
FOUR-FERMION INTERACTION IN THE LOW-ENERGY APPROXIMATION OF QUANTUM CHROMODYNAMICS

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A model with the four-fermion interaction is derived using the self-consistent field method in the low-energy limit of quantum chromodynamics. The resulting Lagrangian contains not only the trivial and chiral terms but also the interaction of the vector and pseudovector currents.

1. Introduction

As is known, low-energy effects related to strong interactions cannot be described perturbatively. That is why effective models gain in importance. Models with the four-fermion interaction that naturally proceed from the calibration field theories are especially promising, because one can consider them as relativistic analogs of the Bardeen–Cooper–Schrieffer (BCS) model and apply the methods successfully used in the theory of superconductivity.

The model developed by Nambu and Jona-Lasinio [1] represents one of the effective theories of this kind. This model was proposed before the appearance of quantum chromodynamics (QCD) and it does not contain colors in its original form. It attracted a renewed interest in the 1980s–1990s. In the framework of the model, a mass spectrum of mesons was obtained in good agreement with experiment. A review of the results can be found in [2].

Work [3] gives the substantiation of the Nambu–Jona-Lasinio model in the framework of QCD. This substantiation is based upon the assumption about the structure of gluon Green functions. The exact form of the gluon propagator is till now unknown, so this reasoning is not too reliable. That is why the search for other ways of substantiation is of special importance.

It turns out that a gluon field, whose Lagrangian includes nonlinear terms of the third and fourth powers with respect to the gluon fields, can be considered, by applying the self-consistent field method (or, what is the same, Bogolyubov’s method of quasiaverages; see, e.g., [4]). Such an approach was first proposed by Kondo [5, 6]. Starting from the Lagrangian for the Yang–Mills free field, Kondo demonstrated the presence of the gluon condensate in the vacuum state and arrived at the Lagrangian quadratic in the gluon fields for the mean field approximation with a mass term caused by the condensate.

The generation of the effective mass of the gluon field results in that, in this approximation, its classical potential becomes the Yukawa one, i.e. quasilocal. That is why, if quarks are added to the model, the four-fermion interaction of the Nambu–Jona-Lasinio type will represent the zero-order approximation of the effective quark Lagrangian.

2. Self-Consistent Field Method for QCD

The BCS model in the theory of superconductivity includes a nonlinearity. For this case, Bogolyubov has developed a special version of the self-consistent field method. It is important that Bogolyubov’s method can be reformulated in terms of the continual integral (see, e.g., [4]), which allows one to apply it to field theories of various kinds.

Bogolyubov’s approach was applied to the Yang–Mills field by Kondo ([5, 6]). We will be based on these results.

According to [5], starting from the standard Yang–Mills Lagrangian and successively introducing the mean fields $\phi, \varphi^a, G_{\mu\nu}, B_{\mu\nu}^a$, and $V_{\mu\nu}^a$, one can obtain a Lagrangian that will be only quadratic with respect to the

gluon fields \mathcal{A}_μ^a :

$$\mathcal{L} = \mathcal{L}_{MF} + \frac{1}{2} \mathcal{A}_\mu^a \mathcal{K}^{\mu\nu ab} \mathcal{A}_\nu^b + \mathcal{A}_\mu \mathcal{J}^\mu. \quad (1)$$

Here, \mathcal{L}_{MF} is the part of the Lagrangian that depends only on the mean fields:

$$\begin{aligned} \mathcal{K}_{\mu\nu}^{ab} = & \delta^{ab} [-(1 - \rho^2)(\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) - \lambda^{-1} \partial_\mu \partial_\nu] - \\ & - i g \sigma f^{abc} B_{\mu\nu}^c + \sigma_\phi \delta^{ab} \eta_{\mu\nu} \phi + \sigma_\varphi d^{abc} \eta_{\mu\nu} \varphi^c + \sigma_G \delta^{ab} \times \\ & \times \left(G_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} G_\rho^\rho \right) + \sigma_V d^{abc} \left(V_{\mu\nu}^c - \frac{1}{2} \eta_{\mu\nu} V_\rho^c \right). \quad (2) \end{aligned}$$

The adding of the fermion part describing quarks to the Yang–Mills Lagrangian will result in the appearance of the quark contribution to the current \mathcal{J}^μ :

$$\mathcal{J}_{q\mu}^a = \bar{\psi}_i \gamma_\mu T_{ij}^a \psi_j. \quad (3)$$

Lagrangian (2) is quadratic with respect to the gluon fields; therefore, they can be integrated. The resulting Lagrangian will include the interaction of the quark currents in the form

$$\mathcal{L}_{qq} = -\frac{1}{2} \mathcal{J}_{q\mu}^a [\mathcal{K}^{-1}]^{\mu\nu ab} \mathcal{J}_{q\nu}^b. \quad (4)$$

In [5, 6], it was shown that ϕ can have a nonzero value of the vacuum average ϕ_0 . It is obvious that this vacuum average is connected with the effective gluon mass:

$$|\sigma_\phi \phi_0| \sim m_{\mathcal{A}}^2. \quad (5)$$

The numerical value of $m_{\mathcal{A}}$ is of the order of 1 GeV.

With regard for this fact, \mathcal{K}^{-1} can be expanded in a series in powers of $(\sigma_\phi \phi_0)^{-1}$:

$$[\mathcal{K}^{-1}]^{\mu\nu ab} = \frac{1}{\sigma_\phi \phi_0} \eta^{\mu\nu} \delta^{ab} + O\left(\frac{1}{\sigma_\phi^2 \phi_0^2}\right). \quad (6)$$

It is evident that the terms of higher orders will be small. Therefore, the expansion will be meaningful if

$$p_\mu < \sqrt{\sigma_\phi \phi_0}, \quad (7)$$

and the mean fields will satisfy similar smallness conditions.

In the zeroth order, we obtain the four-fermion interaction

$$\mathcal{L}_{qq}^{(0)} = -\frac{1}{2\sigma_\phi \phi_0} \bar{\psi}_i \gamma_\mu T_{ij}^a \psi_j \bar{\psi}_k \gamma^\mu T_{kl}^a \psi_l. \quad (8)$$

Condition (7) evidently specifies the low-energy approximation. The above considerations testify to the fact that an adequate description of the low-energy approximation of QCD must include a condition of type (7) in addition to the model with the four-fermion interaction of the Nambu–Jona-Lasinio type. This fact can mean that the momentum cut-off on the scales of the order of $m_{\mathcal{A}}$ is the way of regularization to be preferred. The problem of choosing the regularization is important, because the Nambu–Jona-Lasinio model is not renormalized.

3. Analysis of Four-fermion Interaction

Let us write the quark part of the Lagrangian keeping only the zeroth order of expansion (6):

$$\mathcal{L}_q = \bar{\psi}_i \left(i\hat{\partial} - m_0 \right) \psi_i - \frac{1}{2\sigma_\phi \phi_0} \bar{\psi}_i \gamma_\mu T_{ij}^a \psi_j \bar{\psi}_k \gamma^\mu T_{kl}^a \psi_l. \quad (9)$$

If T^a are the generators of the Lie algebra $su(N)$, then the four-fermion term can be simplified.

Let a bilinear form be specified on $gl(N)$:

$$\langle X, Y \rangle \equiv \text{Tr} [XT^a Y T^a] = X_{ij} T_{jk}^a Y_{kl} T_{li}^a. \quad (10)$$

This form will be invariant with respect to the adjoint action of the $GL(N)$ group:

$$\begin{aligned} \langle \Omega(X), \Omega(Y) \rangle &= \text{Tr} [X\omega^{-1} T^a \omega Y \omega^{-1} T^a \omega] = \\ &= \Omega^{ab} \Omega^{ac} \text{Tr} [XT^b Y T^c], \quad (11) \end{aligned}$$

$$\begin{aligned} \Lambda \delta^{ab} &= \text{Tr} [T^a T^b] = \text{Tr} [\omega^{-1} T^a \omega \omega^{-1} T^b \omega] = \\ &= \Omega^{ac} \Omega^{bd} \text{Tr} [T^c T^d] = \Lambda \Omega^{ac} \Omega^{bc} \Rightarrow \quad (12) \end{aligned}$$

$$\langle \Omega(X), \Omega(Y) \rangle = \langle X, Y \rangle. \quad (13)$$

The basis of the T^a generators can be chosen always in such a way that relation (12) is satisfied.

It is known (see for example [7]) that, for the Lie matrix groups, any Ad-invariant bilinear form can be represented as follows:

$$\langle X, Y \rangle = \lambda \text{Tr} [XY] + \mu \text{Tr} X \text{Tr} Y. \quad (14)$$

It implies that, for form (10),

$$T_{ij}^a T_{kl}^a = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{jk} \delta_{il}. \quad (15)$$

Thus, the Lagrangian can be converted to the form

$$\mathcal{L}_q = \bar{\psi}_i \left(i\hat{\partial} - m_0 \right) \psi_i - \frac{1}{2\sigma_\phi\phi_0} \left[\lambda \bar{\psi}_i \gamma_\mu \psi_i \bar{\psi}_k \gamma^\mu \psi_k + \mu \bar{\psi}_i \gamma_\mu \psi_k \bar{\psi}_k \gamma^\mu \psi_i \right]. \quad (16)$$

As was already noted, the Nambu–Jona-Lasinio model contains no colors. This fact has a physical sense, because we know that particles with free color charges are not observed at low energies. However, Lagrangian (16) includes the last term that is not diagonal, at first sight, with respect to colors as the quadratic or the first four-fermion ones. Applying the mean field method to the Lagrangian written in such a form (by analogy with [1]), one would have to introduce the mean field with free color indices violating the global color symmetry, which would be nonphysical, because all particles observed at low energies are “white”, i.e. have no free color charges. However, it turns out that the last term is actually equivalent to the sum of the terms diagonal with respect to colors similarly to the first four-fermion term.

According to the Fierz theorem [8],

$$\mathcal{L}_q = \bar{\psi}_i \left(i\hat{\partial} - m_0 \right) \psi_i - \frac{1}{2\sigma_\phi\phi_0} \times \left[\mu \bar{\psi}_i \psi_i \bar{\psi}_k \psi_k - \mu \bar{\psi}_i \gamma^5 \psi_i \bar{\psi}_k \gamma^5 \psi_k + \left(\lambda - \frac{\mu}{2} \right) \times \bar{\psi}_i \gamma_\mu \psi_i \bar{\psi}_k \gamma^\mu \psi_k - \frac{\mu}{2} \bar{\psi}_i \gamma^5 \gamma_\mu \psi_i \bar{\psi}_k \gamma^5 \gamma^\mu \psi_k \right]. \quad (17)$$

Let us find λ and μ . For this purpose, we use such a normalization of the generators that

$$T^a T^b = \frac{1}{2N} \delta^{ab} + \frac{1}{2} (i f^{abc} + d^{abc}) T^c, \quad f^{ace} f^{bce} = N \delta^{ab}. \quad (18)$$

Then we obtain

$$\langle I, I \rangle = \text{Tr} \left[\sum_a (T^a)^2 \right] = \frac{N^2 - 1}{2} = \lambda N + \mu N^2,$$

$$\langle T^a, T^b \rangle = \text{Tr} [T^a T^c T^b T^c] =$$

$$\begin{aligned} &= \text{Tr} \left[T^a T^b \sum_c (T^c)^2 \right] + i f^{ace} \text{Tr} [T^b T^c T^e] = \\ &= \frac{N^2 - 1}{2N} \text{Tr} [T^a T^b] - \frac{1}{2} f^{ace} f^{cef} \text{Tr} [T^b T^f] = \\ &= \frac{N^2 - 1}{4N} \delta^{ab} - \frac{N}{4} \delta^{af} \delta^{bf} = \frac{\lambda}{2} \delta^{ab}. \end{aligned} \quad (19)$$

Thus,

$$\lambda = -\frac{3}{2N}, \quad \mu = \frac{1}{2} + \frac{1}{N^2}. \quad (20)$$

The four-fermion Lagrangian takes the form

$$\mathcal{L}_q = \bar{\psi}_i \left(i\hat{\partial} - m_0 \right) \psi_i - \frac{1}{4\sigma_\phi\phi_0} \times \left[\bar{\psi}_i \psi_i \bar{\psi}_k \psi_k - \bar{\psi}_i \gamma^5 \psi_i \bar{\psi}_k \gamma^5 \psi_k - \frac{N+2}{2N} \times \bar{\psi}_i \gamma_\mu \psi_i \bar{\psi}_k \gamma^\mu \psi_k - \frac{1}{2} \bar{\psi}_i \gamma^5 \gamma_\mu \psi_i \bar{\psi}_k \gamma^5 \gamma^\mu \psi_k \right]. \quad (21)$$

4. Conclusions

The application of the mean field method to the gluon dynamics allows one to obtain a model with the four-fermion interaction as a controlled approximation in the low-energy limit of quantum chromodynamics. The structure of the effective Lagrangian is generally similar to that obtained in [3], so our consideration can be treated as a substantiation of the assumptions about the structure of gluon Green functions.

It is important that the structure of the Lagrangian enables one to introduce colorless quark condensates. Their consideration will be the subject of the following work.

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ЧОТИРИФЕРМІОННА
ВЗАЄМОДІЯ У НИЗЬКОЕНЕРГЕТИЧНОМУ
НАБЛИЖЕННІ КВАНТОВОЇ ХРОМОДИНАМІКИ

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Здійснено послідовне виведення моделі з чотириферміонною взаємодією у низькоенергетичному наближенні КХД методом самоузгодженого поля. Результуючий лагранжіан, крім тривіального та кірального доданків, містить взаємодію векторних та псевдовекторних струмів.