), \qquad (5)$$

where the sequence $D(t)$ is a solution of the Cauchy problem of the von Neumann equation [11].

Further, we consider a more general notion, namely, the correlation operators of particle clusters that describe the correlations between clusters of particles.

We introduce the following notations: $Y_{\mathrm{P}} \equiv \{X_1,\dots,X_{|\mathrm{P}|}\}$ is a set, whose elements are $|\mathrm{P}|$ mutually disjoint subsets $X_i \subset Y \equiv \{1,\dots,s\}$ of the partition $\mathrm{P} : Y = \bigcup_{i=1}^{|\mathrm{P}|} X_i$. Since $Y_{\mathrm{P}} \equiv \{X_1,\dots,X_{|\mathrm{P}|}\}$, Y_1 is the set that consists of one element $Y = \{1,\dots,s\}$ of the partition $\mathrm{P}(|\mathrm{P}|=1)$. To underline that the set $\{1,\dots,s\}$ is a connected subset (variables that characterize the cluster of s elements) of a partition $\mathrm{P}(|\mathrm{P}|=1)$, we also denote the set Y_1 by the symbol $\{1,\dots,s\}_1$. In particular case, the cluster can consist of one particle.

We define the declusterization mapping Θ by the following formula:

$$\Theta(Y_P) = Y.$$

Let us consider the set $X_c = \{Y_1, s+1,\dots,s+n\}$. It holds:

$$\Theta(X_c) = X \equiv \{1,\dots,s+n\}.$$

The relation between correlation operators of particle clusters $g(t) = (I, g_1(t,Y_1),\dots,g_{1+n}(X_c),\dots)$ and correlation operators of particles (5) is given by:

$$g_{1+n}(t,X_c) = \sum_{\mathrm{P}:X_c=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\,\times$$

$$\times\mathcal{S}^{\pm}_{s+n}\prod_{X_i\subset\mathrm{P}}\ \sum_{\mathrm{P}':\,\Theta(X_i)=\bigcup_j Z_{ij}}\ \prod_{Z_{ij}\subset\mathrm{P}'} g_{|Z_{ij}|}(t,Z_{ij}). \qquad (6)$$

Using hierarchy (3) from formula (6), we derive the von Neumann hierarchy for the correlation operators of particle clusters:

$$\frac{d}{dt}g_{1+n}(t,X_c) = -\mathcal{N}_{s+n}(X)g_{1+n}(t,X_c)+$$

$$+\mathcal{S}^{\pm}_{s+n}\sum_{\substack{\mathrm{P}:\ X_c=\bigcup_i X_i,\\ |\mathrm{P}|>1}}\ \sum_{\substack{Z_1\subset\Theta(X_1),\\ |Z_1|\geq 1}}\dots\sum_{\substack{Z_{|\mathrm{P}|}\subset\Theta(X_{|\mathrm{P}|}),\\ |Z_{|\mathrm{P}|}|\geq 1}}\times$$

$$\times(-\mathcal{N}^{(\sum_{i=1}^{|\mathrm{P}|}|Z_i|)}_{\mathrm{int}}(Z_1,\dots,Z_{|\mathrm{P}|}))\prod_{X_i\subset\mathrm{P}} g_{|X_i|}(t,X_i), \qquad (7)$$

where $X = \{1,\dots,s+n\}$.

3. The Derivation of the BBGKY Hierarchy from the Dynamics of Correlations

Let us introduce the s-particle (marginal) density operators using the correlation operators that satisfy hierarchy (7) by the formula

$$F_s(t,Y) := \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\dots,s+n}g_{1+n}(t,X_c), \qquad (8)$$

where $Y = \{1,\dots,s\}$, $X_c = \{Y_1, s+1,\dots,s+n\}$. Series (8) is convergent in $\mathfrak{L}^1_{\pm}(\mathcal{F}^{\pm}_{\mathcal{H}})$ if $g_{1+n}(t) \in \mathfrak{L}^1_{\pm}(\mathcal{H}^{\pm}_{s+n})$.

We show that the evolution of marginal density operators defined by expansion (8) is governed by the chain of equations introduced by Bogolyubov [4]. The following derivation is given for the case of a two-body interaction potential, i.e. the terms $\mathcal{N}^{(l)}_{\mathrm{int}}$ with $l > 2$ are equal to zero.

Let us differentiate both sides of expansion (8) with respect to the time variable in the sense of pointwise convergence in $\mathfrak{L}^1_{\pm}(\mathcal{F}^{\pm}_{\mathcal{H}})$ and use equality (7):

$$\frac{d}{dt}F_s(t,Y)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\dots,s+n}\Big(-\mathcal{N}_{s+n}(X)g_{1+n}(t,X_c)+$$

$$+\mathcal{S}^{\pm}_{s+n}\sum_{\mathrm{P}:\ X_c=X_1\bigcup X_2}\ \sum_{i_1\in\Theta(X_1)}\ \sum_{i_2\in\Theta(X_2)}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))\times$$

$$\times g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2)\Big),$$

where $X = \{1,\dots,s+n\}$, and the operators $\mathcal{S}^{\pm}_{s+n}$ are defined by (1).

In view of the fact that

$$\mathcal{N}_{s+n}(X) = \mathcal{N}_s(Y) + \mathcal{N}_n(X\setminus Y)+$$

$$+\sum_{i_1\subset Y}\ \sum_{i_2\subset X\setminus Y}(\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2)),$$

and that, for $g_{1+n}(t) \in \mathfrak{L}^1_{\pm}(\mathcal{H}^{\pm}_{s+n})$, it holds

$$\mathrm{Tr}_{s+1,\dots,s+n}(-\mathcal{N}_n(X\setminus Y))g_{1+n}(t,X_c)=0,$$

we rewrite the last equation in the following form:

$$\frac{d}{dt}F_s(t,Y)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\dots,s+n}\Big(\big(-\mathcal{N}_s(Y)+$$

$$+\sum_{i_1\in Y}\sum_{i_2\in X\setminus Y}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))\big)g_{1+n}(t,X_c)+$$

$$+\mathcal{S}^{\pm}_{s+n}\sum_{\mathrm{P}\,:\,X_c\,=\,X_1\bigcup X_2}\ \sum_{i_1\in\Theta(X_1)}\sum_{i_2\in\Theta(X_2)}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))\times$$

$$\times g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2)\Big).$$

Since the operator $\mathcal{N}_s(Y)$ does not depend on the variables $s+1,\dots,s+n$, we obtain, according to the symmetry properties of operators $g_{1+n}(t)$:

$$\frac{d}{dt}F_s(t,Y)=-\mathcal{N}_s(Y)\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\dots,s+n}g_{1+n}(t,X_c)+$$

$$+\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\dots,s+n}\Big(n\sum_{i\in Y}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i,s+1))g_{1+n}(t,X_c)+$$

$$+\mathcal{S}^{\pm}_{s+n}\sum_{\mathrm{P}\,:\,X_c\,=\,X_1\bigcup X_2}\ \sum_{i_1\in\Theta(X_1)}\sum_{i_2\in\Theta(X_2)}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))\times$$

$$\times g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2)\Big).$$

According to definition (8), the first term on the right-hand side of the equation is equal to $(-\mathcal{N}_sF_s)$. Using the symmetry property of the product $g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2)$ which is the consequence of being under the trace sign, we make the following rearrangement in the last term:

$$\sum_{\mathrm{P}:X_c=X_1\cup X_2}\ \sum_{i_1\in\Theta(X_1)}\sum_{i_2\in\Theta(X_2)}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))\times$$

$$\times g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2)=$$

$$=\sum_{k=1}^{n+1}\ \sum_{\substack{Z\subset\{s+1,\dots,s+n+1\},\\ |Z|=k}}\ \sum_{i_1\in Y}\sum_{i_2\in Z}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))\times$$

$$\times g_k(t,Z)g_{2+n-k}(t,\{X_c,s+1+n\}\setminus Z).$$

As a result, we obtain

$$\frac{d}{dt}F_s(t,Y)=-\mathcal{N}_s(Y)F_s(Y)+\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\dots,s+n+1}\times$$

$$\times\Big(\sum_{i\in Y}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i,s+1))g_{2+n}(t,X_c,s+n+1)+$$

$$+\frac{1}{n+1}\mathcal{S}^{\pm}_{s+n+1}\sum_{k=1}^{n+1}\ \sum_{\substack{Z\subset\{s+1,\dots,s+1+n\},\\ |Z|=k}}\ \sum_{i_1\in Y}\sum_{i_2\in Z}\times$$

$$\times(-\mathcal{N}^{(2)}_{\mathrm{int}}(i_1,i_2))g_k(t,Z)g_{2+n-k}(t,\{X_c,s+1+n\}\setminus Z)\Big).$$

Then we get Tr_{s+1} out of the sum and use the symmetry property to rewrite the last term as

$$\frac{d}{dt}F_s(t,Y)=-\mathcal{N}_s(Y)F_s(t,Y)+$$

$$+\mathrm{Tr}_{s+1}\sum_{i\in Y}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i,s+1))\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+2,\dots,s+n+1}\times$$

$$\times\Big(g_{2+n}(t,X_c,s{+}n{+}1)+\mathcal{S}^{\pm}_{s+n+1}\sum_{k=1}^{n+1}\frac{n!}{(k-1)!(n+1-k)!}\times$$

$$\times g_k(t,s+n+2-k,\dots,s+n+1)\times$$

$$\times g_{2+n-k}(t,Y_1,s+1,\dots,s+n-k+1)\Big).$$

Now, to finish the derivation, we need an auxiliary fact. In view of the symmetry of the operators $g_{1+n}(t)$ with respect to $s+1,\dots,s+1+n$ variables under the corresponding trace signs, it holds:

$$g_{1+n}(t,\{1,\dots,s+1\}_1,s+2,\dots,s+n+1)=$$

$$=g_{n+2}(t,\{1,\dots,s\}_1,s+1,\dots,s+n+1)+$$

$$+\mathcal{S}^{\pm}_{s+n+1}\sum_{k=1}^{n+1}\frac{n!}{(k-1)!(n-k+1)!}g_k(t,s+n+2-k,\ldots,$$

$$s+n+1)g_{2+n-k}(t,\{1,\ldots,s\}_1,s+1,\ldots,s+n-k+1).$$

Thus, we obtain the equality

$$\frac{d}{dt}F_s(t,Y)=-\mathcal{N}_s(Y)F_s(t,Y)+$$

$$+\mathrm{Tr}_{s+1}\sum_{i\in Y}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i,s+1))\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+2,\ldots,s+n+1}\times$$

$$\times g_{1+n}(t,\{1,\ldots,s+1\}_1,s+2,\ldots,s+n+1),$$

and, according to definition (8), we deduce finally:

$$\frac{d}{dt}F_s(t,Y)=-\mathcal{N}_s(Y)F_s(t,Y)+$$

$$+\mathrm{Tr}_{s+1}\sum_{i\in Y}(-\mathcal{N}^{(2)}_{\mathrm{int}}(i,s+1))F_{s+1}(t,Y,s+1).$$

This equality can be treated as the hierarchy of evolution equations for the marginal density operators and was derived by Bogolyubov in work [4] from the von Neumann equation for a system of fixed number of particles. For the case of a n-body interaction potential, we derive the BBGKY hierarchy in a similar way. Its initial-value problem has the form

$$\frac{d}{dt}F_s(t,Y)=-\mathcal{N}_s(Y)F_s(t,Y)+\sum_{n=1}^{\infty}\frac{1}{n!}\times$$

$$\times\mathrm{Tr}_{s+1,\ldots,s+n}\sum_{\substack{Z\subset Y,\\ Z\neq\emptyset}}\Big(-\mathcal{N}^{(|Z|+n)}_{\mathrm{int}}(Z,X\backslash Y)\Big)F_{s+n}(t,X),\quad(9)$$

$$F_s(t)\mid_{t=0}=F_s(0),\quad s\geq 1.\qquad(10)$$

4. On the Solution of the Cauchy Problem for the BBGKY Hierarchy

Let us construct the solution of the Cauchy problem of the BBGKY hierarchy (9)-(10) for one physically motivated example of initial data, namely the case of initial data satisfying the chaos property. A chaos property means that there are no correlations in the system at the initial moment of time (t=0):

$$g(0)=(I,g_1(0,Y_1),0,0,...),\qquad(11)$$

which means in terms of the marginal density operators (8) that

$$F_s(0,Y)=g_1(0,Y_1),\qquad(12)$$

where $Y=\{1,\ldots,s\}$, $Y_1=\{1,\ldots,s\}_1$ is a notation introduced in Section 2.

By $\mathfrak{A}_{1+n}(t,X_c)$, we denote the $(1+n)$-th order, $n\geq 1$, cumulant of groups of operators

$$\mathcal{G}_n(-t)f_n=e^{-\frac{i}{\hbar}tH_n}f_ne^{\frac{i}{\hbar}tH_n}$$

which is defined by the formula

$$\mathfrak{A}_{1+n}(t,X_c):=\sum_{\mathrm{P}:\,X_c=\bigcup_k Z_k}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\times$$

$$\times\prod_{Z_k\subset\mathrm{P}}\mathcal{G}_{|\Theta(Z_k)|}(-t,\Theta(Z_k)),\qquad(13)$$

where H_n is Hamiltonian (2), $f_n\in\mathfrak{L}^1_{\pm}(\mathcal{H}^{\pm}_n)$, Θ is the declusterization mapping defined in Section 2, and $X_c=\{Y_1,s+1,\ldots,s+n\}$.

In the case of the Maxwell–Boltzmann statistics, a solution of the von Neumann hierarchy for the initial data (11) was constructed in [14].

In the case of the Bose or Fermi system of particles, a solution of the initial-value problem of the von Neumann hierarchy (7) for the initial data (11) has the form

$$g_{1+n}(t,X_c)=\mathfrak{A}_{1+n}(t,X_c)\mathcal{S}^{\pm}_{s+n}\prod_{i\in X_c}g_1(0,i),\qquad(14)$$

where $\mathfrak{A}_{1+n}(t,X_c)$ is given by (13), and operators $\mathcal{S}^{\pm}_{s+n}$ are defined by (1).

According to formula (14) and definition (8), we have

$$F_s(t,Y)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\ldots,s+n}\mathfrak{A}_{1+n}(t,X_c)\mathcal{S}^{\pm}_{s+n}\prod_{i\in X_c}g_1(0,i).$$

In view of equality (12), i.e. $g_1(0,i)=\prod_{j\in\Theta(i)}F_1(0,j)$, we obtain finally

$$F_s(t,Y)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{s+1,\ldots,s+n}\mathfrak{A}_{1+n}(t,X_c)\mathcal{S}^{\pm}_{s+n}\prod_{i=1}^{s+n}F_1(0,i),$$

which is the solution of the Cauchy problem of the BBGKY hierarchy (9)-(10) for the Fermi or Bose many-particle system with initial data satisfying the chaos property.

For arbitrary initial data $F(0) \in \mathfrak{L}^1_{\alpha,\pm}(\mathcal{F}^\pm_{\mathcal{H}}) = \bigoplus_{n=0}^{\infty} \alpha^n \mathfrak{L}^1_{\pm}(\mathcal{H}^\pm_n)$, $\alpha > e$, and $t \in \mathbb{R}$, there exists the unique solution of the initial-value problem (9)-(10) given by the series

$$F_s(t,1,\dots,s) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{s+1,\dots,s+n} \mathfrak{A}_{1+n}(t,X_c) \times$$

$$\times F_{s+n}(0,1,\dots,s+n). \tag{15}$$

For initial data $F(0) \in \mathfrak{L}^1_{\alpha,0,\pm}(\mathcal{F}^\pm_{\mathcal{H}}) \in \mathfrak{L}^1_{\alpha,\pm}(\mathcal{F}^\pm_{\mathcal{H}})$, expansion (15) is a strong solution, and, for arbitrary initial data from the space $\mathfrak{L}^1_{\alpha,\pm}(\mathcal{F}^\pm_{\mathcal{H}})$, it is a weak solution.

5. Conclusions

In the present paper, the marginal density operators were defined by means of a solution of the von Neumann hierarchy (7) for correlation operators by formula (8). This definition allowed us to construct the BBGKY hierarchy for the Fermi and Bose particles on the basis of the dynamics of correlations in the space of sequences of trace-class operators.

We note that, using definition (8) of marginal density operators, it is possible to justify the BBGKY hierarchy in other Banach spaces that contain the states of infinite-particle systems, contrary to the definition of the marginal density operators in the framework of a non-equilibrium grand canonical ensemble [11].

We have also defined solution (15) of the BBGKY hierarchy (9) for the Fermi and Bose particles with an n-body interaction potential. Such a solution is represented in the form of expansion (15) over the clusters of particles, whose evolution is governed by cumulants (13) of the groups of operators for the von Neumann equations. These cumulants for the Fermi and Bose particles have the same structure, as in the Maxwell–Boltzmann case [10].

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ІЄРАРХІЯ ББГКІ ТА ДИНАМІКА КОРЕЛЯЦІЙ

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Р е з ю м е

Виведено ієрархію ББГКІ для Фермі та Бозе багаточастинкових систем на основі ієрархії фон Неймана для кореляційних операторів. Побудовано розв'язок задачі Коші сформульованої ієрархії у випадку n-частинкового потенціалу взаємодії для початкових даних із простору послідовностей ядерних операторів.