EXTERNAL-NOISE-SUSTAINED PROCESSES OF PATTERN SELECTION AT THE SPINODAL DECOMPOSITION OF A BINARY SYSTEM

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We consider the processes of pattern selection in a class of nonequilibrium binary stochastic systems subjected to an external influence. A possibility for the process of pattern selection to run on the initial stages of the spinodal decomposition is demonstrated analytically and numerically. It is established that the regular and stochastic components of an external flow play opposite roles at the pattern selection and, in the general case, at the decay of systems. The analytical results are confirmed by a numerical modeling.

1. Introduction

The development of the modern theory of condensed state requires the comprehensive study of the processes of ordering in systems significantly removed from a state of equilibrium [1]. The problem of clarification of both the stability of phases and specific features of their formation due to an external influence on a system becomes more and more actual, since its solution allows one to discover new characteristics of systems and relevant processes used not only in materials science [2] and electronics [3, 4], but generally at the prediction of properties of materials subjected to the action of corrosive media or, for example, to the irradiation. In this case, the problem of determination of the character and specific features of the influence of external factors on the processes of creation of coherent states, microstructural transformations, and the ordering generally is of top priority.

It is well known that the systems which can be considered quasiequilibrium are described in the frame of the hypothesis of local equilibrium, where the basic role is played by the processes on diffusion scales. By this hypothesis, it is assumed that, though the system is nonequilibrium, a local thermodynamical equilibrium is established in infinitely small volumes [5]. However, this hypothesis is violated at a significant deviation from the equilibrium, for example, in the fast processes of spinodal decomposition and in the rapidly running processes of transition from a unstable state to a metastable or stable state [6]. In this case, of great importance are the effects of memory which describe the connection between motive forces and flows in the system. In addition, it is known that the nonequilibrium state can be caused by a contribution of fluctuations of the external medium which can induce the noncompensated flows of a substance [5, 7]. In any case, the stochastic components of the evolution of a system under conditions close to real ones should be considered in a proper way, since they model the effect of microscopic processes at the description of the system on the mesoscopic level [8].

For the recent years, it has been established that the stochastic forces can basically change a behavior of physical systems, by playing a constructive role in this case [8, 9]. In the physics of condensed matter, we may refer the noise-induced phase transitions in the systems with conservative and nonconservative dynamics [10–15], the noise-induced processes of formation of spatial structures [16], the noise-sustained pattern-formation [17–21], *etc.* to such effects. The research of systems which undergo the action of an external stochastic influence allows one to determine new specific features of their behavior for their possible application. In the circle of applied studies, such a problem was posed at the consideration of the stability of phases at the decay of a binary system under conditions of the irradiation with high-energy particles

[22, 23]. In the theoretical analysis of the influence of radiation on the process of spinodal decomposition, the stochasticity of such an influence was assumed on the initial stages of a modeling. The description of specific features of the pattern-formation on the surface of materials and films at the spraying involves, in a natural way, stochastic specific features of a radiation flux [24, 25], in which the incident particles have the Maxwell's distribution [26]. Thus, the theoretical description of relevant processes should take the statistical characteristics of a radiation flux into account.

It is worth noting that the stability of phases at the decay of binary systems under conditions of an external influence (the athermal mixing of atoms caused by the action of radiation) was well studied analytically and by a numerical modeling within the hypothesis of local equilibrium (in the diffusion limit) [27, 28]. However, the transient processes in essentially nonequilibrium systems which possess, for example, a memory were not investigated to a complete extent, though they arouse a stable theoretical interest. We note that the time correlation can lead to such processes as the pattern selection at the spinodal decomposition on characteristic time intervals which are omitted at the consideration of slow (diffusion) processes due to the instantaneous relaxation of diffusion flows [29]. However, such processes were mainly studied under the so-called "noiseless" conditions, where the fluctuations are assumed to be insignificant. In addition, the processes of pattern selection under conditions where the system undergoes the action of a stochastic external influence with high-intensity fluctuations remain unstudied.

Therefore, we will study the processes of pattern selection for the systems, for which the effects of memory play a crucial role in the presence of an external factor which includes both regular and fluctuating components. In our analysis, we restrict ourselves by a model of binary systems which neglects the effects of coherent stresses and anisotropy arising at the phase separation. We use the Cahn–Hilliard–Cook generalized model of phase separation with hyperbolic transport (due to the effects of memory) [30] and introduce, in a standard way, a flow with the athermal mixing of atoms induced by the action of radiation. By assuming the presence of fluctuations of a thermally stimulated diffusion flow and an athermal flow, we propose a general stochastic model. Within the model, we will describe the processes of pattern selection on the initial stages of the decomposition in terms of the mean value of a random concentration field and the structure factor. The results of analytical calculations are confirmed by the independent numerical modeling. In addition, we study the behavior of a system in the stationary case and determine the form of a dynamical phase diagram. The obtained results illustrate the competitive roles of the regular and stochastic components of the athermal flow with atomic mixing at the ordering of the system.

The structure of the present work is as follows. In Section 2, we give the statement of the problem and substantiate the choice of a model of binary systems with conservative dynamics. In Section 3, we study analytically the stability of states of a system and discuss specific features of the process of pattern selection at the spinodal decomposition. We give also a detailed analysis of a behavior of the mean and structure factors, group and phase velocities, and the normalized factor of amplification. To confirm the results of analytical calculations, we present the results of a numerical modeling in Section 4. The main conclusions are contained in Section 5.

2. Model

Let us consider a class of binary systems $A_{\overline{c}}B_{1-\overline{c}}$ described by a scalar field $x = x(\mathbf{r}, t)$ which is a conservative quantity ($\int d\mathbf{r}x(\mathbf{r}, t) = \text{const}$), where $x = c - \overline{c}$ is a deviation from the critical value of the concentration $\overline{c} = 1/2$ of one of the components. The dynamics of the field x is set by the equation of continuity

$$\partial_t x = -\nabla \mathbf{J}_{\text{tot}}. \tag{1}$$

Here, $\mathbf{J}_{\text{tot}}$ is the total diffusion flow composed of two terms, $\mathbf{J}_{\text{tot}} = \mathbf{J}_D + \mathbf{J}_e$, which are the ordinary thermally stimulated diffusion flow $\mathbf{J}_D$ and the flow $\mathbf{J}_e$ with additional athermal mixing of atoms of the system induced by an external influence, for example, by radiation. Considering the system under conditions close to the real ones, we assume that each component of the flow has regular and stochastic components. We will focus the main attention on the study of the processes of competition of the indicated flows in the processes of pattern selection at the spinodal decomposition.

We now determine the components of the total flow $\mathbf{J}_{\text{tot}}$ of a nonequilibrium system. For a wide circle of physical systems far from an equilibrium such as non-Newton liquids, rapidly cooled crystallized alloys and systems, materials strongly frozen in the spinodal region or, in general, systems with memory, the hypothesis of local equilibrium is not valid [31, 32]. In such cases, a diffusion flow is described by the general formula $\mathbf{J}_D = -M \int_0^t M_D(t-t')\nabla\delta\mathcal{F}[x(\mathbf{r},t')]/\delta x(\mathbf{r},t')dt'$, where $M = \text{const}$ is the mobility, $\mathcal{F}$ is the functional of the free energy of a binary system, and $M_D(t-t')$ is the memory function which sets a connection between the motive force and the flow during the memory time τ_D. In the case where the diffusion flow reacts instantaneously to a perturbation ($\tau_D \to 0$), the memory function is reduced to $M_D(t-t') = \delta(t-t')$. This yields the formula for the diffusion flow $\mathbf{J}_D = -M\nabla\delta\mathcal{F}/\delta x$ which is valid under conditions of a local equilibrium. If the effects of memory can play the basic role in the dynamics of corresponding physical processes, for example, at the pattern selection at the initial stages of the decomposition [29], then $\tau_D \neq 0$. As the simplest model involving this peculiarity of the reaction of the diffusion flow, we take an exponential function for $M(t-t')$ in the form $M_D(t-t') = (\tau_D)^{-1}\exp(-|t-t'|/\tau_D)$. Then a time variation of the diffusion flow will be described by the relaxation equation $\tau_D\partial_t\mathbf{J}_D = -\mathbf{J}_D - M\nabla\delta\mathcal{F}/\delta x$, where τ_D characterizes the time of relaxation of the flow to its stationary value. By assuming that fluctuations of the flow ξ exist always under real conditions, we can replace the relaxation equation by a Langevin stochastic equation in the form

$$\tau_D\partial_t\mathbf{J}_D = -\mathbf{J}_D - M\nabla\frac{\delta\mathcal{F}}{\delta x} + \xi(\mathbf{r},t), \qquad (2)$$

where $\xi(\mathbf{r},t)$ is a noise which models thermal fluctuations. It corresponds to the fluctuation-dissipation theorem and possesses the Gauss properties: $\langle\xi(\mathbf{r},t)\rangle = 0$, $\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t)\rangle = 2\theta M\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, where θ is the intensity proportional to the thermostat temperature T/T_c, and T_c is the critical mean-field value. For $M(t-t') = \delta(t-t')$, where $\tau_D \to 0$, we arrive at the Cahn–Hilliard–Cook parabolic model, $\partial_t x = \nabla\cdot M\nabla\delta\mathcal{F}/\delta x + \xi(\mathbf{r},t)$, which is true under conditions of a local equilibrium [33–37]. In the absence of an external influence ($\mathbf{J}_e = 0$) at $\tau_D \neq 0$, Eqs. (1) and(2) describe a stochastic system with hyperbolic transport (the Maxwell–Cattaneo stochastic equation) [39] $\tau_D\partial^2_{tt}x + \partial_t x = \nabla\cdot M\nabla\delta\mathcal{F}/\delta x + \xi(\mathbf{r},t)$. We note that the presence of the effects of memory in nonequilibrium systems allows one to obtain the important conclusion that the propagation velocity $v_D = l_D/\tau_D$ for perturbations of the field x is bounded, where l_D is the diffusion length. For the dimensional quantity τ_D, for example, in the $SiO_2 - 12\%Na_2O$ system, we have the estimation $\tau_D \simeq 10^{-11}$ s with the coefficient of diffusion $D \simeq 2.3\cdot10^{-14}$ cm^2/s [38]. As the time scale, let us choose the duration of the transition of an atom from one position to another one, $\tau_x = \omega_D^{-1}e^{E_a/T}$, where ω_D is the Debye frequency, E_a is the activation energy, and T is the temperature. Then, as $\tau'_D \to 0$, where $\tau'_D = \tau_D/\tau_x$, we pass to the limit of the instantaneous relaxation of a diffusion flow with $v_D \ggg 1$ (in what follows, the prime is dropped). The detailed description of the processes of spinodal decomposition in binary systems with hyperbolic transport is given in [40].

The flow $\mathbf{J}_e$ modeling the additional athermal mixing of atoms which is induced by an external influence is determined by the formula $\mathbf{J}_e = -D_e^0\int_0^t M_e(t-t')\nabla x(\mathbf{r},t')dt'$, where D_e^0 is the effective coefficient of athermal mixing. As a result of the assumption that the action of an external source (e.g., radiation) leads to the instantaneous appearance of a flow of atoms, we set $M_e(t-t') = \delta(t-t')$. Such an assumption is natural, since the duration of a cascade at the interaction of a high-energy particle with atoms of the medium (mixing) is of the order of $\tau_e \simeq 10^{-13}$ *s*. In other words, the mixing flow (formation of a structural disorder, turbulence, *etc.*) arises for the time $\tau_e \ll \tau_D$. This allows us to describe the given flow by the Fick law $\mathbf{J}_e = -D_e^0\nabla x$. By assuming that the incident high-energy particles have the stochastic nature (they are characterized by the Maxwell's distribution over velocities [26]), the stochasticity of such a mixing becomes obvious. This allows us to set $D_e^0 = D_e + \zeta(\mathbf{r},t)$, where D_e determines the regular part of the external flow $\mathbf{J}_e$, and $\zeta(\mathbf{r},t)$ is its stochastic component modeling the fluctuations caused by the formation of structural defects, *etc.* [41]. By its physical content, the quantity D_e characterizes the frequency of atomic jumps induced by the external influence which depends on the radiation flow ϕ, scattering cross-section σ_r, and the mean length of atomic jumps $\langle R\rangle$: $D_e \simeq \phi\sigma_r\langle R\rangle^2$. If the energy characteristics of bombarding particles fluctuate, then it becomes obvious that the lengths of such jumps $\langle(\delta R)^2\rangle$ have a dispersion which characterizes the intensity of the external noise σ^2. Considering the general theoretic problem, we define a stochastic component $\zeta(\mathbf{r},t)$ with the following properties:

$$\langle\zeta(\mathbf{r},t)\rangle = 0,$$

$$\langle\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t)\rangle = 2D_e\sigma^2 C\left(\frac{\mathbf{r}-\mathbf{r}'}{r_c}\right)\delta(t-t'), \qquad (3)$$

where $\sigma_e^2 = \langle(\delta R)^2\rangle/\langle R\rangle^2$ is the intensity of the external noise. The presence of the quantity D_e in correlator (3) testifies that the stochastic component of the flow $\mathbf{J}_e$ arises only if an external source acts. In what follows, we consider a model of the correlation function $C(\mathbf{r}-\mathbf{r}')$ in the form

$$C(\mathbf{r}-\mathbf{r}') = (\sqrt{2\pi}r_c)^{-d}\exp\left(-(\mathbf{r}-\mathbf{r}')^2/2r_c^2\right), \qquad (4)$$

where r_c is the correlation length which determine a linear size of the overlapping of excited atomic configurations due to the interaction with high-energy particles; and d is the space dimension.As $r_c \to 0$, we have the limiting case of white noise in space and in time.

Thus, the full system of equations describing the evolution of a random field reads

$$\begin{cases} \partial_t x = -\nabla \mathbf{J}_D + D_e \Delta x + \nabla(\zeta \nabla x), \\ \tau_D \partial_t \mathbf{J}_D = -\mathbf{J}_D - M\nabla \dfrac{\delta \mathcal{F}}{\delta x} + \xi. \end{cases} \tag{5}$$

In the subsequent theoretical consideration, we will study the ordering in the systems which are described by a functional of free energy in the general form $\mathcal{F} = \int d\mathbf{r} \left(f(x) + \frac{1}{2}(\mathcal{L}x)^2 \right)$, where $f(x)$ is the free energy density, and $\mathcal{L}$ is the operator of a spatial interaction. For the systems undergoing the spinodal decomposition with $x = c - 1/2$, we have a symmetric shape of the free energy density $f(x) = f(-x)$, and $\mathcal{L}$ is determined in the standard way by the expansion in the lowest order of the spatial derivative with respect to the field in the form $\mathcal{L} = \nabla$. Such a term in the free energy testifies that the concentration field gradient is not energy-gained and leads to the surface tension of the boundary of grains separating the regions enriched by that or other component of a binary solution. We also indicate the important fact that the relevant spatial structures which are determined by the minimization of the free energy lose firstly the stability at the zero value of the wave number. Thus, we will consider a model of systems analyzed in the present work which is given by the Ginzburg–Landau functional

$$\mathcal{F} = \int d\mathbf{r} \left(f(x) + \frac{1}{2}(\nabla x)^2 \right) f(x) = \frac{\varepsilon}{2}x^2 + \frac{x^4}{4}, \tag{6}$$

where $\varepsilon = \theta - 1$ is the controlling parameter determined through the temperature T reckoned from the critical mean-field value T_c, $\theta = T/T_c$.

3. Analysis of the Stability and Selection of Structures

Since the system under consideration is stochastic, the information is contained in the measured characteristics which are reduced to statistically averaged quantities such as the mean value of a random field (the volumetric share of a certain component) $\langle x(\mathbf{r}, t)\rangle$ and the structure factor $S_{\mathbf{k}}(t)$, rather than the solutions of the system of equations (5). In this section, we will construct an equation describing the dynamics of these quantities and analyze the corresponding solutions.

3.1. Equation of evolution of the mean and the structure factor

Equation of the dynamics of the mean

First, we consider a behavior of the first statistical moment of a random field $\langle x\rangle$. Performing the averaging of the system of equations (5) over fluctuations, we obtain the following equations for the means:

$$\partial_t \langle x\rangle = -\nabla \cdot \langle \mathbf{J}_D\rangle + D_e \Delta \langle x\rangle + \nabla \cdot \langle \zeta \nabla x\rangle,$$

$$\tau_D \partial_t \langle \mathbf{J}_D\rangle = -\langle \mathbf{J}_D\rangle - \nabla M \left\langle \frac{\delta \mathcal{F}}{\delta x} \right\rangle. \tag{7}$$

The correlator of an external noise is calculated by the Novikov's formula [42]

$$\langle \zeta \nabla x\rangle = D_e \tilde{\sigma}^2 \int\limits_{-\infty}^{\infty} C(\mathbf{r} - \mathbf{r}')\nabla \left\langle \frac{\delta x(\mathbf{r}, t)}{\delta \zeta(\mathbf{r}', t)} \right\rangle d\mathbf{r}'. \tag{8}$$

The derivative with respect to the noise is obtained from a formal solution of the Langevin equation for the field x from (5),

$$\frac{\delta x(\mathbf{r}, t)}{\delta \zeta(\mathbf{r}', t)} = \nabla \left(\delta(\mathbf{r} - \mathbf{r}')\nabla x(\mathbf{r}, t) \right). \tag{9}$$

Substituting (9) in (8), we get [8, 41]

$$\langle \zeta \nabla x\rangle = D_e \sigma_e^2 \left[C(\mathbf{r} - \mathbf{r}')\big|_{\mathbf{r}=\mathbf{r}'} \nabla^3 \langle x\rangle + \right.$$

$$\left. +2 \left(\nabla C(\mathbf{r} - \mathbf{r}')\big|_{\mathbf{r}=\mathbf{r}'} \right) \nabla^2 \langle x\rangle + (\nabla \langle x\rangle)\nabla^2 \, C(\mathbf{r} - \mathbf{r}')\big|_{\mathbf{r}=\mathbf{r}'} \right]. \tag{10}$$

It is worth noting that $C(\mathbf{r} - \mathbf{r}')$ takes the maximum value at $\mathbf{r} = \mathbf{r}'$, which yields

$$\nabla C(\mathbf{r} - \mathbf{r}')\big|_{\mathbf{r}=\mathbf{r}'} = 0; \quad \nabla^2 \, C(\mathbf{r} - \mathbf{r}')\big|_{\mathbf{r}=\mathbf{r}'} < 0. \tag{11}$$

Then, by introducing the notation $\varpi(\nabla^2) = \varepsilon - \mathcal{L}^2 + 3x_0^2$ ($M = 1$), we obtain the system of equations

$$\begin{cases} \partial_t \langle x\rangle = -\nabla \langle \mathbf{J}_D\rangle + D_e \Delta \langle x\rangle + \\ \quad + D_e \sigma^2 (\nabla^2 C(|\mathbf{r}|)|_{\mathbf{r}=\mathbf{0}} \Delta \langle x\rangle + D_e \sigma^2 C(\mathbf{0})\nabla^4 \langle x\rangle, \\ \tau_D \partial_t \langle \mathbf{J}_D\rangle = -\langle \mathbf{J}_D\rangle - \nabla \varpi(\nabla^2)\langle x\rangle. \end{cases} \tag{12}$$

Then the solution can be realized in a Fourier-space. To this end, we use the Fourier-components $\langle x_{\mathbf{k}}(t)\rangle =$

$\int d\mathbf{r}\langle x(\mathbf{r},t)\rangle e^{i\mathbf{kr}}$, $\langle \mathbf{J_k}(t)\rangle = \int d\mathbf{r}\langle \mathbf{J}(\mathbf{r},t)\rangle e^{i\mathbf{kr}}$ and rewrite (12) in the form

$$\begin{cases} \dfrac{d\langle x_\mathbf{k}\rangle}{dt} = -i\mathbf{k}\langle \mathbf{J}_{D\mathbf{k}}\rangle - D_e|\mathbf{k}|^2\langle x_\mathbf{k}\rangle - \\ - D_e\sigma^2\nabla^2 C(\mathbf{r})|_{\mathbf{r}=\mathbf{0}}k^2\langle x_\mathbf{k}\rangle + D_e\sigma^2 C(\mathbf{0})k^4\langle x_\mathbf{k}\rangle, \\ \tau_D\dfrac{d\langle \mathbf{J}_{D\mathbf{k}}\rangle}{dt} = -\langle \mathbf{J}_{D\mathbf{k}}\rangle - i\mathbf{k}\varpi(k^2)\langle x_\mathbf{k}\rangle. \end{cases} \quad (13)$$

The system of ordinary differential equations (13) admits the analytical solution.

The analysis of the stability of a homogeneous state x_0 can be directly carried out for the averaged quantity $\langle x_\mathbf{k}\rangle$, whose study can be performed in a very simple way. To this end, we execute the additional differentiation of the first equation of system (13), by using the evolution equation for the Fourier-transform of a diffusion flow and by taking the flow $\mathbf{J}_{D\mathbf{k}}$ from the first equation of system (13), and get a single differential equation of the second order in time

$$\tau_D\frac{d^2\langle x_\mathbf{k}\rangle}{dt^2} = -\left(1+\tau_D D_e k^2\Xi(k^2)\right)\frac{d\langle x_\mathbf{k}\rangle}{dt} -$$

$$-k^2\left(D_e\Xi(k^2)+\varpi(k^2)\right)\langle x_\mathbf{k}\rangle, \quad (14)$$

where we introduced the notation $\Xi(k^2) \equiv 1 + \sigma^2(\nabla^2 C(|r|)_{r=0} - C(0)k^2)$. We seek a solution of the obtained equation in the form $\langle x_{|\mathbf{k}|}(t)\rangle = \langle x_{|\mathbf{k}|}(0)\rangle\exp(\phi(k)t)$. Substituting it in the second-order equation, we get the formula for the phase $\phi(k) = \Re\phi(k) + i\Im\phi(k)$, where

$$\phi(k)_\pm = -\frac{1+\tau_D D_e k^2\Xi(k^2)}{2\tau_D} \pm$$

$$\pm\frac{1}{2\tau_D}\left[(1+\tau_D D_e k^2\Xi(k^2))^2 -\right.$$

$$\left. - 4\tau_D k^2\left(D_e\Xi(k^2)+\varpi(k^2)\right)\right]^{1/2}. \quad (15)$$

Let us analyze the characteristic specific features of formula (15). It is obvious that the unstable modes appear at $\Re\phi(k)_+ > 0$. It is known that the first instability for the model under consideration arises at the wave number $k = 0$. In this case, the maximum value of k which bounded the region of instability, k_c, is determined by the condition $\Re\phi(k)_+ > 0$ at $k > 0$. In addition, the quantity $\Re\phi(k)_+$ can have a single peak, whose position sets the most unstable mode with a wave number k_m. The value of k_m is calculated from solutions of the equation for the maximum of $\Re\phi(k)_+$ in the interval of instability. It follows from relation (15) that the imaginary part of the phase exists under the condition $(1+\tau_D D_e k^2\Xi(k^2))^2 < 4\tau_D k^2\left(D_e\Xi(k^2)+\varpi(k^2)\right)$. Thus, the evolution of the averaged quantity $\langle x_{|\mathbf{k}|}(t)\rangle$ can run in the general case in the form of damped oscillations with a frequency $\Im\phi(k)$ and a decrement $(1+\tau_D D_e k^2\Xi(k^2))/2\tau_D > 0$. The boundary of the region of damping of the quantity $\langle x_{|\mathbf{k}|}(t)\rangle$ is determined by the wave number

$$k_d^2 = \frac{1}{2\sigma^2 C(0)}\left(1+\sigma^2\nabla^2 C(|r|)_{r=0} + \right.$$

$$\left. + \sqrt{(1+\sigma^2\nabla^2 C(|r|)_{r=0})^2 + \frac{4\sigma^2 C(0)}{\tau_D D_e}}\right). \quad (16)$$

The boundary of the existence of the oscillatory behavior of $\langle x_{|\mathbf{k}|}(t)\rangle$ is determined by the solutions $k_0 = k_0(\theta, D_e, \sigma^2, r_c)$ of the equation

$$(\tau_D D_e k^2\Xi(k^2))^2 < 2\tau_D k^2(D_e\Xi(k^2)+2\varpi(k^2)) - 1. \quad (17)$$

Thus, the phase $\phi(k)$ allows us to find the specific features of a behavior of the solutions $\langle x_{|\mathbf{k}|}(t)\rangle$.

Equation of the dynamics of the structure factor

The full information about the behavior of a system can be obtained, by studying the correlation functions or their Fourier-transforms, namely the structure factor $S_\mathbf{k}(t)$. We now determine the form of a dynamical equation for $S_\mathbf{k}(t)$. To this end, we use the input system of equations (5) rewritten in the Fourier-space:

$$\frac{dx_\mathbf{k}}{dt} = -i\mathbf{k}\mathbf{J}_{D\mathbf{k}} - k^2 D_e x_\mathbf{k} - k^2\zeta_\mathbf{k}x_\mathbf{k},$$

$$\tau_D\frac{d\mathbf{J}_{D\mathbf{k}}}{dt} = -\mathbf{J}_{D\mathbf{k}} - i\mathbf{k}\varpi(k^2)x_\mathbf{k} + \xi_\mathbf{k}. \quad (18)$$

Then the equation for the structure factor $S_\mathbf{k} = \langle x_\mathbf{k}x_{-\mathbf{k}}\rangle$ takes the form

$$\frac{dS_\mathbf{k}}{dt} = -i\mathbf{k}\langle \mathbf{J}_{D\mathbf{k}}x_{-\mathbf{k}}\rangle + i\mathbf{k}\langle \mathbf{J}_{D-\mathbf{k}}x_\mathbf{k}\rangle -$$

$$-2k^2 D_e S_\mathbf{k} - k^2(\langle\zeta_\mathbf{k}x_\mathbf{k}x_{-\mathbf{k}}\rangle + \langle\zeta_{-\mathbf{k}}x_{-\mathbf{k}}x_\mathbf{k}\rangle). \quad (19)$$

The corresponding correlators of the flow and the field can be calculated from the equation

$$\tau_D \frac{d\langle \mathbf{J}_{D\mathbf{k}} x_{-\mathbf{k}}\rangle}{dt} = -\langle \mathbf{J}_{D\mathbf{k}} x_{-\mathbf{k}}\rangle - i\mathbf{k}\varpi(k^2)S_{\mathbf{k}} + \langle \xi_{\mathbf{k}} x_{-\mathbf{k}}\rangle. \quad (20)$$

Then, by calculating the derivative of relation (19) with respect to time, taking the flow correlator from (19), and using (20), we get a second-order equation for the structure factor in the form

$$\tau_D \frac{d^2 S_{\mathbf{k}}}{dt^2} = -(1 + 2k^2\tau_D D_e)\frac{dS_{\mathbf{k}}}{dt} - 2k^2(D_e + \varpi(k^2))S_{\mathbf{k}} -$$

$$-k^2(\langle \zeta_{\mathbf{k}} x_{\mathbf{k}} x_{-\mathbf{k}}\rangle + \langle \zeta_{-\mathbf{k}} x_{-\mathbf{k}} x_{\mathbf{k}}\rangle) -$$

$$-i\mathbf{k}(\langle \xi_{\mathbf{k}} x_{-\mathbf{k}}\rangle + \langle \xi_{-\mathbf{k}} x_{\mathbf{k}}\rangle) -$$

$$-k^2\tau_D \frac{d}{dt}(\langle \zeta_{\mathbf{k}} x_{\mathbf{k}} x_{-\mathbf{k}}\rangle + \langle \zeta_{-\mathbf{k}} x_{-\mathbf{k}} x_{\mathbf{k}}\rangle). \quad (21)$$

Expanding the correlators by the Novikov theorem, where

$$\langle \zeta_{-\mathbf{k}} x_{-\mathbf{k}} x_{\mathbf{k}}\rangle = D_e\sigma^2(\nabla^2 C(|r|)_{r=0} - C(0)k^2)S_{\mathbf{k}},$$

$$\langle \zeta_{-\mathbf{k}} x_{-\mathbf{k}} x_{\mathbf{k}}\rangle \equiv \langle \zeta_{\mathbf{k}} x_{\mathbf{k}} x_{-\mathbf{k}}\rangle,$$

$$\langle \xi_{-\mathbf{k}} x_{\mathbf{k}}\rangle \equiv \langle \xi_{\mathbf{k}} x_{-\mathbf{k}}\rangle = i\mathbf{k}\theta, \quad (22)$$

we arrive at the required dynamical equation

$$\tau_D \frac{d^2 S_{\mathbf{k}}}{dt^2} = -(1 + 2k^2\tau_D D_e \Xi(k^2))\frac{dS_{\mathbf{k}}}{dt} -$$

$$-2k^2(D_e\Xi(k^2) + \varpi(k^2))S_{\mathbf{k}} + 2\theta k^2 -$$

$$-\frac{2k^2 D_e\sigma^2}{(2\pi)^d}\int d\mathbf{k}' C(|\mathbf{k}-\mathbf{k}'|)S_{\mathbf{k}'}(t) -$$

$$-\frac{2k^2\tau_D D_e\sigma^2}{(2\pi)^d}\int d\mathbf{k}' C(|\mathbf{k}-\mathbf{k}'|)\frac{dS_{\mathbf{k}'}(t)}{dt}. \quad (23)$$

As seen from the structure of Eq. (23), it admits a solution in the form $S - S_0 \propto e^{\varphi(k)t}$, where the phase

$$\varphi(k)_\pm = -\frac{1 + 2\tau_D D_e k^2 \Xi(k^2)}{2\tau_D} \pm$$

$$\pm\frac{1}{2\tau_D}\left[(1 + 2\tau_D D_e k^2\Xi(k^2))^2 -\right.$$

$$\left.-4\tau_D k^2\left(2D_e\Xi(k^2) + \varpi(k^2)\right)\right]^{1/2} \quad (24)$$

can possess the real and imaginary parts under certain conditions, i.e., $\varphi = \Re\varphi(k) + \Im\varphi(k)$. In the theory of spinodal decomposition, the real part $\Re\varphi(k)_+$ is known as the amplification rate $R(k) = -\Re\varphi(k)_+$, so that $S - S_0 \propto e^{-R(k)t}$, and the imaginary part $\Im\varphi(k)$ arising only at $\tau_D \neq 0$ is responsible for the process of pattern selection.

3.2. Influence of a noise on the process of pattern selection

Stability of the state $x_0 = 0$

First, we will study the stability of a disordered state $x_0 = 0$ which corresponds to a maximum of the free energy density $f(x)$. To this end, we consider the real and imaginary parts of the phase $\varphi(k)$ which depend on the external conditions and are presented in Fig. 1. In the simple case where no external flow is present (continuous line in Fig. 1,*a*) and the initial density of free energy has a two-well shape ($\theta < 1$), we possess the standard pattern of instability where one of the branches of the real part of the phase ($\Re\varphi(k)_+$) becomes positive in the interval $0 \le k \le k_c$, where k_c bounds the region of unstable modes. The most unstable mode corresponds to the value equal to k_m. If the wave number attains the value of $k = k_0$, two branches of the real part of the phase degenerate, so that the real part has a single value independent of the wave number and determines the damping decrement for solutions of the equation of evolution of the structure factor. At $k > k_0$, the imaginary part of the phase $\Im\varphi(k)$ arises, and a wave behavior of the structure factor is revealed (Fig. 1,*b*). If a determined external influence with $D_e \neq 0$ is switched-on at $\sigma^2 = 0$ (dashed curve), the region of instability becomes narrower (the value of k_c decreases). In this case, we also observe a decrease in the wave number k_0, above which the oscillatory mode is realized. In this case, the instability loss induced by the additional mixing is of importance. As seen from the formula for $\varphi(k)$, the coefficient D_e leads to an increase in the effective temperature of the system by D_e [27]. We note that the external fluctuations ($\sigma^2 \neq 0$ at $D_e \neq 0$) (dashed curve) lead to the appearance of instabilities, extension of the space of unstable modes, and increase in the value of k_0.

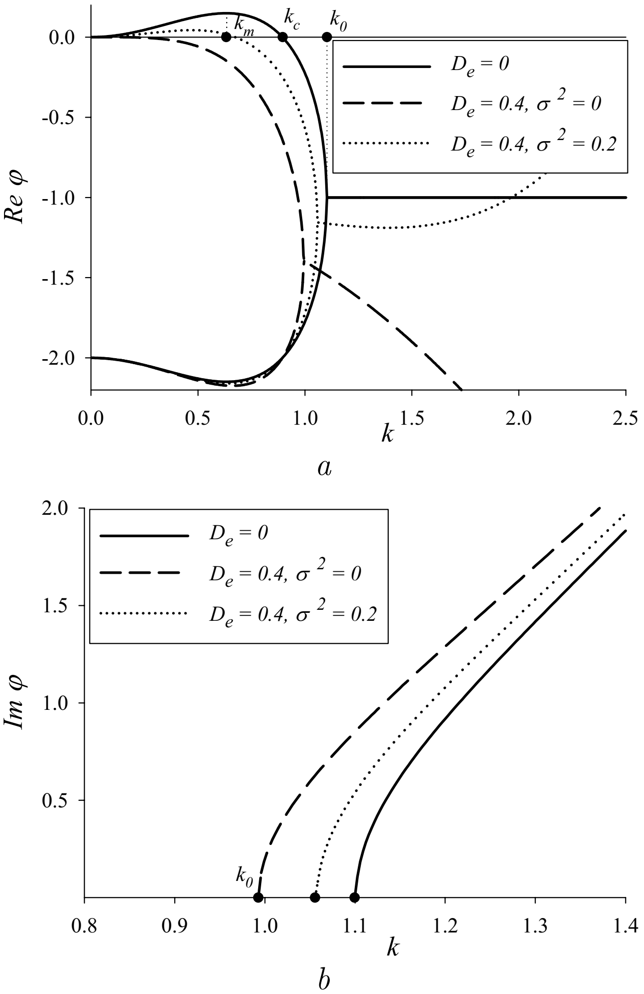


Fig. 1. Real (a) and imaginary (b) parts of the phase $\varphi(k)$ at a variation of parameters of the external flow D_e and σ^2 in a vicinity of the state $x_0 = 0$. Curves are obtained at $\theta = 0.4$, $\tau_D = 0.5$, and $r_c = 0.5$

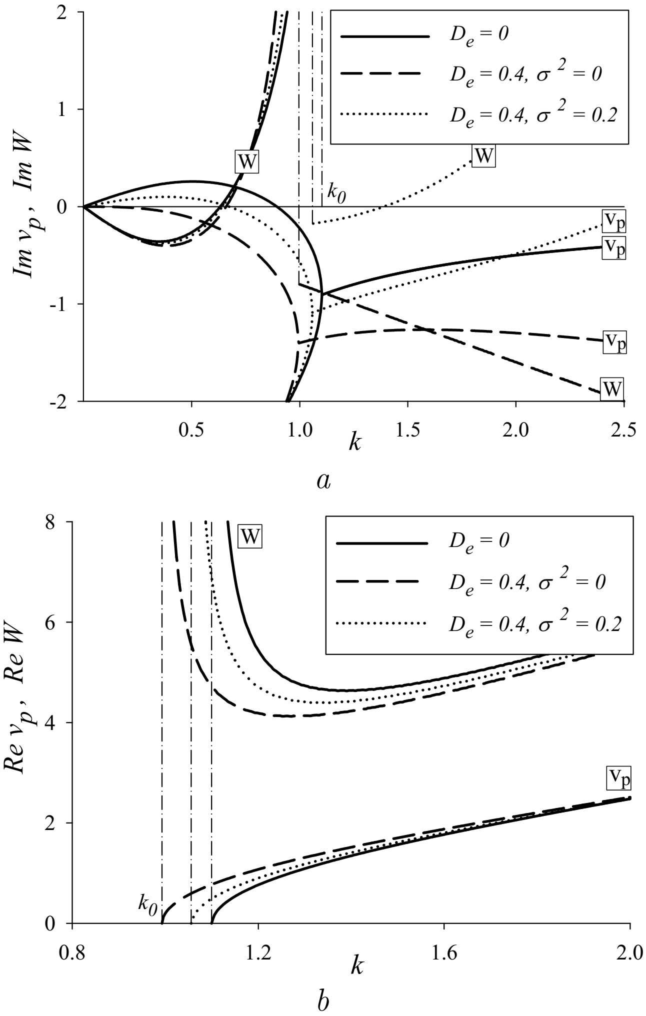


Fig. 2. Real (a) and imaginary (b) parts of the group $W(k)$ and phase $v_p(k)$ velocities at the variation of parameters of the external flow D_e and σ^2 in a vicinity of the state $x_0 = 0$. Curves are obtained at $\theta = 0.4$, $\tau_D = 0.5$, and $r_c = 0.5$

It is worth to note that the imaginary part of the phase in the presence of an external flow behaves itself analogously to the case where the flow is absent (compare the curves in Fig. 1,b. However, the real part of the phase changes its character at the appearance of unstable modes: if only the stable modes are realized (the dashed curve in Fig. 1,a, we observe the descending curve of the real part of the phase at $k > k_0$, whereas the quantity $\Re\varphi(k)$ grows at the appearance of instabilities at small k. In this case, the value of $\Re\varphi(k) = 0$ is realized at $k = k_d$. Therefore, we restrict ourselves in the analytical consideration by choosing the values of parameters of the system to satisfy the condition $k_d \leq \pi$.

Phase and group velocities

We now analyze the character of a behavior of the phase and group velocities in the present model. By definition, the phase velocity is set by the frequency $\omega(k)$ (the law of dispersion). Therefore, we pass to the relation $\varphi(k) = i\omega(k)$ and obtain the following formulas for the real and imaginary components of the phase velocity: $\Re v_p = \Re\omega(k)/\Re k$ or $\Re v_p = \Im\varphi(k)/\Im k$ and $\Im v_p = \Im\omega(k)/\Im k$ or $\Im v_p = \Re\varphi(k)/\Re k$. In the general case, the velocity v_p takes the variation of a single (separated) harmonic into account. The real part of the phase velocity describes the propagation of perturbations in the direct and reverse spatial directions,

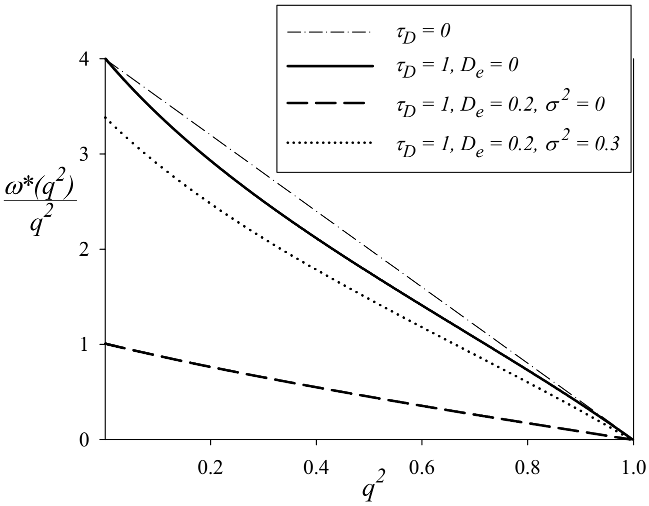


Fig. 3. Normalized amplification factor at early stages at $\tau_D = 1.0$, $r_c = 0.5$, and $\theta = 0.2$

whereas the imaginary part of v_p determines the rate of intensification of the given harmonic. It follows from the calculations executed that, in the case $D_e = 0$ in the high-frequency region $\omega \to \infty$, the real part of the phase velocity is reduced to the propagation velocity, i.e., $\Re v_p = v_D$. At $D_e = 0$ and $D_e \neq 0$, the real part of v_p exists only at $k > k_0$, whereas the imaginary one exists at all allowed k (see Fig. 2,*a, b*). At $k < k_0$, the harmonics do not move at a possible change of their amplitudes, whereas they do at $k > k_0$.

While studying the group velocity $W(k) = \partial\omega(k)/\partial k$ which sets the velocity of motion of a concentration package, we should expect some differences in behaviors of $W(k)$ at $D_e = 0$ and $D_e \neq 0$, because a behavior of the phase changes at $k > k_0$ (see Fig. 1,*a*). In the case where $D_e = 0$, the imaginary part of the group velocity exists only at $k < k_0$, whereas the real part of $W(k)$ is realized at $k > k_0$. On the contrary, the real part of the group velocity at $D_e \neq 0$ takes nonzero positive values at $k > k_0$ (Fig. 2,*b*), and the imaginary one exists at all k and possesses a characteristic singularity of the second kind at $k = k_0$. In this case, at $D_e \neq 0$ and $\sigma^2 = 0$, the real part of the phase $\varphi(k)$ decreases at $k > k_0$. Therefore, $\Im W(k)$ decreases as well. External fluctuations at $D_e \neq 0$ cause the complicated behavior of the phase $\Re\varphi(k)$ in the interval $k > k_0$ (see Fig. 1,*a*). This induces an increase in $\Im W(k)$ at $k > k_0$. A change of the sign of $\Im W(k)$ testifies that the dependence $\Re\varphi(k)$ has a minimum: in the region $k > k_0$, the real part $\Re\varphi(k)$ of the phase decreases firstly due to the determined action of an external flow, attains the minimum, and then begins to increase due to external fluctuations.

Normalized factor of amplification of the decomposition

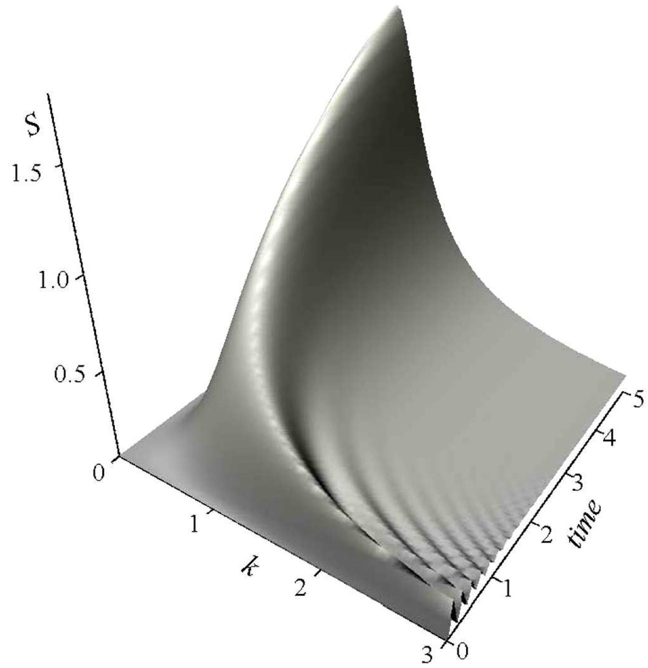
On the basis of the formula for the phase $\varphi(k)$ (or the law of dispersion $\omega(k) = i\varphi(k)$), we can determine the character of a behavior of the function $\Re\omega(k)/k^2$ known as the normalized amplification rate of decomposition. This function characterizes the irreversible growth of the decomposition wavelength. It is known that such a dependence is essentially linear in the Cahn's theory. However, at the spinodal decomposition in glasses, for example, in Na_2OSiO_2, a deviation from the Cahn's linear law is observed [37, 38, 40]. To explain such nonlinear effects, it was proposed to generalize the Cahn–Hilliard–Cook model by introducing an additional relaxation variable with nonconservative dynamics [43–45], whose role can be played by a diffusion flow [39]. In the frame of the hyperbolic model, such a deviation is explained by the presence of the effects of memory according to the hypothesis of a local nonequilibrium: in the case where the relaxation of a diffusion flow is observed, a deviation from the linear law is controlled by the memory time τ_D [38, 40]. Here, we will determine how the external flow can affect such a deviation.

To determine the normalized amplification factor, it is necessary to compare the dispersion relations for the Cahn–Hilliard–Cook parabolic model and for the proposed model. As the normalizing quantity, we use $\omega_{\rm CHC}(k_m)$ obtained within the parabolic model at $D_e = 0$, where $k_m = k_c/\sqrt{2}$. Then the normalized amplification factor takes the form $\omega^*(q)/q^2 = \omega(q)/\omega_{\rm CHC}(k_m)/q^2$, where $q = k/k_c$, and the dispersion $\omega(q)$ is determined from the relation $\omega(q) = \Re\varphi(q)_+$. Thus, within our model, we have $\omega^*(q)/q^2 = 4/q^2\Re\varphi_+(q)/\varepsilon^2$. The normalized amplification factor at early stages is given in Fig. 3. At $D_e = 0$ for the parabolic and hyperbolic models, we have the following limiting values: $\omega^*(q)/q^2 = 0$ at $q = 1$ and $\omega^*(q)/q^2 = 4$ at $q = 0$. As $\tau_D \to 0$ (Cahn–Hilliard–Cook model), we get a linear dependence of $\omega^*(q)/q^2$ on q^2 (thin dash-dotted line). At $\tau_D = 1.0$, we trace deviations from the linear law (continuous line). For $\tau_D = 1.0$, $D_e \neq 0$, and $\sigma^2 = 0$, the athermal mixing leads to the linearization of the dependence of the amplification factor due to the renormalization of the effective temperature (dashed line). However, in the presence of a stochastic external influence which causes the effects of destabilization, we observe the renewal of the nonlinearity of the amplification factor (dotted line). Thus, the deviations from the Cahn's linear law can be reduced by the determined component of the athermal mixing, whereas its stochastic component restores the nonequilibrium state.

Evolution of the structure factor in a vicinity of $x_0 = 0$

The evolution of the structure factor in a vicinity of the state $x_0 = 0$ is shown in Fig. 4. It is seen that the oscillatory behavior of $S(k,t)$ is observed in the course of time and at a change in the wave number. Whereas the oscillatory behavior in the course of time is explained by the form of the equations for the mean and the structure factor, the presence of oscillations in the space of values of the wave number seems to be also important. It is known that, in the Cahn–Hilliard–Cook ordinary parabolic model ($\tau_D = 0$), the unstable modes are realized in the system on early stages. These modes induce the development of concentration waves, so that the structure factor has only one peak depending on k which corresponds to the most unstable mode k_m. Let us now turn to the hyperbolic model ($\tau_D \neq 0$) which considers the relaxation of the diffusion flow. In addition to the oscillatory behavior in the course of time and the principal peak of the structure factor as a function of k, the model admits the presence of accompanying peaks which are responsible for the realization of structures with other values of the wave number. Since the amplitude of such oscillations in k decreases in the course of time, this testifies to the pattern selection during the evolution of a physical system, when the structures with a single value of $k = k_m$ are realized. It is characteristic that, in the case under study, such space-time decaying oscillations are observed at the analysis of the system both in terms of the mean value of the stochastic field and the structure factor. The total time behavior of the corresponding structure factor is standard: on early stages, the main peak of the structure factor shifts to the region of small k, i.e., the size of grains becomes larger, the width of the peak decreases, and the boundaries between phases become more clear.

The dependence of the structure factor on the wave number (right inserts in Fig. 4,*b*) indicates that the external action affects basically the character of the pattern selection. In this case, is characteristic that the regular component of the athermal flow ($D_e \neq 0$, $\sigma^2 = 0$) oppresses the process of pattern selection, whereas the stochastic component $\sigma^2 \neq 0$ causes an increase in values of the structure factor on accompanying peaks, by favoring the processes of selection. We note that the competition of the regular and stochastic components of the external (athermal) flow induces a decrease of the main peak of $S(k)$ at large D_e and an increase in the peak width. This testifies that the boundaries between phases become more diffusive. It is obvious, since the regular component of the external flow causes the additional diffusion inducing the smearing of interfaces. However, the action of the stochastic component of the flow $\mathbf{J}_e$ is op-



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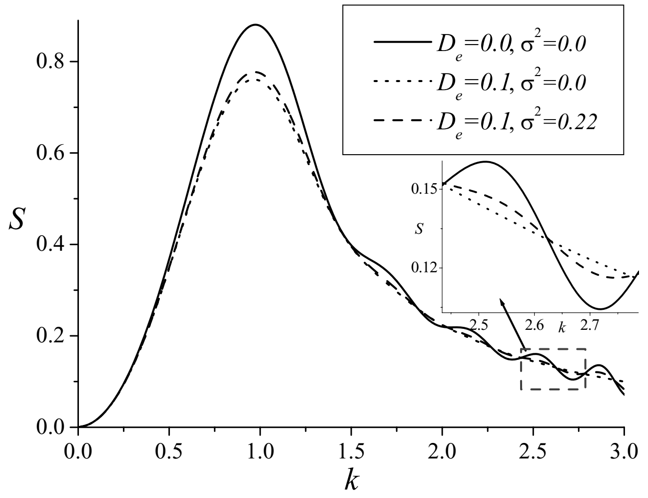


b

Fig. 4. Dynamics of the structure factor on early stages at $\tau_D = 1.0$, $r_c = 1.0$, and $\theta = 0.9$. Parameters of the system on the plot of $S(k,t)$ are as follows: $D_e = 0.1$ and $\sigma^2 = 0.22$ (*a*); the dependence $S(k)$ is constructed at $t = 2$ (*b*)

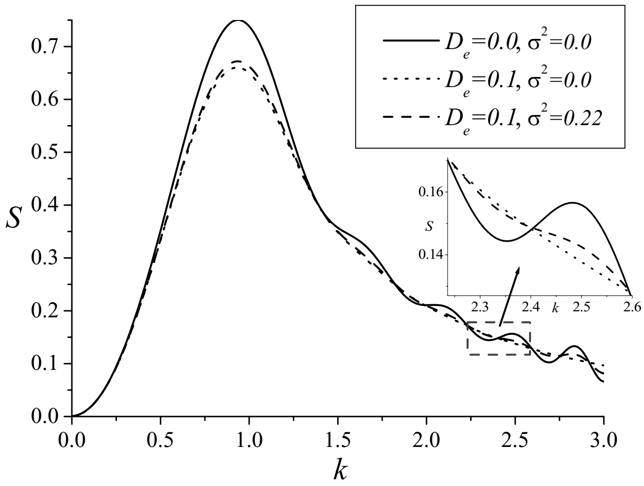


Fig. 5. Structural factor in a vicinity of the state $x_0 = \sqrt{1-\theta}$ at $\tau_D = 1.0$, $r_c = 0.65$, and $\theta = 0.9$

posite to that of the deterministic one. It is worth also noting that the principal peak of the structure factor shifts to the region of large k with decrease in θ, i.e., the size of grains becomes smaller at low temperatures.

Stability of the state $x_0 = \sqrt{1-\theta}$

We now analyze the stability of solutions of the equation of evolution of the structure factor in a vicinity of the state corresponding to a minimum of the free energy density $f(x)$. Even the simplest physical ideas of this process imply that the physical system falls in a minimum of the free energy, where its state must be stable. Therefore, the real part of the phase in a vicinity of $x_0 = \sqrt{1-\theta}$ is negative. Numerical calculations give also the result $\Re\phi(k)_\pm < 0$ at all k in the admissible region of parameters of the system.

Instead of that, the imaginary part of the phase does not disappear and exists at $k > k_0$. The character of the dependence $\Im\phi(k)$ is topologically analogous to that of the curves in Fig. 1,*b*. In this case, an increase in the intensity of the external noise causes an increase in k_0 accompanied by a decrease in the frequency of oscillations in k.

The structure factor in a vicinity of the state $x_0 = \sqrt{1-\theta}$ is given in Fig. 5. It is seen that the corresponding functions reveal the oscillatory behavior. Comparing the dependences $S(k,t)$ in vicinities of two states under consideration, we see that the principal peak shifts to the region $k \to 0$, and its amplitude increases. Analogously to the previous case (in a vicinity of the maximum of the free energy), there occurs the pattern selection in the system in a vicinity of the minimum of $f(x)$. An increase in the determined contribution to the external flow leads to a decrease in both the height of the main peak of the structure factor and the amplitude of its oscillations. On the other hand, the stochastic component of the external flow causes an increase in both the amplitude of oscillations of the function $S(k)$ and the height of the main peak. Thus, due to the action of the external fluctuating source, the spatial structures become clearly pronounced with sharp interfaces. With increase in the time t, the accompanying peaks of the structure factor decrease, and the position of the main peak tends to zero.

4. Modeling

The analytical calculations of the oscillatory behavior of the first statistical moment and the structure factor can be confirmed by an independent numerical modeling of the system on a two-dimensional network $N \times N$ with periodic boundary conditions. The corresponding system of differential equations in a discrete space with a linear size $L = \ell N$ is as follows:

$$\frac{dx_i}{dt} = -(\nabla_R)_{ij} J_j + D_e \Delta_{ij} x_j + (\nabla_R)_{ik} \zeta_k (\nabla_L)_{kl} x_l,$$

$$\tau_D \frac{dJ_i}{dt} = -J_i - M(\nabla_L)_{ij} \frac{\partial F}{\partial x_j} + \xi_i,$$

$$\tau_\zeta \frac{d\zeta_i}{dt} = -(\delta_{ij} - r_c^2 \Delta_{ij})\zeta_j + \tilde{\xi}_i. \qquad (25)$$

Here, ℓ is the size of an elementary cell, and the left- and right-sided gradient operators take the form

$$(\nabla_L)_{ij} = \frac{1}{\ell}(\delta_{i,j} - \delta_{i-1,j}),$$

$$(\nabla_R)_{ij} = \frac{1}{\ell}(\delta_{i+1,j} - \delta_{i,j}),$$

$$(\nabla_L)_{ij} = -(\nabla_R)_{ji},$$

$$(\nabla_L)_{ij}(\nabla_R)_{jl} \equiv \Delta_{il} = \frac{1}{\ell^2}(\delta_{i,l+1} - 2\delta_{i,l} + \delta_{i,l-1}), \qquad (26)$$

where δ_{ij} is the Kronecker delta-symbol. In the modeling in the case of a quasiwhite external noise, we assume $\tau_\zeta \ll 1$; $\langle \tilde{\xi}_i(t) \rangle = 0$, and $\langle \tilde{\xi}_i(t) \tilde{\xi}_j(t') \rangle = \delta_{ij}\delta(t-t')$. In such a case, we arrive at the limiting case of a stochastic

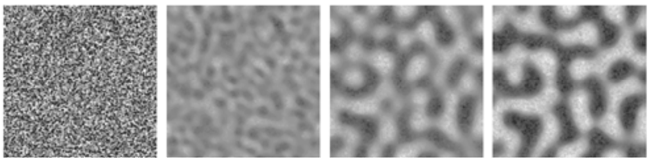


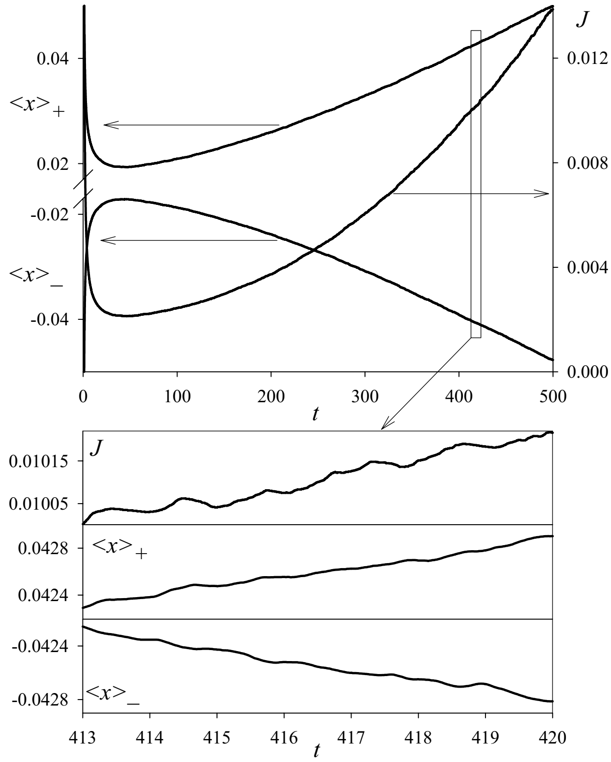
Fig. 6. Typical pattern of the evolution of a system at $\theta = 0.4$, $D_e = 0.5$, $\sigma^2 = 1.0$, $\tau_D = 0.5$, and $r_c = 1.0$. Time cuts correspond to the time moments $t = 0$, 50, 500, 1000

process with properties (3) and the spatial correlation function (4).

The modeling was performed with the integration step $\delta = 10^{-3}$ at $\ell = 1.0$ on the network 128×128 in size. A typical snapshot of the evolution of the stochastic system under the initial conditions $\langle x(\mathbf{r}, t = 0)\rangle = 0$ and $\langle(\delta x)^2\rangle = 0.3$ is shown in Fig. 6. It is seen that the system in the course of time decays into two domains which are equivalent by density by the mechanism of spinodal decomposition.

In order to confirm the oscillatory behavior of the first statistical moment and the structure factor in time, we calculated the evolution of averaged quantities. Since the system under study is referred to a class of systems with conservative dynamics ($\int d\mathbf{r}x(\mathbf{r}, t) = \text{const}$, where const $= 0$ in our case), we chose the means to be measured in a numerical experiment such that they correspond separately positive and negative values of the field x. That is, $\langle x\rangle_+ = \langle N^{-2}\sum_i x_i^>\rangle$, where $x_i^>$ corresponds to the i-th node of the grid with $x_i > 0$, and $\langle x\rangle_- = \langle N^{-2}\sum_i x_i^<\rangle$, where $x_i^<$ corresponds to $x_i < 0$, and $\langle\ldots\rangle$ means the averaging over the realizations. It is obvious that the means $\langle x\rangle_+$, $\langle x\rangle_-$ must increase during the evolution, by satisfying the condition of conservation $\langle x\rangle \equiv \langle x\rangle_+ + \langle x\rangle_- = 0$. If the oscillations of the first moment arise in the system at a deviation from a certain stationary value, then they must be presented on the dependences $\langle x(t)\rangle_\pm$. It is known that the order parameter in the systems with conservative dynamics is the second statistical moment $J = \langle N^{-2}\sum_i x_i^2\rangle$, whose increase testifies to the process of ordering. By definition, $J(t) = \sum_k S(k, t)$, where $S(k, t)$ is the spherically averaged structure factor, and it is equal to the area under the function $S(k, t)$ at the time moment t. Thus, oscillations of the order parameter $J(t)$ testify to oscillations of the structure factor in the course of time. In order to clarify the process of pattern selection, we calculated the spherically averaged structure factor $S(k, t)$ by the formula $S(k, t) = (N_k)^{-1}\sum_{k\le\mathbf{k}\le k+\Delta k} S_{\mathbf{k}}(t)$.

We established in the numerical modeling that the components $\langle x(t)\rangle_\pm$ of the total mean increase, indeed,



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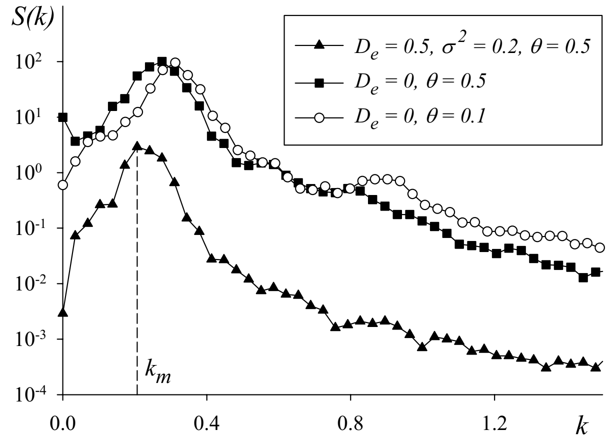

b

Fig. 7. Evolution of the mean values of the field $\langle x\rangle_+$ and $\langle x\rangle_-$ and the second statistical moment (the order parameter) $J = \langle x^2\rangle$ at $\tau_D = 0.5$, $r_c = 1.0$, $\theta = 0.5$, $D_e = 0.5$, and $\sigma^2 = 1.0$ (*a*) and the dependence of the structure factor on the wave number at $\tau_D = 1.0$ and $r_c = 1.0$ (*b*)

up to their stationary values and reveal the oscillatory behavior (see Fig. 7,*a*). In this case, the oscillations of $\langle x\rangle_+$ and $\langle x\rangle_-$ occur in the opposite phases, which yields the fulfilment of the law of conservation. The increasing order parameter $J(t)$ presented in Fig. 7,*a* testifies

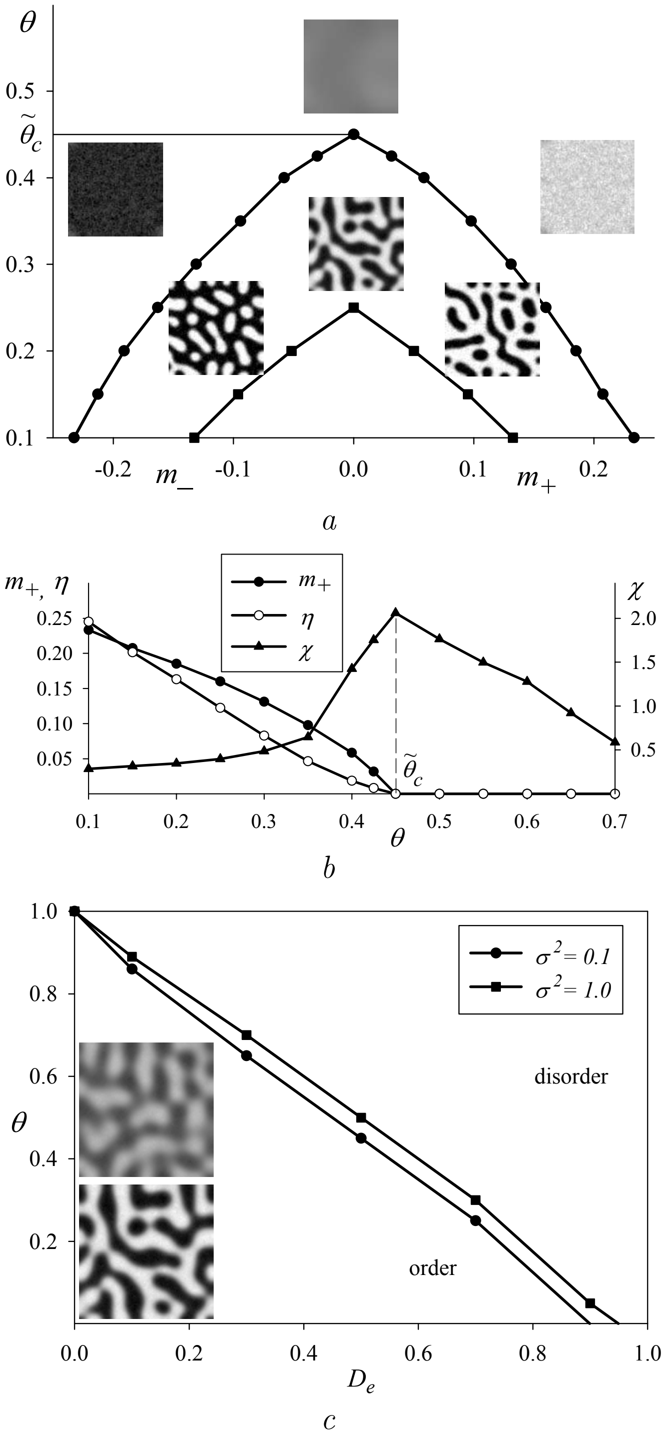

Fig. 8. Typical dependences of the means m_- and m_+ on the temperature θ at $D_e = 0.5$ and 0.7 (circles and squares) (a), the quantity m_+, the order parameter η, and the generalized susceptibility χ at $D_e = 0.5$ and $\sigma^2 = 0.1$ (b), and the ordering diagram (c) (inserts are given at $D_e = 0.5$, σ^2=0.1, and $\theta = 0.3$ and 0.4); the rest of parameters: $\tau_D = 0.5$ and $r_c = 1.0$

to the realization of the ordering in the system, and the relevant oscillations on this dependence are the reflection of the corresponding behavior of the structure factor in time. The dependence of the structure factor on the wave number is given in Fig. 7,b.

It is seen that if the additional athermal mixing of atoms ($D_e = 0$) caused by an external influence is absent, the dependence $S(k)$ possesses a clearly pronounced main peak which corresponds to the most unstable mode ($k = k_m$) and has the oscillatory decrease at $k > k_m$. This testifies that the process of pattern selection occurs in the system. In the presence of an external flow ($D_e \neq 0$) of the stochastic character ($\sigma^2 \neq 0$), the main peak of the structure factor decreases, which indicates the diffusivity of interfaces. Moreover, the additional oscillations of the structure factor occur with a smaller amplitude, i.e., the processes of pattern selection are oppressed. It is characteristic that, as the temperature decreases, the size of grains becomes smaller (the main peak of $S(k)$ shifts to the region of large k). The dependences of the structure factor obtained in the computer modeling confirm qualitatively the dependences given by the stability analysis.

In order to establish the character of a change of critical values of the basic parameters of a system under an external influence in the frame of generally known positions, we consider a behavior of the measured statistical quantities in the stationary case $t \to \infty$. In this case, the information is presented by the quantities $m_+ \equiv \lim_{t\to\infty} \overline{\langle x(t)\rangle_+}$ and $\eta \equiv \lim_{t\to\infty} \overline{\langle J\rangle}$ which are additionally averaged over a large time interval (the notation $\overline{\ldots}$) as $t \to \infty$, when the relaxation processes have already stopped. Since η is the order parameter averaged over time (by assuming that the ergodic hypothesis is valid), it is expedient to define the generalized susceptibility χ in the standard way as $\chi = N^{-2}\overline{(\langle J^2\rangle - \langle J\rangle^2)/\langle J\rangle^2}$. In this case, we have $\eta = 0$ in the disordered phase and $\eta \neq 0$ in the ordered one. In a vicinity of the transition point (e.g., at $\theta \simeq \tilde{\theta}_c$), a growth of fluctuations must induce an increase in the generalized susceptibility $\chi \propto \overline{\langle \delta J^2\rangle}$, where $\tilde{\theta}_c$ is the critical temperature of a transition induced by the action of the external noise. in Fig. 8,a, we present the temperature dependence of the means m_- and m_+ in the limit $t \to \infty$. It is seen that an increase in the determined part of the flow at a fixed value of the noise intensity σ^2 causes a decrease in the critical temperature $\tilde{\theta}_c$. The inserts in Fig. 8,a present typical patterns of ordering at various values of the temperature. In Fig. 8,b, we give the temperature dependences of the quantity m_+, the order parameter η, and the susceptibility χ at $D_e = 0.5$ and $\sigma^2 = 0.1$. It follows from the figure that, as the temperature increases to the critical value $\tilde{\theta}_c$, the quantity m_+ and the order parameter η drop to zero, and the generalized susceptibility increases significantly at $\theta \simeq \tilde{\theta}_c$.

In addition, it is seen that, as the noise intensity σ^2 increases, the value of $\tilde{\theta}_c$ at a fixed D_e approaches its mean-field value $\theta_c = 1$, which induces a change of the modality of the free energy density $f(x)$. In Fig. 8,*c*, we give a phase diagram illustrating the influence of two components of an external flow on the ordering pattern. It is seen that the increase in D_e leads to a decrease in the critical value $\tilde{\theta}_c$ which lies on the corresponding lines, and the increase in σ^2 causes the instability of a disordered state at higher temperatures (increase in $\tilde{\theta}_c$). This conclusion agrees well with the analysis of the system at early stages. We also note that fluctuations of the field x become large in a region close to the critical one. Therefore, the structures are eroded, whereas the spatial structures become clearly pronounced at a deviation from $\tilde{\theta}_c$ (see the insets in the phase diagram at $D_e = 0.5$, $\theta = 0.3$ and 0.4).

5. Conclusions

Within the hyperbolic model of spinodal decomposition, we studied the processes of pattern selection at initial stages of the decomposition of a binary system in the presence of an external flow endowed by stochastic properties. It is shown in the frame of the linear analysis for stability that the process of pattern selection in the model with a hyperbolic transport can be controlled by the regular and stochastic components of the external flow. We have established the competing influence of such components on the processes of pattern selection, where external fluctuations support selective processes. It is shown that, due to an external noise in the system, the nonlinearity of the dependence of the amplification factor on the square of the wave number is conserved with increase in the athermal mixing intensity. The analytical results obtained in the analysis for the stability of states of the system on early stages agree with the results of an independent numerical modeling. Studying the system at late stages allows us to get the dynamical phase diagrams demonstrating the competition of the regular and stochastic components of the external flow. The results of numerical calculations agree with the known analytical data on the increase in the effective temperature of a binary system due to the influence of an athermal mixing flow.

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ПІДТРИМУВАНІ ЗОВНІШНІМ ШУМОМ ПРОЦЕСИ ВІДБОРУ СТРУКТУР ПРИ СПІНОДАЛЬНОМУ РОЗПАДІ БІНАРНОЇ СИСТЕМИ

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Р е з ю м е

Розглянуто процеси відбору структур у класі нерівноважних бінарних стохастичних систем, підданих зовнішньому впливові. Аналітично та чисельно показана можливість проходження процесу відбору структур на початкових стадіях спінодального розпаду. Встановлено, що регулярна та стохастична компоненти зовнішнього потоку відіграють протилежну роль при відборі структур та загалом при розпаді системи. Результати, одержані аналітично, підтверджено чисельним моделюванням.