COMPARISON BETWEEN THE VARIATIONAL EQUATIONS OF THE AVERAGED LAGRANGIAN METHOD AND A NONLINEAR SCHRÖDINGER EQUATION FOR WAVES ON THE SURFACE OF A FLUID LAYER

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By the example of waves on the surface of a fluid layer, the variational equations obtained in the averaged Lagrangian method of Whitham are demonstrated to become equivalent to a nonlinear Schrödinger equation, if the construction of the Lagrangian involves a term related to fundamental harmonic in addition to the terms from the second and zero harmonics considered by Whitham to get linear corrections to the field functions.

1. Introduction

The nonlinear Schrödinger equation (NSE)

$$i\,(A_t+\omega_0' A_x)+\frac{1}{2}\omega_0'' A_{xx}+qA|A|^2=0 \tag{1}$$

describes a slow evolution of the first-harmonic amplitude $A(x,t)$ of an almost linear wave $\eta(x,t)$,

$$\eta(x,t)=b(x,t)+$$

$$+\left(\frac{1}{2}A(x,t)\exp i\theta_0+A_2(x,t)\exp 2i\theta_0+\ldots+\text{c.c.}\right),$$

$$\theta_0=k_0x-\omega_0 t \tag{2}$$

in a weakly nonlinear low-dispersion nondissipative medium. Other harmonics, namely the zero, $b(x,t)$, the second, $A_2(x,t)$, and the higher-order ones, can be calculated by applying their expressions in terms of $A(x,t)$ obtained at the reduction of the equations of motion in a medium to the NSE. While deriving the NSE, the expansions in small wave amplitudes can be made either immediately in the equations of motion or in the functions used for constructing the Hamiltonian or Lagrangian of waves and, therefore, in themselves. Averaging over fast oscillations $\exp i\theta_0$ is carried out either by formally introducing slow and fast variables, or by intuitively choosing the most crucial variables in the derivatives, or by integrating over the period of fast oscillations. The expansion and averaging of the Hamiltonian or the Lagrangian are made only once for their subsequent use: e.g., at constructing the evolutionary equations, finding the spectra, and so forth. Therefore, these procedures look more attractively than starting the expansion in the equations of motion afresh for every specific problem. For instance, the coefficients of the expansion of Hamiltonians in series in a nonlinearity obtained by Zakharov for waves on a fluid surface and in plasma as early as at the end of the 1960s became standard parameters for those waves and are in use for subsequent applications [1]. Among other approaches, it is worth separating the method of asymptotic expansion of the derivative A_t, which can be well algorithmized for symbolic transformations [2].

The procedure of description of nonlinear waves with the help of a Lagrangian averaged over fast oscillations was developed by Whitham for waves on a fluid surface [3–6] and extended to other media [7]. However, the Lagrangian obtained in work [4] contains only the expansion in powers of the amplitude a^2 and a^4, but does not includes derivatives of a. The variational equations do contain derivatives of a, but only the first-order ones.

The absence of a_{xx}, which is responsible for the pulse dispersion [8] or the beam diffraction, is amazing even in the linear case [9] and complicates our understanding of the relations between Whitham's theory and the NSE. An inconsistency of the variational equations with the NSE was marked in work [9] for waves on a fluid surface and in work [10] for plasma. Strictly speaking, the NSE can be derived in the framework of the averaged Lagrangian method of Whitham, but in an indirect way. In particular, the NSEs, obtained using this method for waves of various nature in works [11–13], are based upon the rule that the coefficient in the nonlinear term of NSE coincides with that in the nonlinear term of the nonlinear dispersion law, and it is the averaged Lagrangian method of Whitham that allows the nonlinear dispersion law to be calculated elegantly with the help of a single variational equation, by varying the Lagrangian with respect to the amplitude (see Eq. (14)). In work [14], the NSE was derived from the averaged Lagrangian using perturbation theory. The perturbations were introduced into the averaged Lagrangian, when the multiple-scale approach was incorporated into Whitham's method. A direct relation between the combination of two variational equations in Whitham's theory and the NSE was found in works [15, 16] for waves on a fluid surface in the specific case of the basin with infinite depth. It was done by appending the trial functions and Whitham's Lagrangian with a term that contained the derivatives of the amplitude a. This work generalizes the approach of works [15, 16] to the case of the basin with any depth, when the number of variational equations is four.

2. Extension of Trial Functions and Changes in the Lagrangian for a Fluid Layer. Modifications in Variational Equations. NSE for a Fluid Layer

In the multiple-scale method, the description of an almost sinusoidal nonlinear wave is formalized in such a way that it is represented by a series of slowly varying harmonics of fast oscillations $\exp i\theta_0$ and the slowly evolving zero harmonic $b(x,t)$ which is a real-valued function (2). In Whitham's works, it is described in terms of real-valued amplitudes:

$$\eta(x,t) = b(x,t) + a(x,t)\cos\theta + a_2(x,t)\cos 2\theta,$$

$$\theta = k(x,t)x - \omega(x,t)t. \tag{3}$$

Let us express the complex-valued amplitudes $A(x,t)$ and $A_2(x,t)$ of harmonics in the exponential form in terms of the real-valued amplitudes a and a_2, as well as the corresponding phases $\tilde{\theta}$ and $2\tilde{\theta}$:

$$A(x,t) = a(x,t)\exp i\widetilde{\theta}(x,t), \tag{4}$$

$$A_2(x,t) = 2a_2(x,t)\exp 2i\widetilde{\theta}(x,t)$$

Comparing between Eqs. (3) and (2), we see that

$$\widetilde{\theta} = (k\,(x,t) - k_0)x - (\omega\,(x,t) - \omega_0)t. \tag{5}$$

The explanations given above for the notations used were dictated by the fact that the NSE was written down for the complex-valued amplitude $A(x,t)$ of the first harmonic of a wave profile, whereas Whitham's Lagrangian and the corresponding variational equations include three real-valued functions $a(x,t)$, $\widetilde{\theta}(x,t)$, and $b(x,t)$ which are the characteristics of a wave profile (the deviations from the equilibrium surface) (10) and the fourth, slowly evolving real-valued quantity $\psi(x,t)$ which is the zero harmonic of the velocity potential in (11). The connection of $A(x,t)$ with $a(x,t)$ and $\widetilde{\theta}(x,t)$ is described by Eq. (4) with regard for Eq. (5).

Variations of the phase $\widetilde{\theta}$ in the form of $k\,(x,t)$ and $\omega\,(x,t)$ are assumed to be slower than those in the fast filling oscillations $\exp i\theta_0$. In practice, this means that the theory cannot be applied, if the internal filling by $\exp i\theta_0$ gives only a few oscillations per one oscillation of the external envelope A. Whence,

$$\widetilde{\theta}_x = k(x,t) - k_0, \quad \widetilde{\theta}_t = -(\omega(x,t) - \omega_0).. \tag{6}$$

According to the results of works [3–6], the variational principle

$$\delta \int\int \mathcal{L}\left(a, \widetilde{\theta}, b, \psi\right) dxdt = 0, \quad \mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\theta, \tag{7}$$

where L is the Lagrange function, brings about the variational equations for the functions $a(x,t)$, $\widetilde{\theta}(x,t)$, $b(x,t)$, and $\psi(x,t)$. According to the Hamilton principle, the Lagrangian for waves on a fluid surface is equal to the difference between the kinetic and potential energies of the fluid, and the boundary conditions are introduced by the Lagrange multipliers

$$L = \int_{-h_0}^{\eta(x,t)} [\frac{1}{2}(\varphi_x^2 + \varphi_y^2) - gy]dy + \ldots \text{constraints}. \tag{8}$$

Here, $\eta(x,t)$ is a profile of the fluid surface (a wave), φ is the fluid velocity potential, x and y are the horizontal and vertical coordinates, respectively, h_0 is the fluid depth, and g is the gravitational acceleration.

Luke [17] demonstrated that, for a vortex-free fluid and provided that averaging is carried out over the period of fast oscillations (7), the following, more convenient form for the Lagrange function can be used instead of formula (8):

$$L = \int\limits_{-h_0}^{\eta(x,t)} [\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2) + gy]dy. \qquad (9)$$

The key moment of this work consists in the following. In addition to the amplitudes a and ϕ of the fundamental (the first) harmonics and the harmonics nearest to them with respect to the order of smallness (these are the second and the zero one) used by Whitham, we consider also the corrections $a^{(1)}$ and $\phi^{(1)}$ to the first harmonics in the trial functions for the wave profile $\eta(x,t)$ and the velocity potential $\varphi(x,t)$, because they have the same order of smallness (appear in the same approximation) [18]:

$$\eta = \varepsilon^2 b + \varepsilon \left(a\cos\theta + \varepsilon a^{(1)} \sin\theta \right) + \varepsilon^2 a_2 \cos 2\theta,$$

$$a^{(1)} = \frac{\omega_0'}{\omega_0} a_x, \quad \omega_0' = \frac{\partial\omega}{\partial k}|_{k=k_0} \qquad (10)$$

$$\varphi = \varepsilon^2 \psi + \varepsilon \left(\phi \sin\theta + \varepsilon \phi^{(1)} \cos\theta \right) + \varepsilon^2 \phi_2 \sin 2\theta,$$

$$\phi^{(1)} = \left(h_0 \frac{\cosh k_0 \,(y+h_0)}{\cosh k_0 h_0} - \right.$$

$$\left. -(y+h_0) \frac{\sinh k_0\,(y+h_0)}{\sinh k_0 h_0} \right) \frac{\omega_0}{k_0} a_x. \qquad (11)$$

Here, b is the zero harmonic of a wave profile, and ψ is the zero harmonic of the velocity potential; both are the quantities, which slowly vary in time and space similarly to the amplitudes of other harmonics and describe the modulations of fast oscillations of waves in the medium. Their fundamental harmonics are as follows: $\varepsilon a \cos\theta$ for η and $\varepsilon\phi\sin\theta$ for φ. The derivatives designated by the prime sign, ω_0' and ω_0'' (see below), are calculated by differentiating the linear dispersion law $\omega^2 = gk\tanh kh_0$ with respect to the wave vector k at the point of "carrier" wave vector k_0, and ε is the formal small parameter.

The reproduction of Whitham's calculations [4], taking into account the corrections to the trial functions discussed above and the fact that not only the phases, but also the amplitudes in Eqs. (10) and (11) slowly depend on the coordinates x and t, brings about the corrected Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{Whitham}} + c_1 a_x^2 + c_2 a a_{xx},$$

$$c_1 = -\frac{\omega_0^2}{16 k_0^3 \sigma^3} \left(k_0^2 h_0^2 (\sigma^2-1)^2 - \sigma^2 \right),$$

$$c_2 = -\frac{\omega_0^2}{8 k_0^3 \sigma^2} \left(2k_0^2 h_0^2 \sigma (\sigma^2-1) - k_0 h_0 (\sigma^2-1) - \sigma \right),$$

$$\sigma = \tanh k_0 h_0, \quad c_1 - c_2 \equiv \frac{\omega_0}{4\sigma k_0} \omega_0'', \qquad (12)$$

where

$$\mathcal{L}_{\text{Whitham}} = \frac{1}{4} g \left(1 - \frac{(\omega - \psi_x k)^2}{gk\tanh kh} \right) a^2 +$$

$$+\frac{gD}{8} k^2 a^4 + (\frac{1}{2}\psi_x^2 + \psi_t) h + \frac{1}{2} g b^2 \qquad (13)$$

is Whitham's Lagrangian [4,7]. Here, we kept untouched the notations [4]

$$D = \frac{9\sigma^4 - 10\sigma^2 + 9}{8\sigma^4}, \qquad h = h_0 + b.$$

Note that Eqs. (10) and (12) take into account that, according to the linear theory, the envelope amplitude propagates with the group velocity $a_t = -\omega_0' a_x$, and the coefficients in expressions (10) and (11) obtained in the framework of the multiple-scale approach [18] were verified, for the approach to be closed, by varying the averaged Lagrangian (7), (9) directly with respect to those coefficients considered as the unknown variables.

Lagrangian (12) depends on four functions: a, θ, b, and ψ. Therefore, we have to construct four Euler's equations.

1. In contrast to work [4], the variational equation with respect to a,

$$\frac{\partial\mathcal{L}}{\partial a} = \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial a_x} - \frac{\partial^2}{\partial x^2}\frac{\partial\mathcal{L}}{\partial a_{xx}}$$

includes the right-hand side different from zero, because Lagrangian (12) depends also on the derivatives a_x and a_{xx}. The chain of transformations yields

$$(\omega - \psi_x k)^2 = gk\tanh kh \left(1 + Dk^2 a^2 - \frac{\omega_0 \omega_0''}{g\sigma k_0} \frac{a_{xx}}{a} \right) + o(a^2),$$

$$\omega = \sqrt{gk\tanh kh_0} + \frac{1}{2}D\omega_0 k^2 a^2 + \frac{1-\sigma^2}{2\sigma}\omega_0 kb -$$

$$-\frac{1}{2}\omega_0''\frac{a_{xx}}{a} + \psi_x k + o(a^2, \frac{a_{xx}}{a}, b),$$

$$\omega - \omega_0 = \omega_0'(k-k_0) + \frac{1}{2}\omega_0''(k-k_0)^2 - \frac{1}{2}\omega_0''\frac{a_{xx}}{a} +$$

$$+\frac{1}{2}D\omega_0 k_0^2 a^2 + \omega_0 k_0\frac{1-\sigma^2}{2\sigma}b + k_0\psi_x + o(a^2, \frac{a_{xx}}{a}, b, k-k_0).$$

Hence, we obtain the following evolution equation for $\tilde{\theta}$:

$$-\tilde{\theta}_t = \omega_0'\tilde{\theta}_x + \frac{1}{2}\omega_0''\left(\tilde{\theta}_x^2 - \frac{a_{xx}}{a}\right) +$$

$$+\frac{1}{2}\omega_0 k_0^2 D a^2 + \omega_0 k_0 \frac{1-\sigma^2}{2\sigma}b + k_0\psi_x. \tag{14}$$

The difference with the result of work [4] consists in the appearance of the second term in the parentheses on the right-hand side of formula (14).

2. In order to construct the variational Euler's equation with respect to $\tilde{\theta}$,

$$\frac{\partial\mathcal{L}}{\partial\tilde{\theta}} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\tilde{\theta}_t} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\tilde{\theta}_x} \tag{15}$$

we have to take into account that, in accordance with expression (6), the dependence on $\tilde{\theta}$ in formula (12) is contained in the variables $\omega = \omega_0 - \tilde{\theta}_t$ and $k = k_0 + \tilde{\theta}_x$ (13). Then, from Eq. (15), we obtain the evolution equation for a:

$$a_t + \omega_0' a_x + \frac{1}{2}\omega_0''\left(\tilde{\theta}_{xx}a + 2\tilde{\theta}_x a_x\right) = 0. \tag{16}$$

Here, the first term arises from the first and the others from the second term on the right-hand side of formula (15).

3. Euler's equation for b,

$$\frac{\partial\mathcal{L}}{\partial b} = 0, \tag{17}$$

yields

$$gb + \frac{1}{2}\psi_x^2 + \psi_t + \frac{1}{4}a^2(\omega - \psi_x k)^2\frac{1-\tanh^2(k(h_0+b))}{\tanh^2(k(h_0+b))} = 0.$$

After simplifications, we obtain the evolution equation for ψ:

$$\psi_t + gb + \frac{1}{4}\omega_0^2\frac{1-\sigma^2}{\sigma^2}a^2 = 0. \tag{18}$$

4. Euler's equation for ψ,

$$\frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\psi_t} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\psi_x}, \tag{19}$$

results in the evolution equation for b,

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x}\left(\psi_x h_0 + \frac{1}{2}\frac{k_0}{\omega_0}ga^2\right) = 0, \tag{20}$$

according to the first and second terms on the right-hand side of Eq. (19). Equations (18) and (20) are identical to the corresponding equations obtained in work [4], because the new term in Lagrangian (12) does not manifest itself in Eqs. (17) and (19), as well as to the equations obtained in work [20] within the multiple-scale method. Similarly to what was done in work [4], using the approximation concerning the evolution of zero harmonics of the velocity potential ψ and the wave height b with the group velocity of linear waves $\psi_t = -\omega_0'\psi_x$ and $b_t = -\omega_0' b_x$, we express b and ψ_x from Eqs. (18) and (20:

$$b = -\frac{\omega_0^2}{4\sigma^2 k_0}\frac{\left(1-\sigma^2\right)k_0 h_0 + 2\sigma\frac{\omega_0}{k_0}\omega_0'}{gh_0 - {\omega_0'}^2}a^2,$$

$$\psi_x = -\frac{\omega_0^2}{4\sigma^2}\frac{\left(1-\sigma^2\right)\omega_0' + 2\frac{\omega_0}{k_0}}{gh_0 - {\omega_0'}^2}a^2.$$

Substituting these formulas into Eq. (14), we obtain

$$\tilde{\theta}_t + \omega_0'\tilde{\theta}_x + \frac{1}{2}\omega_0''\left(\tilde{\theta}_x^2 - \frac{a_{xx}}{a}\right) - qa^2 = 0, , \tag{21}$$

where

$$q = \frac{\omega_0 k_0^2}{16\sigma^2}\left\{-\frac{9\sigma^4 - 10\sigma^2 + 9}{\sigma^2} +\right.$$

$$\left. +2\left[(1-\sigma^2)^2 + \frac{1}{gh_0 - {\omega_0'}^2}\left(2\frac{\omega_0}{k_0} + (1-\sigma^2)\omega_0'\right)^2\right]\right\}. \tag{22}$$

Hence, the real-valued quantities – the amplitude and the phase of the complex-valued amplitude of envelope A – evolve in accordance with the system of equations (16) and (21). Adding Eqs. (16) and (21) multiplied by $i\exp i\tilde{\theta}$ and $-a\exp i\tilde{\theta}$, respectively, we obtain an equation for the complex-valued amplitude of the envelope A—the nonlinear Schrödinger equation (1). For the first time, this equation was obtained in such a context in work [21], using the multiple-scale method. Expression (22) coincides with the coefficient in the nonlinear term of NSE [21, 22].

3. Conclusion

A relation between Whitham's variational equations and the nonlinear Schrödinger equation has been demonstrated. The real-valued variational equations of Whitham's theory for the absolute value and the phase of the complex-valued amplitude are shown to be equivalent to the NSE, if, besides the terms considered by Whitham [4, 7], all other terms of the same order of smallness ((10), (11)) are additionally taken into account in the functions used for constructing Lagrangian (13).

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ПОРІВНЯННЯ ВАРІАЦІЙНИХ РІВНЯНЬ МЕТОДА УСЕРЕДНЕНОГО ЛАГРАНЖІАНА І НЕЛІНІЙНОГО РІВНЯННЯ ШРЕДІНГЕРА ДЛЯ ХВИЛЬ НА ПОВЕРХНІ ШАРУ РІДИНИ

Ю.В. Седлецький

Р е з ю м е

На прикладі хвиль на поверхні шару рідини продемонстровано, що варіаційні рівняння методу усередненого лагранжіана Уізема стають еквівалентними нелінійному рівнянню Шредінгера, якщо при побудові лагранжіана врахувати, крім перших після лінійних покращень польових функцій – другої і нульової гармонік, власне залучених Уіземом, ще й того ж порядку малості доданок в основній гармоніці.