ON THE PLASTICITY OF NONLOCAL QUANTUM CORRELATIONS

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The quantum correlations of two or more entangled particles present the possibility of stronger-than-classical outcome coincidences. We investigate binary correlations of quanta with spin one, three-half, and higher in a state satisfying a uniqueness property in the sense that knowledge of an outcome of one particle observable entails the certainty that, if this observable were measured on the other particle(s) as well, the outcome of the measurement would be a unique function of the outcome of the measurement performed. We also investigate correlations of four particles with spin one-half.

1. Introduction

The possibility of a peculiar and "mind-boggling" type of connectedness between two or more spatially separated particles beyond classical correlations surprised the quantum pioneers in their early exploration of quanta. Already Schrödinger noted that a state of several quantized particles or quanta could be *entangled* (in Schrödinger's own German terminology *"verschränkt"*) in the sense that it cannot be represented as the product of states of the isolated, individual quanta, but is rather defined by the *joint* or *relative* properties of the quanta involved [1, 2]. Typical examples of such joint properties of entangled states are the propositions, "when measured along two or more different directions, two spin one-half particles have opposite spin" (defining the Bell singlet state), or "when measured along a particular direction, three spin one-half particles have identical spin" (one of the three defining properties of the Greenberger–Horne–Zeilinger–Mermin state).

With respect to the outcome of certain measurements on the individual particles in an entangled state, the observation of stronger-than-classical correlations for nonlocal, i.e., spatially and even causally separated, quanta in "delayed choice" measurements has been experimentally verified [3]. A typical phenomenological criterion of such correlations is the *increased* or *decreased* frequency of the occurrence of certain coincidences of outcomes, such as the more- or less-often-than-classically expected recordings of joint spin up and down measurements labeled by "++," "+−," "−+" or "−−," respectively.

Stated pointedly, the "magic" behind the quantum correlations as compared to classical correlations resides in the fact that, for almost all measurement directions (despite the collinear or orthogonal ones), an observer "Alice," when recording some outcome of a measurement, can be sure that her partner "Bob," although spatially and causally disconnected from her, is either more or less likely to record a particular measurement outcome on his side. However, because of the randomness and the uncontrollability of individual events, and because of the no-cloning theorem, no classically useful information can be transferred from Alice to Bob, or *vice versa*: The parameter independence and the outcome dependence of otherwise random events ensure that the nonlocal correlations among quanta cannot be directly used to communicate classical information. The correlations of joint outcomes on Alice's and Bob's sides can only be verified by collecting all the different outcomes *ex post facto*, recombining joint events one-by-one. Nevertheless, there are the hopes and the visions to utilize nonlocal quantum correlations for a wide range of explanations and applications; for instance, in quantum information theory [4] and life sciences [5].

In what follows, a few known and novel quantum correlations will be systematically enumerated. We will derive the correlations between two and four two-state particles in singlet states, as wel as the correlations of two three-, four- and general d-state particles in a singlet state. In doing so, we attempt to "sharpen" the non-classical behavior beyond the standard quantum correlations.

2. Two Particle Correlations

We now consider two particles or quanta. On each of the two quanta, certain measurements (such as the spin state or the polarization) of (dichotomic) observables $O(a)$ and $O(b)$ along the directions a and b, respectively, are performed. The individual outcomes are encoded or labeled by the symbols "$-$" and "$+$," or values "-1" and "$+1$" are recorded along the directions a for the first particle, and b for the second particle, respectively. A two-particle correlation function $E(a,b)$ is defined by averaging over the product of the outcomes $O(a)_i, O(b)_i \in \{-1,1\}$ in the ith experiment for the total of N experiments; i.e.,

$$E(a,b) = \frac{1}{N}\sum_{i=1}^{N} O(a)_i O(b)_i. \tag{1}$$

Quantum-mechanically, we follow the standard procedure for obtaining the probabilities, upon which the correlation coefficients are based. We start from the angular momentum operators defined, for instance, in Schiff's *"Quantum Mechanics"* [7, Chap. VI, Sec.24] in arbitrary directions given by the spherical angular momentum coordinates θ and φ, as defined above. Then, the projection operators corresponding to the eigenstates associated with the different eigenvalues are derived from the dyadic (tensor) product of normalized eigenvectors. In the Hilbert-space-based quantum logic, every projector corresponds to a proposition that the system is in a state corresponding to that observable. The quantum probabilities associated with these eigenstates are derived from the Born rule, assuming singlet states for the physical reasons discussed above. These probabilities contribute to the correlation coefficients.

2.1. Three-state particles

Observables

The angular momentum operator in an arbitrary direction θ, φ is given by its spectral decomposition

$$S_1(\theta,\varphi) = -F_-(\theta,\varphi) + 0\cdot F_0(\theta,\varphi) + F_+(\theta,\varphi), \tag{2}$$

where F_-, F_0, and F_+ are the orthogonal projectors associated with the eigenstates of $S_1(\theta,\varphi)$. The generalized one-particle observable with the previous outcomes of spin state measurements "coded" into the map $-1 \mapsto \lambda_-$, $0 \mapsto \lambda_0$, $+1 \mapsto \lambda_+$ can be written as $R_1(\theta,\varphi) = \lambda_- F_-(\theta,\varphi) + \lambda_0 F_0(\theta,\varphi) + \lambda_+ F_+(\theta,\varphi)$.

For the operationalization of 117 contexts contained in their proof, Kochen and Specker [8] introduced an observable based on spin one with degenerate eigenvalues corresponding to $\lambda_+ = \lambda_- = 1$ and $\lambda_0 = 0$, or its "inverted" form $\lambda_+ = \lambda_- = 0$ and $\lambda_0 = 1$. The corresponding correlation functions will be discussed below.

Singlet state

Consider the singlet state of two spin-one particles $|\Psi_{3,2,1}\rangle = \frac{1}{\sqrt{3}}\left(-|00\rangle + |-+\rangle + |+-\rangle\right)$. Its vector space representation can be explicitly enumerated by taking the direction $\theta = \varphi = 0$ and recalling that $|+\rangle \equiv (1,0,0)$, $|0\rangle \equiv (0,1,0)$, and $|-\rangle \equiv (0,0,1)$; i.e., $|\Psi_{3,2,1}\rangle \equiv \frac{1}{\sqrt{3}}(0,0,1,0,-1,0,1,0,0)$.

Results

The general computation of the quantum correlation coefficient yields

$$E_{\Psi_{3,2,1}\,\lambda_-\lambda_0\lambda_+}(\hat\theta,\hat\varphi) = \text{Tr}\left[\rho_{\Psi_{3,2,1}}\cdot R_{11}(\hat\theta,\hat\varphi)\right] =$$

$$= \frac{1}{192}\Big\{24\lambda_0^2 + 40\lambda_0(\lambda_- + \lambda_+) + 22(\lambda_- + \lambda_+)^2 -$$

$$-32(\lambda_- - \lambda_+)^2\cos\theta_1\cos\theta_2 +$$

$$+2(-2\lambda_0 + \lambda_- + \lambda_+)^2\cos(2\theta_2)\times$$

$$\times\left[(3+\cos(2(\varphi_1-\varphi_2)))\cos(2\theta_1) + 2\sin(\varphi_1-\varphi_2)^2\right] +$$

$$+2\left(-2\lambda_0 + \lambda_- + \lambda_+\right)^2\times$$

$$\times\left[\cos\left(2\left(\varphi_1-\varphi_2\right)\right) + 2\cos\left(2\theta_1\right)\sin\left(\varphi_1-\varphi_2\right)^2\right] -$$

$$-32\left(\lambda_- - \lambda_+\right)^2\cos\left(\varphi_1-\varphi_2\right)\sin\theta_1\sin\theta_2 +$$

$$+8\left(-2\lambda_0 + \lambda_- + \lambda_+\right)^2\cos\left(\varphi_1-\varphi_2\right)\times$$

$$\times \sin(2\theta_1)\sin(2\theta_2)\Big\}. \quad (3)$$

For the sake of comparison, let us relate the rather lengthy correlation coefficient in Eq. (3) to the standard quantum mechanical correlations based on the dichotomic outcomes, by setting $\lambda_0 = 0$, $\lambda_+ = +1$, and $\lambda_- = -1$. With these identifications,

$$E_{\Psi_{3,2,1}\,-1,0,+1}(\hat{\theta},\hat{\varphi}) = -\frac{2}{3}\left[\cos\theta_1\cos\theta_2 + \right.$$

$$\left. + \cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2\right] = \frac{2}{3}E_{\Psi_{2,2,1}\,-1,+1}(\hat{\theta},\hat{\varphi}). \quad (4)$$

This correlation coefficient is functionally identical with the spin one-half (two outcomes) correlation coefficients.

The correlation coefficient resulting from the Kochen–Specker observable corresponding to $\lambda_+ = \lambda_- = 1$ and $\lambda_0 = 0$ or its inverted form $\lambda_+ = \lambda_- = 0$ and $\lambda_0 = 1$ is

$$E_{\Psi_{3,2,1}\,+1,0,+1}(\hat{\theta},\hat{\varphi}) = \tfrac{1}{24}\{11 + \cos[2(\varphi_1-\varphi_2)] +$$
$$+4\cos(\varphi_1-\varphi_2)\sin(2\theta_1)\sin(2\theta_2) +$$
$$+2\left[\cos(2\theta_1)+\cos(2\theta_2)\right]\sin^2(\varphi_1-\varphi_2) +$$
$$+\cos(2\theta_1)\cos(2\theta_2)\left[\cos(2(\varphi_1-\varphi_2))+3\right]\},$$
$$E_{\Psi_{3,2,1}\,0,+1,0}(\hat{\theta},\hat{\varphi}) = \tfrac{1}{3}\left[\cos\theta_1\cos\theta_2\right) +$$
$$+\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2]^2, \quad (5)$$
$$E_{\Psi_{3,2,1}\,+1,0,+1}(\tfrac{\pi}{2},\tfrac{\pi}{2},\hat{\varphi}) = \tfrac{1}{6}\{\cos[2(\varphi_1-\varphi_2)]+3\},$$
$$E_{\Psi_{3,2,1}\,0,+1,0}(\tfrac{\pi}{2},\tfrac{\pi}{2},\hat{\varphi}) = \tfrac{1}{3}\cos^2(\varphi_1-\varphi_2),$$
$$E_{\Psi_{3,2,1}\,+1,0,+1}(\hat{\theta},0,0) = \tfrac{1}{6}\{\cos[2(\theta_1-\theta_2)]+3\},$$
$$E_{\Psi_{3,2,1}\,0,+1,0}(\hat{\theta},0,0) = \tfrac{1}{3}\cos^2(\theta_1-\theta_2).$$

By comparing the quantum correlation coefficient $E_{\Psi_{3,2,1}\,-1,0,+1}(\hat{\theta},0,0) \propto -\cos(\theta_1-\theta_2)$ of the spin operators in Eq. (4) with the quantum correlation coefficient of the Kochen–Specker operators $E_{\Psi_{3,2,1}\,+1,0,+1}(\hat{\theta},0,0) \propto \cos[2(\theta_1-\theta_2)]$ of Eq. (5), one could envision an "enhancement" of the quantum correlation coefficient for higher-than one-half angular momentum observables, by adding the weighted correlation coefficients generated from different labels λ_i. Indeed, in the domain $0 < |\theta_1-\theta_2| < \frac{\pi}{3}$, the plasticity of $E_{\Psi_{l,2,1}\,\lambda_{-1},\lambda_0,\lambda_{+1}}$ can be used to build up "enhanced" quantum correlations *via*

$$\frac{1}{2}\Big\{E_{\Psi_{3,2,1}\,-1,0,+1}(\hat{\theta},0,0) +$$
$$+3\left[2E_{\Psi_{3,2,1}\,+1,0,+1}(\hat{\theta},0,0)-1\right]\Big\} =$$
$$= \frac{1}{2}\left[-\cos(\theta_1-\theta_2)+\cos 2(\theta_1-\theta_2)\right] <$$
$$< -\cos(\theta_1-\theta_2) = E_{\Psi_{2,2,1}\,-1,+1}(\hat{\theta},0,0). \quad (6)$$

2.2. Four-state particles

Observables

The angular momentum operator in an arbitrary direction θ, φ can be written in a spectral form

$$S_{\frac{3}{2}}(\theta,\varphi) = -\frac{3}{2}F_{-\frac{3}{2}}(\theta,\varphi) - \frac{1}{2}F_{-\frac{1}{2}}(\theta,\varphi) +$$
$$+\frac{1}{2}F_{+\frac{1}{2}}(\theta,\varphi) + \frac{3}{2}F_{+\frac{3}{2}}(\theta,\varphi). \quad (7)$$

If we are only interested in spin state measurements with the associated outcomes of spin states in units of $\hbar$, the associated two-particle operator is given by $S_{\frac{3}{2}}(\theta_1,\varphi_1)\otimes S_{\frac{3}{2}}(\theta_2,\varphi_2)$. More generally, one could define a two-particle operator by

$$F^2_{\lambda_{-\frac{3}{2}},\lambda_{-\frac{1}{2}},\lambda_{+\frac{1}{2}},\lambda_{+\frac{3}{2}}}(\hat{\theta},\hat{\varphi}) = F_{\lambda_{-\frac{3}{2}},\lambda_{-\frac{1}{2}},\lambda_{+\frac{1}{2}},\lambda_{+\frac{3}{2}}}(\theta_1,\varphi_1)\otimes$$
$$\otimes F_{\lambda_{-\frac{3}{2}},\lambda_{-\frac{1}{2}},\lambda_{+\frac{1}{2}},\lambda_{+\frac{3}{2}}}(\theta_2,\varphi_2), \quad (8)$$

where

$$F_{\lambda_{-\frac{3}{2}},\lambda_{-\frac{1}{2}},\lambda_{+\frac{1}{2}},\lambda_{+\frac{3}{2}}}(\theta,\varphi) = \lambda_{-\frac{3}{2}}F_{-\frac{3}{2}}(\theta,\varphi) +$$
$$+\lambda_{-\frac{1}{2}}F_{-\frac{1}{2}}(\theta,\varphi) + \lambda_{\frac{1}{2}}F_{+\frac{1}{2}}(\theta,\varphi) + \lambda_{\frac{3}{2}}F_{+\frac{3}{2}}(\theta,\varphi). \quad (9)$$

For the sake of the physical interpretation of this operator (8), let us consider as the specific example of a spin state measurement on two quanta: $F_{\lambda_{-\frac{3}{2}}}(\theta_1,\varphi_1)\otimes F_{\lambda_{+\frac{3}{2}}}(\theta_2,\varphi_2)$ stands for the proposition

> *'The outcome of the first particle measured along θ_1,φ_1 is "$\lambda_{-\frac{3}{2}}$" and the outcome of the second particle measured along θ_2,φ_2 is "$\lambda_{+\frac{3}{2}}$".'*

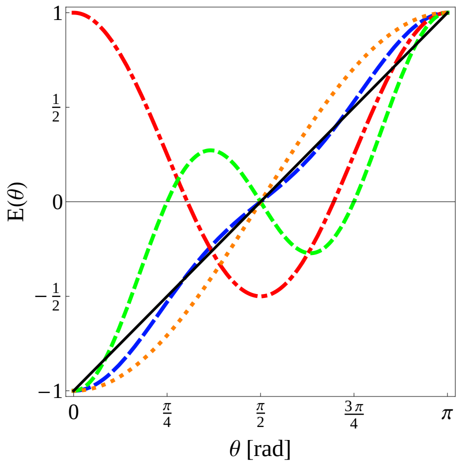

Fig. 1. Plasticity of the correlation coefficient of two spin three-half quanta in a singlet state. (a) $E_{\Psi_{4,2,1}\;-1,-1,+1,+1}$ is represented by the long-dashed curve, (b) $E_{\Psi_{4,2,1}\;-1,+1,+1,-1}$ is represented by the dashed-dotted curve, (c) $E_{\Psi_{4,2,1}\;+1,-1,+1,-1}$ is represented by the short-dashed curve, (d) $\frac{4}{5}E_{\Psi_{4,2,1}\;-\frac{3}{2},-\frac{1}{2},+\frac{1}{2},+\frac{3}{2}}$ is represented by the dotted curve, and (e) $E_{\mathrm{cl},2,2}(\theta)$ is represented by the linear line

Singlet state

The singlet state of two spin-3/2 observables can be found by the general methods developed in Ref. [9]. In this case, this amounts to summing all possible two-partite states yielding the zero angular momentum multiplied by the corresponding Clebsch–Gordan coefficients $\langle j_1 m_1 j_2 m_2 | 00\rangle = \delta_{j_1,j_2}\delta_{m_1,-m_2}\frac{(-1)^{j_1-m_1}}{\sqrt{2j_1+1}}$ of mutually negative single particle states resulting in the zero total angular momentum. More explicitly, for $j_1 = j_2 = \frac{3}{2}$, $|\psi_{4,2,1}\rangle = \frac{1}{2}\left(\left|\frac{3}{2},-\frac{3}{2}\right\rangle - \left|-\frac{3}{2},\frac{3}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|-\frac{1}{2},\frac{1}{2}\right\rangle\right)$. Again, this two-partite singlet state satisfies the uniqueness property. The four different spin states can be identified with a Cartesian basis of the four-dimensional Hilbert space $\left|\frac{3}{2}\right\rangle \equiv (1,0,0,0)$, $\left|\frac{1}{2}\right\rangle \equiv (0,1,0,0)$, $\left|-\frac{1}{2}\right\rangle \equiv (0,0,1,0)$, and $\left|-\frac{3}{2}\right\rangle \equiv (0,0,0,1)$, respectively.

Results

For the sake of comparison, let us again specify the rather lengthy correlation coefficient in the case of general observables with arbitrary outcomes λ_i, $i = 1,\ldots,4$ to the standard quantum mechanical correlations (4), by setting $\lambda_{+\frac{3}{2}} = +\frac{3}{2}$, $\lambda_{+\frac{1}{2}} = +\frac{1}{2}$, $\lambda_{-\frac{1}{2}} = -\frac{1}{2}$ and $\lambda_{-\frac{3}{2}} = -\frac{3}{2}$; i.e., by substituting the general outcomes with spin state observables in units of $\hbar$. With these identifications, the correlation coefficients can be directly calculated *via* $S_{\frac{3}{2}\frac{3}{2}}$; i.e.,

$$
\begin{aligned}
&E_{\Psi_{4,2,1}\;-\frac{3}{2},-\frac{1}{2},+\frac{1}{2},+\frac{3}{2}}(\hat{\theta},\hat{\varphi}) = \\
&= \mathrm{Tr}\left\{\rho_{\Psi_{4,2,1}}\left[S_{\frac{3}{2}}(\theta_1,\varphi_1)\otimes S_{\frac{3}{2}}(\theta_2,\varphi_2)\right]\right\} = \\
&= -\frac{5}{4}\left[\cos\theta_1\cos\theta_2 + \cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2\right] = \\
&= \frac{8}{15}E_{\Psi_{2,3,1}\;-1,+1}(\hat{\theta},\hat{\varphi}) = \\
&= 5E_{\Psi_{2,2,1}\;-\frac{1}{2},+\frac{1}{2}}(\hat{\theta},\hat{\varphi}) = \frac{5}{4}E_{\Psi_{2,2,1}\;-1,+1}(\hat{\theta},\hat{\varphi}). \qquad (10)
\end{aligned}
$$

This correlation coefficient is again functionally identical with the spin one-half and spin one (two and three outcomes) correlation coefficients.

The plasticity of the general correlation coefficient

$$
\begin{aligned}
&E_{\Psi_{4,2,1}\;\lambda_{-\frac{3}{2}},\lambda_{-\frac{1}{2}},\lambda_{+\frac{1}{2}},\lambda_{+\frac{3}{2}}}(\hat{\theta},\hat{\varphi}) = \\
&= \mathrm{Tr}\left[\rho_{\Psi_{4,2,1}}F^2_{\lambda_{-\frac{3}{2}},\lambda_{-\frac{1}{2}},\lambda_{+\frac{1}{2}},\lambda_{+\frac{3}{2}}}(\hat{\theta},\hat{\varphi})\right] \qquad (11)
\end{aligned}
$$

can be demonstrated by enumerating the special cases; e.g.,

$$
\begin{aligned}
&E_{\Psi_{4,2,1}\;-1,-1,+1,+1}(\theta,0,0,0) = \frac{1}{8}\left[-7\cos\theta - \cos(3\theta)\right], \\
&E_{\Psi_{4,2,1}\;-1,+1,+1,-1}(\theta,0,0,0) = \frac{1}{4}\left[3\cos(2\theta) + 1\right], \\
&E_{\Psi_{4,2,1}\;+1,-1,+1,-1}(\theta,0,0,0) = \frac{1}{2}\left[-\cos\theta - \cos(3\theta)\right]. \qquad (12)
\end{aligned}
$$

These functions are drawn in Fig. 1, together with the spin state correlation coefficient $\frac{4}{5}E_{\Psi_{4,2,1}\;-\frac{3}{2},-\frac{1}{2},+\frac{1}{2},+\frac{3}{2}}(\theta,0,0,0) = -\cos\theta$ and the classical linear correlation coefficient $E_{\mathrm{cl},2,2}(\theta) = 2\theta/\pi - 1$.

2.3. *General case of two spin j particles*

The general case of spin correlation values of two particles with an arbitrary spin j (see Appendix in [10] for a

Probabilities and correlation coefficients for finding an odd or even number of spin-"−"-states for both four-partite singlet states. Omitted arguments are zero

$P_{\text{even}} = \frac{1}{2}[1+E]$, $P_{\text{odd}} = \frac{1}{2}[1-E]$, $E = P_{\text{even}} - P_{\text{odd}}$
$E_{\Psi_{2,4,1}-1,+1}(\hat{\theta},\hat{\varphi}) = \frac{1}{3}\{\cos\theta_3\sin\theta_1[-\cos\theta_4\cos(\varphi_1-\varphi_2)\sin\theta_2 + 2\cos\theta_2\cos(\varphi_1-\varphi_4)\sin\theta_4] +$ $+\sin\theta_1\sin\theta_3[2\cos\theta_2\cos\theta_4\cos(\varphi_1-\varphi_3)+$ $+(2\cos(\varphi_1+\varphi_2-\varphi_3-\varphi_4)+\cos(\varphi_1-\varphi_2)\cos(\varphi_3-\varphi_4))\sin\theta_2\sin\theta_4]+$ $+\cos\theta_1[2\sin\theta_2(\cos\theta_4\cos(\varphi_2-\varphi_3)\sin\theta_3+\cos\theta_3\cos(\varphi_2-\varphi_4)\sin\theta_4)+$ $+\cos\theta_2(3\cos\theta_3\cos\theta_4-\cos(\varphi_3-\varphi_4)\sin\theta_3\sin\theta_4)]\}$ $E_{\Psi_{2,4,1}-1,+1}(\hat{\theta}) = \frac{1}{3}[2\cos(\theta_1+\theta_2-\theta_3-\theta_4)+\cos(\theta_1-\theta_2)\cos(\theta_3-\theta_4)]$. $E_{\Psi_{2,4,1}-1,+1}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\hat{\varphi}) = \frac{1}{3}[2\cos(\varphi_1+\varphi_2-\varphi_3-\varphi_4)+\cos(\varphi_1-\varphi_2)\cos(\varphi_3-\varphi_4)]$
$E_{\Psi_{2,4,2}-1,+1}(\hat{\theta},\hat{\varphi}) = [\cos\theta_1\cos\theta_2+\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2]\times$ $\times[\cos\theta_3\cos\theta_4+\cos(\varphi_3-\varphi_4)\sin\theta_3\sin\theta_4]$ $E_{\Psi_{2,4,2}-1,+1}(\hat{\theta}) = \cos(\theta_1-\theta_2)\cos(\theta_3-\theta_4)$, $E_{\Psi_{2,4,2}-1,+1}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\hat{\varphi}) = \cos(\varphi_1-\varphi_2)\cos(\varphi_3-\varphi_4)$,

group-theoretic derivation) can be directly calculated in an analogous way as before, yielding

$$E_{\Psi_{2,2j+1,1}\,-j,-j+1,\ldots,+j-1,+j}(\hat{\theta},\hat{\varphi}) =$$

$$= \operatorname{Tr}\left\{\rho_{\Psi_{2,2j+1,1}}\left[S_j(\theta_1,\varphi_1)\otimes S_j(\theta_2,\varphi_2)\right]\right\} =$$

$$= -\frac{j(1+j)}{3}\left[\cos\theta_1\cos\theta_2+\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2\right]. \tag{13}$$

Thus, the functional form of the two-particle correlation coefficients based on spin state observables is *independent* of the absolute spin value.

3. Four Spin One-Half Particle Correlations

To begin with the analysis of four-partite correlations, we now consider four spin-$\frac{1}{2}$ particles in one of the two singlet states [9] $|\Psi_{2,4,1}\rangle = \frac{1}{\sqrt{3}}\Big[|++--\rangle+|--++\rangle-\frac{1}{2}\big(|+-\rangle+|-+\rangle\big)\big(|+-\rangle+|-+\rangle\big)\Big]$, and $|\Psi_{2,4,2}\rangle = \big(|\Psi_{2,2,1}\rangle\big)^2 = \frac{1}{2}\big(|+-\rangle-|-+\rangle\big)\big(|+-\rangle-|-+\rangle\big)$, where $|\Psi_{2,2,1}\rangle = \frac{1}{\sqrt{2}}\big(|+-\rangle-|-+\rangle\big)$ is the two particle singlet "Bell" state. In what follows, we concentrate on the first state $|\Psi_{2,4,1}\rangle$, since $|\Psi_{2,4,2}\rangle$ is just a product of two two-partite singlet states, thus presenting the entanglement merely among two pairs of two quanta.

The projection operators F corresponding to a joint measurement of four spin one-half particles aligned ("+") or antialigned ("−") along those angles are

$$F_{\pm\pm\pm\pm}(\hat{\theta},\hat{\varphi}) = \frac{1}{2}\left[\mathbb{I}_2\pm\sigma(\theta_1,\varphi_1)\right]\otimes\frac{1}{2}\left[\mathbb{I}_2\pm\sigma(\theta_2,\varphi_2)\right]\otimes$$

$$\otimes\frac{1}{2}\left[\mathbb{I}_2\pm\sigma(\theta_3,\varphi_3)\right]\otimes\frac{1}{2}\left[\mathbb{I}_2\pm\sigma(\theta_4,\varphi_4)\right]. \tag{14}$$

To demonstrate the physical interpretation, let us consider a specific example: $F_{-+-+}(\hat{\theta},\hat{\varphi})$ stands for the proposition

> *'The spin state of the first particle measured along θ_1,φ_1 is "−", the spin state of the second particle measured along θ_2,φ_2 is "+", the spin state of the third particle measured along θ_3,φ_3 is "−", and the spin state of the fourth particle measured along θ_4,φ_4 is "+".'*

The joint probability to register the spins of the four particles in state $\Psi_{2,4,1}$ aligned or anti-aligned along the directions defined by (θ_1,φ_1), (θ_2,φ_2), (θ_3,φ_3), and (θ_4,φ_4) can be evaluated by a straightforward calculation of

$$P_{\Psi_{2,4,1}\pm1,\pm1,\pm1\pm1}(\hat{\theta},\hat{\varphi}) =$$

$$= \operatorname{Tr}\left[\rho_{\Psi_{2,4,1}}\cdot F_{\pm\pm\pm\pm}\left(\hat{\theta},\hat{\varphi}\right)\right]. \tag{15}$$

The correlation coefficients and the joint probabilities to find the four particles in an even or odd number of spin-"−"-states when measured along (θ_1,φ_1), (θ_2,φ_2), (θ_3,φ_3), and (θ_4,φ_4) obey $P_{\text{even}}+P_{\text{odd}}=1$, as well as $E=P_{\text{even}}-P_{\text{odd}}$; hence, $P_{\text{even}}=\frac{1}{2}[1+E]$ and $P_{\text{odd}}=\frac{1}{2}[1-E]$. Thus, the four particle quantum correlation is given by (cf. Table)

$$E_{\Psi_{2,4,1}-1,+1}(\hat{\theta},\hat{\varphi}) =$$

$$= \frac{1}{3}\{\cos\theta_3\sin\theta_1[-\cos\theta_4\cos(\varphi_1-\varphi_2)\times$$

$$\times\sin\theta_2+2\cos\theta_2\cos(\varphi_1-\varphi_4)\sin\theta_4]+$$

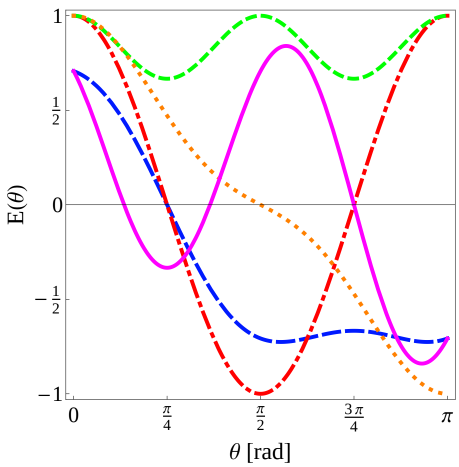

Fig. 2. Plasticity of the correlation coefficient of four spin one-half quanta in a singlet state. (a) $E_{\Psi_{2,4,1}-1,+1}(\theta,\frac{\pi}{4},-\theta,\theta)$ is represented by the long-dashed curve, (b) $E_{\Psi_{2,4,1}-1,+1}(\theta,\theta,-\theta,\theta)$ is represented by the dashed-dotted curve, (c) $E_{\Psi_{2,4,1}-1,+1}(\theta,-\theta,-\theta,\theta)$ is represented by the short-dashed curve, (d) $E_{\Psi_{2,4,1}-1,+1}(\theta,-\theta,-\theta,0)$ is represented by the dotted curve, and (e) $E_{\Psi_{2,4,1}-1,+1}(-\theta,-\theta,\frac{\pi}{4},\theta)$ is represented by the solid line

$$+\sin\theta_1\sin\theta_3\,[2\cos\theta_2\cos\theta_4\cos(\varphi_1-\varphi_3)+$$

$$+(2\cos(\varphi_1+\varphi_2-\varphi_3-\varphi_4)+$$

$$+\cos(\varphi_1-\varphi_2)\cos(\varphi_3-\varphi_4))\sin\theta_2\sin\theta_4]+$$

$$+\cos\theta_1\,[2\sin\theta_2\,(\cos\theta_4\cos(\varphi_2-\varphi_3)\sin\theta_3+$$

$$+\cos\theta_3\cos(\varphi_2-\varphi_4)\sin\theta_4)+$$

$$+\ \cos\theta_2\,(3\cos\theta_3\cos\theta_4-\cos(\varphi_3-\varphi_4)\sin\theta_3\sin\theta_4)]\}\,. \tag{16}$$

If all the polar angles $\hat{\theta}$ are set to $\pi/2$, then this correlation function yields

$$E_{\Psi_{2,4,1}-1,+1}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\hat{\varphi})=$$

$$=\frac{1}{3}\,[2\cos(\varphi_1+\varphi_2-\varphi_3-\varphi_4)+$$

$$+\cos(\varphi_1-\varphi_2)\cos(\varphi_3-\varphi_4)]\,. \tag{17}$$

Likewise, if all the azimuthal angles $\hat{\varphi}$ are set to zero, we obtain

$$E_{\Psi_{2,4,1}-1,+1}(\hat{\theta})=\frac{1}{3}\,[2\cos(\theta_1+\theta_2-\theta_3-\theta_4)+$$

$$+\cos(\theta_1-\theta_2)\cos(\theta_3-\theta_4)]\,. \tag{18}$$

The plasticity of the correlation coefficient $E_{\Psi_{2,4,1}-1,+1}(\hat{\theta})$ of Eq. (18) for various parameter values θ is depicted in Fig. 2.

4. Summary

Compared with the two-partite quantum correlations of two-state particles, the plasticity of the quantum correlations of states of *more than two particles* originates in the dependence of the *multitude of angles* involved, as well as the *multitude of singlet states* in this domain. For the states related to the particles of *more than two mutually exclusive outcomes,* the plasticity is also increased by *various values associated with the outcomes.*

We have explicitly derived the quantum correlation functions of two- and four-partite spin one-half systems, as well as two-partite systems of higher spin. All quantum correlation coefficients of the two-partite spin observables have identical form, all being proportional to $\cos\theta_1\cos\theta_2+\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2$. We have also argued that, by utilizing the plasticity of the quantum correlation coefficients for spins higher that one-half, this well-known correlation function can be "enhanced" by defining sums of quantum correlation coefficients, at least in some domains of the measurement angles.

It would be interesting to know whether this plasticity of the quantum correlations $E_{\Psi_{l,2,1}\,\lambda_{-l},\ldots,\lambda_{+l}}$ for "very high" angular momentum l observables could be pushed to the point of the maximal violation of the Clauser–Horne–Shimony–Holt inequality *without* an insignificant exchange such as that with the use of the "buildup" of a step function from the individual correlation coefficients [10]; e.g., for $0\le\theta\le\pi$,

$$\operatorname{sgn}(x)=\begin{cases}-1 & \text{for } 0\le x<\frac{\pi}{2}\\ 0 & \text{for } x=\frac{\pi}{2}\\ +1 & \text{for } \frac{\pi}{2}<\theta\le\pi\end{cases}=$$

$$=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos\left[(2n+1)\left(\theta+\frac{\pi}{2}\right)\right]}{2n+1}\quad. \tag{19}$$

Any such violation of the Boole–Bell-type "conditions of possible experience" beyond the maximal quantum violations as those, for instance, derived by Tsirelson [11] and generalized in Ref. [12] not necessarily generalizes to the multipartite nondichotomic cases. Note also that such a strong or even the maximal violation of the Boole–Bell-type "conditions of possible experience" beyond the maximal quantum violations needs not necessarily violate the relativistic causality [13, 14] or be associated with "sharpening" the angular dependence of the joint occurrence of certain elementary dichotomic outcomes such as "++," "+−," "−+" or "−−," respectively.

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ЩОДО ПЛАСТИЧНОСТІ НЕЛОКАЛЬНИХ КВАНТОВИХ КОРЕЛЯЦІЙ

К. Свозіл

Р е з ю м е

Квантові кореляції двох і більше переплутаних частинок надають можливість для кращої збіжності результатів, ніж у класичному випадку. Досліджено парні кореляції частинок зі спіном 1, 3/2 та вище у стані, що задовольняє вимогу однозначності в тому сенсі, що знання результату для параметра однієї частинки дає впевненість у тому, що, якщо цей параметр було би виміряно для другої частинки, результат виміру буде однозначною функцією попереднього результату. Також досліджено кореляції чотирьох частинок зі спіном 1/2.