INSTABILITY OF THE FUNDAMENTAL HARMONIC OF STOKES WAVES TO TWO-DIMENSIONAL PERTURBATIONS WITH ALLOWANCE FOR THE ZEROTH HARMONIC

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The results on an additional, not long-wave region of instability of weakly non-linear waves on the surface of a liquid layer with finite depth h obtained earlier in one-dimensional case (JETP Letters **86**, 502 (2007)) have been extended on a two-dimensional geometry. The consideration is carried out in the framework of Zakharov's Hamiltonian approach for a system of coupled Fourier amplitudes of the first harmonic with the wave vector k_0 and the non-oscillating wave component (the zeroth harmonic). When analyzing the linear stability of a weakly non-linear Stokes solution of this system, the dispersion equation of the fourth order, obtained earlier by Zakharov and studied analytically in the range of wave perturbations with small wave vectors, gives, besides the known instability region, a new one for not too large angles θ between the main and perturbation waves, in contrast to the quadratic equation usually obtained in the long-wave region, which gives only the ordinary Benjamin–Feir instability. The additional instability region grows with a reduction of the liquid depth, whereas the region of ordinary instability becomes narrower and disappears at $k_0h = 1.363$ for $\theta = 0$ or does not exist at any θ, if $k_0h = 0.38$.

1. Introduction

Researches dealing with the stability of the exact solution of the equations of motion with respect to small modulations be carried out by obtaining the straightforward solution of the corresponding Cauchy problem, with the initial condition being given as a sum of this solution and a perturbation. But such an approach cannot be applied in the case of nonlinear waves on the surface of a liquid of finite depth, because 1) there is a zeroth harmonic B in the system, 2) the perturbation is two-dimensional, and 3) the spectral width of a perturbation is large. Due to the last circumstance, it is impossible to use a spectrally narrow pulse of the nonlinear Schrödinger equation (NSE), for which the method for the solution of the one-dimensional Cauchy problem has been elaborated, for the description of amplitude A of the first harmonic. In this regard, the stability is studied by an indirect method, namely, by the linear expansion near the solution, the stability of which is analyzed. It is worth recalling the terminology and the known results and indicating the accents of this work.

In a nonlinear conservative medium, which admits wave motion, the account of only linear terms in the equations of motion gives rise to a linear dispersion law $\omega(k)$ and a solution in the form of a sine wave (the first harmonic) with constant amplitude A, the frequency $\omega_0 = \omega(k_0)$, and the wave vector $k_0 = 2\pi/\lambda$, where λ is the wavelength. Making allowance for the nonlinear terms of the second and higher orders brings about the appearance of the second and higher harmonics in the wave profile $\eta(x,t)$:

$$\eta(x,t)=\left(\frac{1}{2}A(x,t)\exp i\theta+A_2(x,t)\exp 2i\theta+\ldots+\text{c.c.}\right)+$$
$$+B(x,t), \qquad \theta = k_0x-\omega_0t. \tag{1}$$

We also obtain the corresponding system of evolution equations for the amplitudes of those harmonics. In the narrow-spectrum approximation and applying a simplified dependence on time, it is possible to eliminate the zeroth and higher harmonics. In this fashion, we obtain an NSE for the description of the time evolution of the first harmonic amplitude A,

$$i\left(A_t+\omega' A_x\right)+\frac{1}{2}\omega'' A_{xx}+qA|A|^2=0, \tag{2}$$

or its higher generalizations [1]. The reduction of the system of equations to a single closed equation – the NSE – becomes more substantiated in problems, where the zero harmonic is absent in the system or appears in higher approximations. For the waves on the liquid surface, the formulas for the amplitudes of the zeroth, B, and the second, A_2, harmonics are given in Appendix in terms of the first harmonic amplitude, A. The formulas demonstrate that the neglect of the zeroth harmonic corresponds to the limiting case of infinite liquid depth to the accuracy order concerned.

Being written down in terms of the variables

$$\tau = \frac{1}{2}\omega'' t, \quad \xi = x - \omega' t, \quad \text{and} \quad u = \sqrt{\left|\frac{q}{\omega''}\right|}A,$$

the NSE takes a canonical form,

$$iu_\tau + u_{\xi\xi} \pm 2u|u|^2 = 0. \tag{3}$$

The sign $\pm$ is determined by that of $q\omega''$. If $q\omega'' > 0$, Eq. (3), along with other solutions – for example, the soliton one $u = e^{i\tau}/\cosh(\xi)$ – has a trivial plane-wave solution:

$$u = e^{2i\tau}. \tag{4}$$

Taking into account the fact that, provided that $u(\xi,\tau)$ is a solution of Eq. (3), the function $au(a\xi, a^2\tau)$ also satisfies this equation, choosing

$$a = \sqrt{\left|\frac{q}{\omega''}\right|}A_0$$

and returning back to the variables t, x, and A, we express solution (4) as

$$A(x,t) = A_0 \exp iqA_0^2 t. \tag{5}$$

Solution (5) implies that the linear dispersion law is appended with the dependence of the frequency on the squared amplitude,

$$\omega_0 \to \omega_0 - qA_0^2. \tag{6}$$

This is a non-linear correction found by Stokes [2] for a liquid of infinite depth,

$$\omega_0 \to \omega_0(1 + \frac{1}{2}k_0^2 A_0^2). \tag{7}$$

Really, in this case $q = -\frac{1}{2}\omega_0 k_0^2$ for the NSE [3]. For the finite-depth case [4], the expression for q is presented in Appendix.

Weakly nonlinear solution (1), where the amplitude $A(x,t)$ is given by expression (5), is conventionally referred to as a Stokes wave in the case of waves on the ideal liquid surface. This wave, similarly to the sine one, has a constant amplitude, but a wider trough and a narrower crest. Its time evolution corresponds to the plane-wave solution of NSE, and the evolution of a weak perturbation of Stokes wave corresponds to the solution of NSE, provided that the perturbed plane-wave solution is its initial condition.

The procedure of consecutive account of nonlinear terms, which was put forward by Stokes, looks regular, and it was proved to be convergent. However, it turned out that the Stokes solution can be unstable with respect to a harmonic perturbation with a small, in comparison with k_0, wave vector $\varkappa$. It was found that such a modulation instability (MI) manifests itself for waves of various origins. First, it was studied for waves on the ideal liquid surface, namely, by Zakharov, making use of the Hamiltonian method in the Fourier plane (as well for ion-acoustic oscillations in plasma) [3, 5]; by Benjamin, by analyzing harmonic series [6]; by Whitham, in the framework of the Lagrange method [7]; by Zakharov [3] and, later, by Hasimoto and Ono [4], with the help of NSE. The instability was studied at the linear stage of its development, by linearizing the relevant equations and analyzing the imaginary part of roots of the dispersion equation for the perturbation frequency. The condition $\varkappa \ll k_0$ was selected to simplify calculations.

How does the excited Stokes solution evolve? This question can be answered, in principle, in the framework of the first and fourth indicated approaches, because they contain the corresponding evolution equations. In the approach that uses the NSE, the evolution was studied both by the method of inverse scattering transform (IST) for a periodic boundary condition, which is valid for a harmonically excited Stokes wave [8], and by direct methods as the evolution of the plane-wave limit of exact NSE solutions at $t \to \pm\infty$. It was shown [9, 10] that, in the course of a long-term evolution, the Stokes solution can transform its profile into a spatially or temporally repetitive cnoidal wave, breather, Kuznetsov–Ma soliton, or other soliton- or quasisoliton solutions of the NSE, depending on the wave amplitude A_0 and the ratio between the wave, k_0, and perturbation, $\varkappa$, wave vector moduli.

The source of the fundamental harmonic instability in a nonlinear medium is the presence of higher-order harmonics in the system. Namely, if two perturbations with wave vectors $\boldsymbol{k_0} \pm \boldsymbol{\varkappa}$ act on the first harmonic A_0 with the wave vector $\boldsymbol{k_0}$, a resonance with the second harmonic with the wave vector $2\boldsymbol{k_0}$ is possible,

$$(\boldsymbol{k_0} + \boldsymbol{\varkappa}) + (\boldsymbol{k_0} - \boldsymbol{\varkappa}) = 2\boldsymbol{k_0}, \tag{8}$$

and, therefore, the perturbation excitation (the instability of the first harmonic) arises. The specific values of the parameters A_0, k_0, and $\varkappa$, at which the perturbation increases, are determined by the equations of motion for harmonics. Such a mechanism was considered in the majority of previous works, where the inequality $|\boldsymbol{\varkappa}| \ll |\boldsymbol{k_0}|$ was supposed—mainly, in order that analytical transformations be possible. In works [3, 5], the higher analogs of expression (8) were also studied – e.g.,

$$(2\boldsymbol{k_0} + \boldsymbol{\varkappa}) + (\boldsymbol{k_0} - \boldsymbol{\varkappa}) = 3\boldsymbol{k_0},$$

which describes the instability of the second and first harmonics, if, being perturbed, they come into resonance with the third harmonic – and the corresponding instability increment was found. Now, it is coined as class II instability [11]. For the time evolution of such an instability to be described, higher-order terms have to be included into the NSE [1, 12].

In this work, quantitative changes in the MI description are considered in the case where there is the zeroth harmonic in the expansion. For waves on the surface of a finite-depth liquid, it corresponds to a flow induced by waves. In nonlinear optics, the zeroth harmonic is associated with the so-called optical detecting. At a qualitative level, an additional possibility for the instability is supposed as a result of the development of the process (8) of resonance of two first perturbed harmonics together with the second and zeroth ones, provided the validity of the relation

$$(\boldsymbol{k_0} + \boldsymbol{\varkappa}) + (\boldsymbol{k_0} - \boldsymbol{\varkappa}) = 2\boldsymbol{k_0} + \boldsymbol{0}, \tag{9}$$

between their momenta and keeping in mind that the excitation of the zeroth harmonic can change the energy resonance relations (8) and intensify the perturbation. One may suggest that the interaction will be more effective at $\boldsymbol{\varkappa} \approx \boldsymbol{k_0}$, when the first pair in expression (9) resonates with the second harmonic, and, simultaneously, the second pair with the zeroth one.

The quantitative consideration is carried out, when the zeroth harmonic contribution to the evolution equation for the complex amplitude of the first harmonic and the additional evolution equation of the second order for the actual amplitude of the zeroth harmonic with a term given by the first harmonic are taken into account. In this case, if the Stokes solution is modulated by a harmonic wave with the wave vector $\boldsymbol{\varkappa}$, the dispersion equation for the perturbation frequency becomes of the fourth order – instead of the second one – and its new roots, being considered in a wide range of $\boldsymbol{\varkappa}$, contain an imaginary component that corresponds to new instabilities in such a system. Since $\boldsymbol{\varkappa}$ is not supposed to vary in a narrow range with respect to $\boldsymbol{k}_0$, the analysis is carried out, provided a complete account of linear dispersion.

The account of the zeroth harmonic in the form of an additional evolution equation was carried out in a number of works and by various methods: the Hamiltonian [13], multiscale [14–16], and variational [7, 17, 18] ones, with the total linear dispersion being considered only in work [13]. However, in those works, in order to obtain analytical estimations in the range $|\boldsymbol{\varkappa}| \ll |\boldsymbol{k}_0|$, (i) the corresponding dispersion equation of the fourth order for the perturbation frequency was simplified to the quadratic one [13], (ii) two of four roots in the interval $|\boldsymbol{\varkappa}| \ll |\boldsymbol{k}_0|$ were *a priori* assumed to be real-valued [7, 14] (or of "no interest" [17]), (iii) a simplified time dependence for the zeroth harmonic was adopted in the range $|\boldsymbol{\varkappa}| \ll |\boldsymbol{k}_0|$, so that the dispersion equation became quadratic [4, 15], (iv) the case of infinite depth was analyzed only [16]. Anyway, no other instabilities but a long-wave one were considered. The equation of the fourth order, provided that only the parabolic dispersion is taken into account, was obtained in the framework of the variational method in work [18], where the author indicated that two roots associated with the zeroth harmonic are complex. In work [19], an equation of the fourth order for the perturbation frequency was obtained in the case of one-dimensional perturbations, making use of the multiscale method in the parabolic approximation for the linear dispersion, and, besides an ordinary MI at $\varkappa \ll k_0$, an instability band at $\varkappa \approx k_0$ was found, when the total linear dispersion in the equations for the first and zeroth harmonics was made allowance for. Since the account of the total linear dispersion seems not conventional, when the consideration is carried out in the coordinate space, in work [21], the author used the Hamiltonian approach developed by Zakharov. In this approach, such an account is made in a natural way in terms of the function $\omega(\boldsymbol{\varkappa})$ at the transformation into the Fourier plane.

The efficiency of the Hamiltonian approach consists in that the similarity between the waves of different origins is described in it by a formally identical expansion of the Hamiltonian into an integro-power series in the nonlinearities of canonical variables, in terms of which the wave field of a specific nonlinear medium is succeeded to be presented. As soon as the coefficients of this expansion are calculated for the given type of nonlinear waves in one problem, they become the passport characteristic of those waves and can be applied to other problems. This idea was formulated and implemented by Zakharov in works [3, 5], with nonlinear waves in plasma and on the deep liquid surface being taken as examples. The case of finite-depth liquid was considered in work [13], where the equations of motion for harmonics were derived as Hamilton equations, by varying the Hamiltonian, from which higher harmonics were eliminated beforehand. Concerning the MI problem of Stokes waves in a liquid layer, in work [13], there was obtained an equation of the fourth order for the frequency of a harmonic perturbation by linearizing the system of two equations for the zeroth and first harmonics near the Stokes solution. This equation, however, was reduced in work [13] to the quadratic one, in order to make analytical estimations possible. It allowed the instability increment to be calculated analytically for two roots in the interval $|\boldsymbol{\varkappa}| \ll |\boldsymbol{k}_0|$. In work [21], it was demonstrated numerically that, in the case where the wave vectors of the perturbation and the first harmonic are directed identically, the dispersion equation, which was derived in work [13] and reproduced in work [21], has two more, complex roots at $\varkappa \sim k_0$, in accordance with the results of work [20].

In this work, in the two-dimensional geometry and in the case where the perturbation is directed at a certain angle to the wave vector of the first harmonic, new roots have been separated analytically in the range $|\boldsymbol{\varkappa}| = |\boldsymbol{k}_0|$, and the instability increment has been calculated numerically in a wider, in comparison with that of the work [13], interval of perturbation wave vectors (the internal surfaces in Fig. 5). The first numerical tabulations for the case of two-dimensional perturbations were carried out in work [22].

2. Hamiltonian, Its Formal Expansion into Integro-Power Series, and Equations of Motion in the Fourier Representation

In the Hamiltonian approach to the description of potential surface nonlinear waves [3, 5], the equations of motion for the "normal complex coordinate" $a(\boldsymbol{k}, t)$ are written down in the form of the Hamilton equation

$$\frac{\partial a(\boldsymbol{k}, t)}{\partial t} = -i \frac{\delta H}{\delta \overline{a}(\boldsymbol{k}, t)}$$

and its conjugate variant. Here, $\boldsymbol{k}$ is the horizontal wave vector $\boldsymbol{k} = (\boldsymbol{k}_x, \boldsymbol{k}_y)$, and H is the wave Hamiltonian, which is a functional of $a(\boldsymbol{k})$ and $\overline{a}(\boldsymbol{k})$. For waves on the surface of the ideal liquid, the deviation $\eta(\boldsymbol{x}, t)$ from the equilibrium surface is expressed in terms of $a(\boldsymbol{k}, t)$-quantities as the two-dimensional normalized Fourier transform

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi} \int \left(\frac{\omega(\boldsymbol{k})}{2g}\right)^{1/2} (a(\boldsymbol{k}, t) + \overline{a}(-\boldsymbol{k}, t))\, e^{i\boldsymbol{k}\boldsymbol{x}} d\boldsymbol{k},$$

$$\omega(\boldsymbol{k}) = \sqrt{g|\boldsymbol{k}| \tanh(|\boldsymbol{k}|h)}, \tag{10}$$

where g is the free fall acceleration, h is the liquid depth, and $\boldsymbol{x} = (x, y)$ is the two-dimensional vector of horizontal coordinates.

The Hamiltonian expanded into an integro-power series in the variables $a(\boldsymbol{k})$ and $\overline{a}(\boldsymbol{k})$ looks like [13]

$$H = \int\limits_{-\infty}^{\infty} \omega(\boldsymbol{k})\, a(\boldsymbol{k})\, \overline{a}(\boldsymbol{k})\, d\boldsymbol{k} +$$

$$+ \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} (\overline{a}(\boldsymbol{k})\, a(\boldsymbol{k}_1)\, a(\boldsymbol{k}_2) + a(\boldsymbol{k})\, \overline{a}(\boldsymbol{k}_1)\, \overline{a}(\boldsymbol{k}_2)) \times$$

$$\times V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, d\boldsymbol{k}\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2 +$$

$$+ \frac{1}{3} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} (a(\boldsymbol{k})\, a(\boldsymbol{k}_1)\, a(\boldsymbol{k}_2) + \overline{a}(\boldsymbol{k})\, \overline{a}(\boldsymbol{k}_1)\, \overline{a}(\boldsymbol{k}_2)) \times$$

$$\times U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)\ \delta(\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2)\, d\boldsymbol{k}\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2 +$$

$$+ \frac{1}{2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \overline{a}(\boldsymbol{k})\, \overline{a}(\boldsymbol{k}_1)\, a(\boldsymbol{k}_2)\, a(\boldsymbol{k}_3) \times$$

$$\times W(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\, \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\, d\boldsymbol{k}\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2\, d\boldsymbol{k}_3. \tag{11}$$

Here, the terms of the fourth order, which are of the forms $\overline{a}(\boldsymbol{k})a(\boldsymbol{k}_1)a(\boldsymbol{k}_2)a(\boldsymbol{k}_3)$, $a(\boldsymbol{k})\overline{a}(\boldsymbol{k}_1)\overline{a}(\boldsymbol{k}_2)\overline{a}(\boldsymbol{k}_3)$, $\overline{a}(\boldsymbol{k})\overline{a}(\boldsymbol{k}_1)\overline{a}(\boldsymbol{k}_2)\overline{a}(\boldsymbol{k}_3)$, $a(\boldsymbol{k})a(\boldsymbol{k}_1)a(\boldsymbol{k}_2)a(\boldsymbol{k}_3)$, as well as the terms with higher a-powers, are not included, because they do not manifest themselves in the considered order of accuracy.

The common physical character of the approach consists in the universality of expansion (11) and the formulas that follow from it (the equations of motion, instability increments, statistical characteristics, and so on) for different media. The importance of expansion (11) for the description of different wave phenomena in a specific physical medium is defined by the formulas for the expansion coefficients $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, and $W(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$. For waves on the surface of a liquid of infinite or finite depth, they were calculated in works [3,5] and [13], respectively, when the approach itself was formulated. Here, only those coefficients are given, the explicit form of which is necessary for what follows:

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = -V_0(-\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) -$$

$$-V_0(-\boldsymbol{k},\,\boldsymbol{k}_2,\,\boldsymbol{k}_1)+V_0(\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k}\,),$$

$$U(\boldsymbol{k},\,\boldsymbol{k}_1,\,\boldsymbol{k}_2)=V_0(\boldsymbol{k},\,\boldsymbol{k}_1,\,\boldsymbol{k}_2)+$$

$$+V_0(\boldsymbol{k},\,\boldsymbol{k}_2,\boldsymbol{k}_1\,)+V_0(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}),$$

$$V_0(\boldsymbol{k},\,\boldsymbol{k}_1,\,\boldsymbol{k}_2)=-\mathcal{N}(\boldsymbol{k})\mathcal{N}(\boldsymbol{k}_1)\mathcal{M}(\boldsymbol{k}_2)E^{(3)}(\boldsymbol{k},\boldsymbol{k}_1),$$

$$E^{(3)}=-\frac{1}{4\pi}\left((\boldsymbol{k}\cdot\boldsymbol{k}_1)+q(\boldsymbol{k})q(\boldsymbol{k}_1)\right), \tag{12}$$

$$W(\boldsymbol{k},\,\boldsymbol{k}_1,\,\boldsymbol{k}_2,\,\boldsymbol{k}_3)=$$

$$=W_0(-\boldsymbol{k},-\boldsymbol{k}_1,\,\boldsymbol{k}_2,\,\boldsymbol{k}_3)+W_0(\boldsymbol{k}_2,\;\boldsymbol{k}_3,-\boldsymbol{k},\,-\boldsymbol{k}_1)-$$

$$-W_0(-\boldsymbol{k},\boldsymbol{k}_2,-\boldsymbol{k}_1,\;\boldsymbol{k}_3)-W_0(-\boldsymbol{k}_1,\boldsymbol{k}_2,-\boldsymbol{k},\;\boldsymbol{k}_3)-$$

$$-W_0(-\boldsymbol{k},\,\boldsymbol{k}_3,\,-\boldsymbol{k}_1,\,\boldsymbol{k}_2)-W_0(-\boldsymbol{k}_1,\,\boldsymbol{k}_3,-\boldsymbol{k},\;\boldsymbol{k}_2),$$

$$W_0(\boldsymbol{k},\,\boldsymbol{k}_1,\,\boldsymbol{k}_2,\boldsymbol{k}_3)=$$

$$=-2\mathcal{N}(\boldsymbol{k})\mathcal{N}(\boldsymbol{k}_1)\mathcal{M}(\boldsymbol{k}_2)\mathcal{M}(\boldsymbol{k}_3)E^{(4)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3),$$

$$E^{(4)}=-\frac{1}{32\pi^2}\Big\{2|\boldsymbol{k}|^2q(\boldsymbol{k}_1)+2|\boldsymbol{k}_1|^2q(\boldsymbol{k})-q(\boldsymbol{k})q(\boldsymbol{k}_1)\times$$

$$\times(q(\boldsymbol{k}+\boldsymbol{k}_2)+q(\boldsymbol{k}_1+\boldsymbol{k}_2)+q(\boldsymbol{k}+\boldsymbol{k}_3)+q(\boldsymbol{k}_1+\boldsymbol{k}_3))\Big\}$$

$$\mathcal{N}(\boldsymbol{k})=\left(\frac{\omega(\boldsymbol{k})}{2q(\boldsymbol{k})}\right)^{1/2},\quad \mathcal{M}(\boldsymbol{k})=\left(\frac{q(\boldsymbol{k})}{2\omega(\boldsymbol{k})}\right)^{1/2},$$

$$q(\boldsymbol{k})=|\boldsymbol{k}|\tanh(|\boldsymbol{k}|h).$$

The expressions for all coefficients with regard for the results of discussion [23] on their symmetry properties, which preserve the Hamiltonian character of the theory at the reduction of Hamiltonian (11), are presented in the most compact form in works [24, 25], (in particular, formula (12) was taken from work [24]).

By varying Hamiltonian (11), we obtain the equation of motion for $a(\boldsymbol{k},t)$:

$$\frac{\partial}{\partial t}\,a(\boldsymbol{k},\,t)+i\Bigg[\omega(\boldsymbol{k})\,a(\boldsymbol{k})+$$

$$+\int\limits_{-\infty}^{\infty}V(\boldsymbol{k},\,\boldsymbol{k}-\boldsymbol{\xi},\,\boldsymbol{\xi})a(\boldsymbol{\xi})\,a(\boldsymbol{k}-\boldsymbol{\xi})\,d\boldsymbol{\xi}+$$

$$+2\int\limits_{-\infty}^{\infty}V(\boldsymbol{k}+\boldsymbol{\xi},\,\boldsymbol{k},\boldsymbol{\xi})\overline{a}(\boldsymbol{\xi})\;a(\boldsymbol{k}+\boldsymbol{\xi})\,d\boldsymbol{\xi}+$$

$$+\int\limits_{-\infty}^{\infty}U(-\boldsymbol{k}-\boldsymbol{\xi},\,\boldsymbol{k},\,\boldsymbol{\xi})\,\overline{a}(\boldsymbol{\xi})\,\overline{a}(-\boldsymbol{k}-\boldsymbol{\xi})\,d\boldsymbol{\xi}+$$

$$+\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}W(\boldsymbol{\xi}+\boldsymbol{\zeta}-\boldsymbol{k},\,\boldsymbol{k},\,\boldsymbol{\xi},\,\boldsymbol{\zeta})\times$$

$$\times a(\boldsymbol{\xi})\,a(\boldsymbol{\zeta})\,\overline{a}(\boldsymbol{\xi}+\boldsymbol{\zeta}-\boldsymbol{k})\,d\boldsymbol{\xi}d\boldsymbol{\zeta}\Bigg]=0. \tag{13}$$

3. Equations of Motion for Fourier Amplitudes of the First and Zeroth Harmonics

Consider, as in work [13], a wave packet with the central wave vector $\boldsymbol{k}_0$. In this case, the Fourier amplitude of the first harmonic is concentrated around the wave vector $\boldsymbol{k}_0$:

$$a_1=a_1(\boldsymbol{k},\,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_0). \tag{14}$$

Nonlinear terms in the equations of motion give rise to the appearance – in the next order of the formal smallness parameter ε – of the zeroth and second harmonics:

$$a=\varepsilon a_1+\varepsilon^2(b+a_2). \tag{15}$$

Substituting expression (15) into Eq. (13) and combining the coefficients of the terms with $\boldsymbol{k}=\pm2\boldsymbol{k}_0$, we obtain the equations of motion for components of the second harmonic, a_2. Those equations are used to express the second harmonic in terms of the first one. The latter expression together with formula (15) is substituted into Hamiltonian (11), which becomes free of a_2 now. In work [13], by varying the Hamiltonian obtained with respect to a_1 and b, two coupled equations of motion for the Fourier amplitudes of the first and zeroth harmonics were obtained:

$$\frac{\partial}{\partial t}\,a_1(\boldsymbol{k})+i\omega(\boldsymbol{k})\,a_1(\boldsymbol{k})+$$

$$+i\int_{-\infty}^{\infty} a_1(\xi)\,\left[\,f(\boldsymbol{k}_0-\boldsymbol{\xi})\,b(\boldsymbol{k}-\boldsymbol{\xi})+f(\boldsymbol{\xi}-\boldsymbol{k}_0)\,\bar{b}(\boldsymbol{\xi}-\boldsymbol{k})\,\right]\times$$

$$\times d\boldsymbol{\xi}+i\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1(\zeta)\,a_1(\boldsymbol{\xi})\,\overline{a_1}(\boldsymbol{\zeta}+\boldsymbol{\xi}-\boldsymbol{k})\,d\boldsymbol{\xi}\,d\boldsymbol{\zeta}=0, \quad (16)$$

$$\frac{\partial}{\partial t}\,b(\boldsymbol{k})+i\,\omega(\boldsymbol{k})\,b(\boldsymbol{k})+i\,f\,(\boldsymbol{k})\int_{-\infty}^{\infty}\overline{a}_1(\boldsymbol{\xi})\,a_1(\boldsymbol{k}+\boldsymbol{\xi})\,d\boldsymbol{\xi}=0, \quad (17)$$

where the notations

$$f(\boldsymbol{k})=2V(\boldsymbol{k},\,\boldsymbol{k}_0,\,\boldsymbol{k}_0), \quad (18)$$

$$\lambda=W(\boldsymbol{k}_0,\,\boldsymbol{k}_0,\,\boldsymbol{k}_0,\,\boldsymbol{k}_0)-$$

$$-2\left(\frac{V^2(2\,\boldsymbol{k}_0,\,\boldsymbol{k}_0,\,\boldsymbol{k}_0)}{\omega(2\,\boldsymbol{k}_0)-2\,\omega(\boldsymbol{k}_0)}+\frac{U^2(-2\,\boldsymbol{k}_0,\,\boldsymbol{k}_0,\,\boldsymbol{k}_0)}{\omega(2\,\boldsymbol{k}_0)+2\,\omega(\boldsymbol{k}_0)}\right) \quad (19)$$

are used. In work [21], those results were reproduced in a somewhat different way. Namely, after the substitution of expression (15) into formula (13), the combined coefficients of $\boldsymbol{k}=\boldsymbol{k}_0$, $\boldsymbol{k}=0$, and $\boldsymbol{k}=\pm 2\boldsymbol{k}_0$ present the equations of motion for the first, a_1, zeroth, b, and second, a_2, harmonics, respectively (a_2 should be excluded from the equation for $a_1(\boldsymbol{k})$). It is worth noting that, in work [21], the first and third arguments in the analog of formula (18) for $f(\boldsymbol{k})$ are arranged in the inverse order in comparison with the original work [13]:

$$f(\boldsymbol{k})=2V(\boldsymbol{k}_0,\,\boldsymbol{k}_0,\,\boldsymbol{k}). \quad (20)$$

4. Modulation Instability

The discovery of the IST method for the NSE, including the periodic initial condition, as well as the development of direct methods for the NSE solution, allow the MI to be considered not only at its linear – in a small perturbation – stage, but also in the course of its subsequent development [9,10]. The Stokes wave evolution was studied as a development in time of the plane-wave solution (4) of NSE (3) with a small harmonic additive given at the initial moment in the form

$$u(x,0)=1+\varepsilon\cos(px). \quad (21)$$

The stability and the character of a long-term evolution of distribution (21) are governed by the modulation coefficient p. Such researches are of interest, because they explicitly demonstrate various evolution scenarios which depend on the initial condition parameters and could bring us closer to the understanding of the physical mechanism underlying the generation and the development of the so-called "freak or killer waves" really existing in the open ocean. However, the satellite-assisted observations testify that such waves do not contain too many oscillations [26], for the description of their envelope making use of a single NSE to be eligible. At the same time, they are not isolated waves described by the Korteweg–de Vries (KdV) equation. Their somewhat intermediate character and the fact that they exist at large depths characteristic of the NSE, as well as at finite and even small depths which are characteristic of the isolated KdV waves [this fact is confirmed by the witnesses of huge waves that destroyed coastal constructions in Katsiveli (Ukraine) some years ago] suggest that the Zakharov equation, which describes a general wave field, should be used as the evolution one, rather than the NSE or the KdV equation. In this case, it is possible to retain the coupled first (taking the influence of the second harmonic into account) and zeroth harmonics in the expansions, the evolution of which can be described separately by the NSE and the KdV equation, respectively, only approximately.

The system of equations of motion (16), (17) has Stokes form (5) after the inverse Fourier transformation of the first harmonic $a(\boldsymbol{k})$ is made. Its solution looks like

$$a(\boldsymbol{k})=\mathcal{A}_0\,e^{-i\,(\omega(\boldsymbol{k}_0)+\lambda\mathcal{A}_0^2)\,t}\delta(\boldsymbol{k}-\boldsymbol{k}_0),\qquad b(\boldsymbol{k})=0, \quad (22)$$

where, taking formulas (10), (5), and (1) into account, the physical amplitude A_0 is connected with the quantity $\mathcal{A}_0$ by the relation

$$\mathcal{A}_0^2=\frac{2\pi^2}{\sigma}\frac{\omega_0}{k_0}A_0^2,\quad \sigma=\tanh k_0h, \quad (23)$$

and

$$qA_0^2=-\lambda\mathcal{A}_0^2. \quad (24)$$

The calculation of λ by formula (19) gives [27–29]

$$\lambda=\frac{k_0^3}{32\pi^2}\,\frac{9\sigma^4-10\sigma^2+9}{\sigma^3} \quad (25)$$

for waves on the surface of a finite-depth liquid. Hence, we have the following nonlinear correction to the frequency before the zeroth harmonic excitation was taken into account:

$$qA_0^2 = -\frac{9\sigma^4 - 10\sigma^2 + 9}{16\sigma^4}\omega_0 k_0^2 A_0^2. \tag{26}$$

The specification of λ after the account of the zeroth harmonic excitation is made is given below by expressions (29) and (35). In the limiting case of infinite depth (the absence of the zeroth harmonic), corresponding expression (24) for qA_0^2, as well as expression (26), transforms into the Stokes result (7).

Following work [13], let us introduce the perturbation

$$a(\boldsymbol{k}) = e^{-i\,t\,(\omega(\boldsymbol{k}_0)+\lambda\mathcal{A}_0^2)}\Big(\mathcal{A}_0\,\delta(\boldsymbol{k}-\boldsymbol{k}_0) + \varepsilon\,\alpha(\boldsymbol{k})\times$$

$$\times e^{-i\Omega\,t}\delta(\boldsymbol{k}-\boldsymbol{k}_0-\boldsymbol{\varkappa}) + \varepsilon\,\alpha(\boldsymbol{k})\,e^{i\,\Omega\,t}\,\delta(\boldsymbol{k}-\boldsymbol{k}_0+\boldsymbol{\varkappa})\Big), \tag{27}$$

$$b(\boldsymbol{k}) = \varepsilon\,\beta(\boldsymbol{k})\,e^{-i\,\Omega\,t}\,\delta(\boldsymbol{k}-\boldsymbol{\varkappa}) + \varepsilon\,\beta(\boldsymbol{k})\,e^{i\,\Omega\,t}\,\delta(\boldsymbol{k}+\boldsymbol{\varkappa}),$$

where $\alpha(\boldsymbol{k})$ and $\beta(\boldsymbol{k})$ are real-valued quantities. Here, the vector $\boldsymbol{\varkappa}$, which forms an angle with $\boldsymbol{k}_0$, is the quantity, by which the wave vectors of the excited first and zeroth harmonics differ from their unexcited counterparts $\boldsymbol{k}_0$ and $\mathbf{0}$. Let the wave vector $\boldsymbol{k}_0$ of the carrier wave be directed along the x-axis in the horizontal coordinate system (x, y): $\boldsymbol{k}_0 = (k_0, 0)$, and let the vector $\boldsymbol{\varkappa}$ have both horizontal coordinates: $\boldsymbol{\varkappa} = (\kappa, \chi)$. Then, the wave vectors of the perturbed first and zeroth harmonics have the coordinates $\boldsymbol{k}_0 \pm \boldsymbol{\varkappa} = (k_0 \pm \kappa, \pm\chi)$ and $\mathbf{0} \pm \boldsymbol{\varkappa} = (\pm\kappa, \pm\chi)$. Note that not only the first but also the zeroth harmonics are excited. Since, in what follows, $\boldsymbol{\varkappa}$ is considered to be not necessarily small in comparison with $\boldsymbol{k}_0$, we should make a comment. The addition of considerable deviations of $\boldsymbol{\varkappa}$ from $\boldsymbol{k}_0$ and $\mathbf{0}$ in the arguments of the first, $a(\boldsymbol{k})$, and zeroth, $b(\boldsymbol{k})$, harmonics is carried out only under their small perturbations, the development of which is considered at the initial linear stage only, with a single purpose to determine the possibility of their growth. Therefore, in our opinion, it does not contradict the assumption on the concentration of arguments of the first, $a(\boldsymbol{k})$, and zeroth, $b(\boldsymbol{k})$, harmonics. Nevertheless, we should emphasize that only $|\boldsymbol{\varkappa}|$ satisfying the condition $|\boldsymbol{\varkappa}| \ll |\boldsymbol{k}_0|$ were studied in work [13]. Arbitrary $|\boldsymbol{\varkappa}|$ were considered in the fundamental works [3, 5], where the expression for instability increment was derived (it was used as the basis for numerical calculations in works [27, 30–32]), but for the perturbation in a single region only (near the wave vector of the first harmonic). The peculiarity of this work is studying the MI under a broadband perturbation in two regions (for the wave vectors of the first and zeroth harmonics).

Let us analyze the possibility for the perturbation frequency Ω to have an imaginary part at certain $\boldsymbol{\varkappa}$ depending on the normalized liquid depth $|\boldsymbol{k}_0|\,h$, which means the instability of the unperturbed wave with respect to a perturbation wave with such $\boldsymbol{\varkappa}$. After substituting expression (27) into the linearized equations of motion (16) and (17), we obtain the following system of homogeneous equations for $\alpha(\boldsymbol{k}_0+\boldsymbol{\varkappa})$, $\alpha(\boldsymbol{k}_0-\boldsymbol{\varkappa})$, $\beta(\boldsymbol{\varkappa})$, and $\beta(-\boldsymbol{\varkappa})$:

$$\left(\Omega + \omega(\boldsymbol{k}_0-\boldsymbol{\varkappa}) - \omega(\boldsymbol{k}_0) + \lambda\mathcal{A}_0^2\right)\alpha(\boldsymbol{k}_0-\boldsymbol{\varkappa}) +$$

$$+\lambda\mathcal{A}_0^2\alpha(\boldsymbol{k}_0+\boldsymbol{\varkappa}) + \mathcal{A}_0\left[f(-\boldsymbol{\varkappa})\beta(-\boldsymbol{\varkappa}) + f(\boldsymbol{\varkappa})\beta(\boldsymbol{\varkappa})\right] = 0,$$

$$\left(\Omega - \omega(\boldsymbol{k}_0+\boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0) - \lambda\mathcal{A}_0^2\right)\alpha\boldsymbol{k}_0+\boldsymbol{\varkappa}) -$$

$$-\lambda\mathcal{A}_0^2\alpha(\boldsymbol{k}_0-\boldsymbol{\varkappa}) - \mathcal{A}_0\left[f(-\boldsymbol{\varkappa})\beta(-\boldsymbol{\varkappa}) + f(\boldsymbol{\varkappa})\beta(\boldsymbol{\varkappa})\right] = 0,$$

$$(\Omega + \omega(\boldsymbol{\varkappa}))\,\beta(-\boldsymbol{\varkappa}) + \mathcal{A}_0 f(-\boldsymbol{\varkappa})\times$$

$$\times\left[\alpha(\boldsymbol{k}_0-\boldsymbol{\varkappa}) + \alpha(\boldsymbol{k}_0+\boldsymbol{\varkappa})\right] = 0,$$

$$(\Omega - \omega(\boldsymbol{\varkappa}))\,\beta(\boldsymbol{\varkappa}) - \mathcal{A}_0\,f(\boldsymbol{\varkappa})\left[\alpha(\boldsymbol{k}_0-\boldsymbol{\varkappa}) + \alpha(\boldsymbol{k}_0+\boldsymbol{\varkappa})\right] = 0.$$

Zeroing the corresponding determinant gives a necessary equation for the perturbation frequency Ω,

$$(\Omega-\delta)^2 = \Delta\left(\Delta - 2\lambda(\Omega)\mathcal{A}_0^2\right) \tag{28}$$

where

$$\lambda(\Omega) = -\lambda + \lambda^{(0)}(\Omega), \tag{29}$$

$$\lambda^{(0)}(\Omega) = \frac{f^2(-\boldsymbol{\varkappa})}{\omega(\boldsymbol{\varkappa})+\Omega} + \frac{f^2(\boldsymbol{\varkappa})}{\omega(\boldsymbol{\varkappa})-\Omega} \tag{30}$$

$$\Delta = \frac{1}{2}\left(\omega(\boldsymbol{k}_0+\boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0-\boldsymbol{\varkappa})\right) - \omega(\boldsymbol{k}_0), \tag{31}$$

$$\delta = \frac{1}{2}\left(\omega(\boldsymbol{k}_0+\boldsymbol{\varkappa}) - \omega(\boldsymbol{k}_0-\boldsymbol{\varkappa})\right). \tag{32}$$

The subscript in $\lambda^{(0)}(\Omega)$ indicates that it is a contribution of the zeroth harmonic to the nonlinear interaction. In the expanded form, expression (28) looks like

$$(\Omega + \omega(\boldsymbol{k}_0-\boldsymbol{\varkappa}) - \omega(\boldsymbol{k}_0))\,(\Omega - \omega(\boldsymbol{k}_0+\boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0)) =$$

$$= -2\lambda(\Omega)\mathcal{A}_0^2\,\Delta \tag{33}$$

and coincides with that from work [13]. In formulas (28) and (33), the transition to the physical amplitude A_0 by formula (23) can be made.

Let us calculate quantity (30). In order to obtain, according to formula (20), $f(\boldsymbol{\varkappa}) = 2V(\boldsymbol{k}_0,\, \boldsymbol{k}_0,\, \boldsymbol{\varkappa})$, let us simplify the coefficient $V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ (12) of $\boldsymbol{k}_0 = (k_0, 0)$ and $\boldsymbol{\varkappa} = (\kappa, \chi)$. We have

$$V(\boldsymbol{k}_0,\, \boldsymbol{k}_0, \boldsymbol{\varkappa}) =$$

$$= -V_0(-\boldsymbol{k}_0,\, \boldsymbol{k}_0,\, \boldsymbol{\varkappa}) - V_0(-\boldsymbol{k}_0,\, \boldsymbol{\varkappa},\, \boldsymbol{k}_0) + V_0(\boldsymbol{k}_0, \boldsymbol{\varkappa}, -\boldsymbol{k}_0),$$

where

$$V_0(-\boldsymbol{k}_0,\, \boldsymbol{k}_0,\, \boldsymbol{\varkappa}) = \frac{1}{8\sqrt{2\pi}}\sqrt{\frac{\omega(\boldsymbol{\varkappa})}{k_0\sigma}}\left(-k_0^2 + k_0^2\sigma\right),$$

$$V_0(-\boldsymbol{k}_0,\, \boldsymbol{\varkappa},\, \boldsymbol{k}_0) = \frac{1}{8\sqrt{2\pi}}\frac{\omega_0}{\sqrt{k_0\sigma\omega(\boldsymbol{\varkappa})}}\times$$

$$\times\left(-k_0\kappa + k_0|\boldsymbol{\varkappa}|\tanh|\boldsymbol{\varkappa}|h\right),$$

$$V_0(\,\boldsymbol{k}_0,\, \boldsymbol{\varkappa}, -\, \boldsymbol{k}_0) =$$

$$= \frac{1}{8\sqrt{2\pi}}\frac{\omega_0}{\sqrt{k_0\sigma\omega(\boldsymbol{\varkappa})}}\left(k_0\kappa + k_0|\boldsymbol{\varkappa}|\tanh|\boldsymbol{\varkappa}|h\right).$$

Whence,

$$f(\boldsymbol{\varkappa}) = \frac{k_0^{3/2}\omega_0^{1/2}}{4\sqrt{2\pi}\sqrt{\sigma}}\left(2\frac{\kappa}{k_0}\sqrt{\frac{\omega_0}{\omega(\boldsymbol{\varkappa})}} + (1-\sigma^2)\sqrt{\frac{\omega(\boldsymbol{\varkappa})}{\omega_0}}\right). \tag{34}$$

Note that, in the case of two-dimensional perturbations, the transverse component χ of the vector $\boldsymbol{\varkappa} = (\kappa, \chi)$ appears only in $\omega(\boldsymbol{\varkappa})$, whereas the numerator of the first term includes only the longitudinal component κ, as it occurs in the one-dimensional case. Substituting expression (34) into formula (30), we obtain the second term in formula (29):

$$\lambda^{(0)}(\Omega) = \frac{k_0^3}{16\pi^2\sigma}\times$$

$$\times\left(\frac{\kappa^2}{\omega^2(\boldsymbol{\varkappa}) - \Omega^2}\left(2\frac{\omega_0}{k_0} + (1-\sigma^2)\frac{\Omega}{\kappa}\right)^2 +\right.$$

$$\left. +(1-\sigma^2)^2\right). \tag{35}$$

Our task is to analyze all four roots of Eq. (33). The dispersion equations of high orders [33] were mainly solved numerically, when studying the interaction of nonlinear waves. Under condition of small non-linearity, some factorization is possible in two cases. In both of them, the crucial role is played by only two of those four roots. Therefore, if the remaining two roots are replaced in the first iteration by their approximate values, the interaction between the "main" roots can be described by quadratic equations. Such an approach gives only analytical estimations, and it will be verified by a straightforward numerical solution of the fourth-order equation (33) in a wide range of $\boldsymbol{\varkappa} = (\kappa, \chi)$ for specific values of k_0h and k_0A_0.

4.1. Instability of the first harmonics associated with its transformation into the second one. Comparison with available results

One can see from Eq. (28) that two of four roots Ω have an imaginary parts. The perturbation with such a frequency Ω is unstable, if the right-hand side of Eq. (28) becomes negative. The boundaries of this instability region in the plane (κ, χ) are found by zeroing each of the multipliers on the right-hand side of Eq. (28).

$$1.\quad \Delta = 0. \tag{36}$$

This curve is known in the theory of instability of infinitesimal amplitude waves as the figure-of-eight of Phillips [34] or the resonance curve

$$\omega(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - 2\omega(\boldsymbol{k}_0) = 0. \tag{37}$$

$$2.\quad \Delta = 2\lambda(\delta)\mathcal{A}_0^2\,. \tag{38}$$

where

$$\lambda(\delta) = \frac{k_0^3}{16\pi^2\sigma}\left(\nu + \frac{\kappa^2}{\omega^2(\boldsymbol{\varkappa}) - \delta^2}\left(2\frac{\omega_0}{k_0} + (1-\sigma^2)\frac{\delta}{\kappa}\right)^2\right),$$

$$\nu = -\frac{9\sigma^4 - 10\sigma^2 + 9}{2\sigma^2} + (1-\sigma^2)^2.$$

The region of instability is confined by curves (37) and (38). In Figs. 1 and 2, this region is shown in the coordinates defined by the longitudinal, $p = \kappa/k_0$, and transversal, $q = \chi/k_0$, components of the normalized

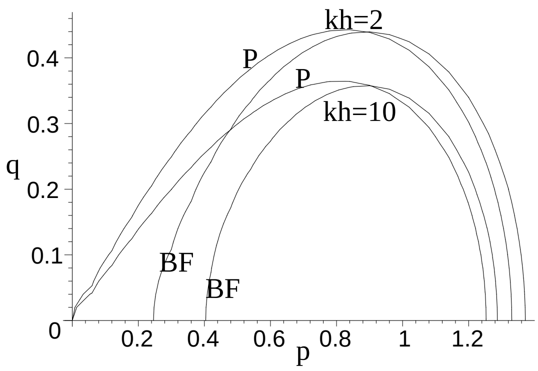

Fig. 1. Instability region according to Eqs. (37) and (38) at $k_0h = 2$ and 10. $k_0A_0 = 0.2$

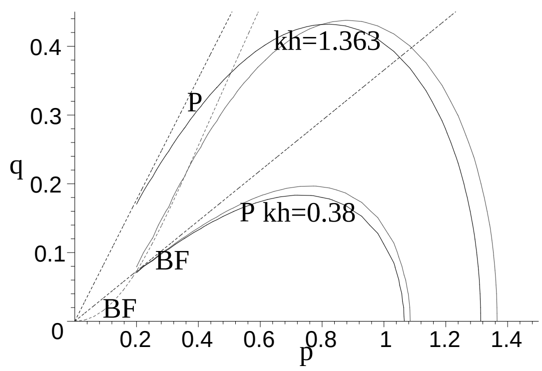

Fig. 2. The same as in Fig. 1, but at $k_0h = 1.363$ and $k_0h = 0.38$. Dotted lines are approximations by formulas (39), (38), and (40)

vector $\boldsymbol{\varkappa} = (\kappa, \chi)$ for several k_0h-values [curves (37) and (38) are designated as P and BF, respectively].

In the limiting case $(\kappa, \chi) \ll k_0$, we have

$$\delta \to c_g\kappa, \quad c_g = \left.\frac{\partial\omega}{\partial\kappa}\right|_{\kappa=k_0,\chi=0} = \frac{1}{2}\left(1 + \frac{1-\sigma^2}{\sigma}k_0h\right)\frac{\omega_0}{k_0},$$

$$\Delta \to \frac{1}{2}(\alpha\kappa^2 + \beta\chi^2),$$

$$\alpha \equiv \left.\frac{\partial^2\omega}{\partial\kappa^2}\right|_{\kappa=k_0,\chi=0} =$$

$$= -\frac{1}{4}\left(1 - 2\frac{1-\sigma^2}{\sigma}k_0h + \frac{\left(3\sigma^2+1\right)\left(1-\sigma^2\right)}{\sigma^2}k_0^2h^2\right)\frac{\omega_0}{k_0^2},$$

$$\beta \equiv \left.\frac{\partial^2\omega}{\partial\chi^2}\right|_{\kappa=k_0,\chi=0} = \frac{c_g}{k_0},$$

$$\omega^2(\boldsymbol{\varkappa}) = gh\left(\kappa^2+\chi^2\right) \equiv k_0h\left(\kappa^2+\chi^2\right)\frac{\omega_0^2}{k_0^2},$$

where c_g is the group velocity of linear waves. Hence, in this case, the left border (37) of the instability region becomes a straight line

$$\chi = \sqrt{-\frac{\alpha}{\beta}}\kappa, \tag{39}$$

whereas, for the right border (38),

$$\lambda(\delta)|_{\kappa,\chi\ll k_0} = \frac{k_0^3}{16\pi^2\sigma}\times$$

$$\times\left(\nu + \frac{\kappa^2}{gh\left(\kappa^2+\chi^2\right) - c_g^2\kappa^2}\left(2\frac{\omega_0}{k_0} + (1-\sigma^2)c_g\right)^2\right). \tag{40}$$

As is seen from the comparison of expressions (38) and (40) with formulas (39) and (30) in work [6], in the one-dimensional case ($\chi = 0$) and for small κ, border (40) of the instability region transforms into the right end of the interval of κ-instability obtained for the first time by Benjamin and Feir for this case [6] (it also follows from the Zakharov general formula for the instability region, provided small $|\boldsymbol{\varkappa}|$ [3, 5]). Moreover, this case of small κ corresponds to the NSE approximation [4], so that expression (40) is identical, to within a constant factor, to the nonlinear coefficient q (48) in NSE (2). At $k_0h = 1.363$, expression (40) changes its sign from the negative, at $k_0h > 1.363$, to positive one. Since $\frac{\partial^2\omega}{\partial\kappa^2}|_{\kappa=k_0,\chi=0} < 0$, equality (38) is impossible, if $k_0h < 1.363$. Hence, two lines, which confine the instability region, converge at the point ($\kappa = 0, \chi = 0$), so that instability BF is absent in the one-dimensional case if $k_0h \le 1.363$ for any perturbation wave vector κ [6,7,14]. In the NSE approach [4], such a situation corresponds to the absence of soliton solutions for NSE (2), $q\omega'' < 0$.

In the two-dimensional case and for small $\kappa, \chi \ll k_0$, we obtained the results for the instability region that are identical to those of works [14,15,17]. The corresponding region is shown in Fig. 2 for specific values $k_0h = 1.363$ and 0.38. In the range of small κ and χ, the instability region is described by the simplified formulas (39), (38), and (40) shown by dotted asymptotes. These lines were presented in work [14]; here, they are given as test ones.

If $k_0h = 0.38$, curves (38),(40) coincide with the straight line (39), and the instability region degenerates into the latter. This conclusion on the disappearance of the instability at $k_0h = 0.38$ [14, 15, 17] corresponds in

Fig. 2 to the convergence of its two borders into a single straight asymptote in the range of small $|\boldsymbol{\varkappa}|$.

From the simplified formulas (38), (40), and (39), it also follows that, if $k_0 h < 1.363$ in the range of small κ, and $\chi \neq 0$, curves (38),(40) lie below the straight line (39) (in this connection, see work [35]).

In the case where κ and χ are not small, the borders of the instability region (36) and (38) are depicted in Figs. 1 and 2 by solid curves. Another intersection point (κ, χ) of curves (36) and (38) was found for the first time. It is a solution of the system of equations $\lambda = 0$ and $\Delta = 0$. At arbitrary κ and χ, the curves concerned are only qualitatively similar to those obtained in works [27] (the finite depth) and [30, 31] (the infinite depth) – a quantitative agreement is attained only at small κ and χ – which used the general expression for the instability region obtained, as was indicated above, without perturbation with the center at the zeroth harmonic (only at the fundamental one) in works [3, 5]. Given the zeroth harmonic (the finite depth), the equation of motion for it has to be taken into account explicitly, as was done in work [13]. In this case, we obtain a dispersion equation of the fourth order (33) which contains both the influence of the zeroth harmonic on the known instability region at small $|\boldsymbol{\varkappa}|$ and the emergence of additional instability section.

4.2. Additional region of the first harmonic instability associated with its transformation into the second and zeroth harmonics

To make analytical estimations of the possibility that some roots of Eq. (33) have an imaginary part at $\kappa \simeq k_0$, let us divide it by $\Omega - \omega(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0)$ and multiply by $\Omega - \omega(\boldsymbol{\varkappa})$. Now, let us change over to the physical amplitude by formula (23). As a result, Eq. (33) reads

$$(\Omega + \omega(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - \omega_0)\,(\Omega - \omega(\boldsymbol{\varkappa})) =$$

$$= -\left(\nu\,(\Omega - \omega(\boldsymbol{\varkappa})) - \frac{\kappa^2}{\omega(\boldsymbol{\varkappa}) + \Omega}\left(2\frac{\omega_0}{k_0} + (1 - \sigma^2)\frac{\Omega}{\kappa}\right)^2\right) \times$$

$$\times \frac{(\omega(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - 2\omega_0)}{(\Omega - \omega(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \omega_0)}\frac{\omega_0}{8\sigma^2}k_0^2 A_0^2. \quad (41)$$

Since the quantity $k_0^2 A_0^2$ is small, two of four roots of Eq. (41) are close to the corresponding values obtained, if the right-hand side equals zero. Since $\omega(0) = 0$, those two roots are equal to $\omega_0 \equiv \omega(k_0)$ at the point ($\kappa = k_0, \chi = 0$). Therefore, in the first iteration in the range of small χ, let us approximately substitute Ω by ω_0 and κ by k_0 on the right-hand side of Eq. (41):

$$(\Omega + \omega(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - \omega_0)\,(\Omega - \omega(\boldsymbol{\varkappa})) =$$

$$= -\frac{\left(3 - \sigma^2\right)^2}{8\sigma^2}\frac{\omega_0^3\,(\omega(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \omega(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - 2\omega_0)}{(\omega(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) - 2\omega_0)\,(\omega(\boldsymbol{\varkappa}) + \omega_0)}k_0^2 A_0^2. \quad (42)$$

Now, on the basis of Eq. (42), let us analyze whether those two roots can have a small imaginary part in the region ($\kappa \simeq k_0, \chi \simeq 0$) in the next approximation. Let us change over to frequencies normalized by ω_0 and denote them by a hat. Equation (42) looks like

$$\left(\widehat{\Omega} + \widehat{\omega}(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - 1\right)\left(\widehat{\Omega} - \widehat{\omega}(\boldsymbol{\varkappa})\right) =$$

$$= -\frac{\left(3 - \sigma^2\right)^2}{8\sigma^2}\frac{(\widehat{\omega}(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \widehat{\omega}(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - 2)}{(\widehat{\omega}(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) - 2)\,(\widehat{\omega}(\boldsymbol{\varkappa}) + 1)}k_0^2 A_0^2. \quad (43)$$

At $\kappa = k_0$ and $\chi = 0$, we obtain

$$\left(\widehat{\Omega} - 1\right)^2 = -\frac{\left(3 - \sigma^2\right)^2}{16\sigma^2}k_0^2 A_0^2,$$

i.e. a perturbation leads to the instability with the increment

$$\mathrm{Im}\widehat{\Omega} = \frac{3 - \sigma^2}{4\sigma}k_0 A_0. \quad (44)$$

In this case, $\mathrm{Re}\,\widehat{\Omega} = 1$. In Fig. 5 obtained by direct numerical calculations by formula (33), value (44) corresponds to the vertex of the internal surface which is really located at ($\kappa \simeq k_0, \chi = 0$).

In the region $\kappa = k_0$ and $\chi \simeq 0$, the instability increment is

$$\mathrm{Im}\widehat{\Omega} = \left\{-\frac{1}{4}(1 - \widehat{\omega}(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - \widehat{\omega}(\boldsymbol{\varkappa}))^2 + \right.$$

$$\left. + \frac{\left(3 - \sigma^2\right)^2}{8\sigma^2}\frac{\widehat{\omega}(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) + \widehat{\omega}(\boldsymbol{k}_0 - \boldsymbol{\varkappa}) - 2}{(\widehat{\omega}(\boldsymbol{k}_0 + \boldsymbol{\varkappa}) - 2)\,(\widehat{\omega}(\boldsymbol{\varkappa}) + 1)}k_0^2 A_0^2\right\}^{1/2}. \quad (45)$$

In Fig. 3, the plots of the function $\mathrm{Im}\,\widehat{\Omega}$ depending on two variables, κ and χ, calculated by formula (45) are shown for $k_0 h = 10$ and 2 and $k_0 A_0 = 0.2$. They are depicted as the curves of the $\mathrm{Im}\,\widehat{\Omega}$-dependence on the transversal component $q = \chi/k_0$ of the wave vector $\boldsymbol{\varkappa}$ for several values of its longitudinal component, $p = \kappa/k_0$. The additional two-dimensional regions of instability ((43) for the same values of parameters are shown in Fig. 4.

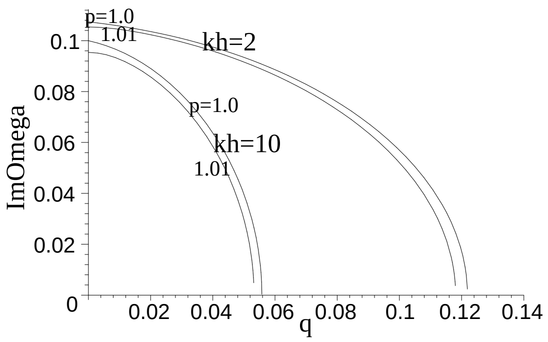

Fig. 3. Instability increment $\mathrm{Im}\,\widehat{\Omega}$ (formula (45)) as a function of q for several p-values at $k_0h = 10$ and 2. $k_0A_0 = 0.2$

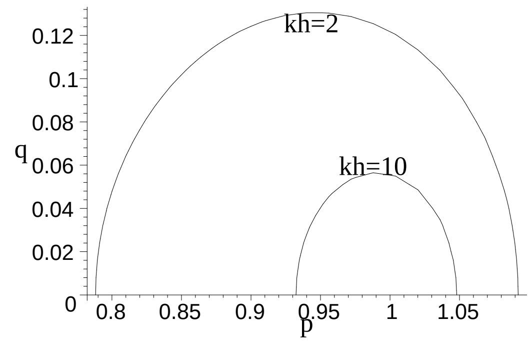

Fig. 4. Instability region (according to Eq. (43)) in the coordinates set by the longitudinal, p, and transverse, q, components of the normalized vector $\boldsymbol{\varkappa}$ at $k_0h = 10$ and 2. $k_0A_0 = 0.2$

4.3. Common numerical description of both instability regions

In the case of arbitrary $\boldsymbol{\varkappa}$ and the non-linearity k_0A_0, Eq. (28) must be solved numerically. The fourth-order equation obtained in work [13] was not solved numerically, but, in order to obtain analytical estimations for the instability increment, it was reduced to a quadratic one in $\boldsymbol{\varkappa}$-regions weakly different from the resonance surface (37) ($\Delta = 0$). The results of such numerical calculations are shown in Fig. 5. The normalized instability increment $\widehat{\Omega} = \frac{\Omega}{\omega_0}$ is shown in the plane of two normalized components, $p = \frac{\kappa}{k_0}$ and $q = \frac{\chi}{k_0}$, of the vector $\boldsymbol{\varkappa}$. One can distinctly see the external and internal regions of instability described in subsections 4.1 and 4.2, respectively, with some approximations. The external instability region at small $|\boldsymbol{\varkappa}|$ in the vicinity of the coordinate origin is known as the Benjamin–Feir long-wave instability. Both regions are described by the dispersion law (33) obtained by Zakharov in his pioneer work [13] and analytically studied there in the case of small $|\boldsymbol{\varkappa}|$. The possibility for the existence of an additional instability region in the one-dimensional case was pointed out for the first time in work [20], where the corresponding dispersion law was obtained in the framework of the multiscale method. First numerical tabulations in the case of two-dimensional perturbations were made in work [22].

The additional (internal) instability region broadens out at the reduction of k_0h and, as the results of calculations testify, reaches the known (external) region at $k_0h = 0.38$ irrespective of the k_0A_0-value (usually, owing to the weak non-linearity of the theory, k_0A_0-values substantially smaller than unity were considered). However, we do not present this effect in Fig. 5 at $k_0h = 0.38$, because, as is seen from the condition that A_2 is small in comparison with A in expression (46), Stokes expansions into harmonic series are valid provided that $(k_0h)^3 \gg k_0A_0$; so that, for $k_0A_0 = 0.2$ adopted for Fig. 5, we confine ourselves by the value $k_0h = 1$. The value $k_0h = 0.38$ was indicated in works [14,15,17], when studying the instability in the range of small $|\boldsymbol{\varkappa}|$, as such, at which the region degenerates into a straight line, i.e. the instability disappears. From our results – see the tendency at a depth reduction in Fig. 5 (see also Fig. 2 for $k_0h = 0.38$) – it follows that, if the depth decreases, the instability disappears only in the range small $|\boldsymbol{\varkappa}|$.

Direct numerical calculations of the instability region for two-dimensional perturbation wave vectors were carried out in work [11], proceeding from the Euler equations of motion for an ideal liquid; however, the additional instability region described above was not revealed there. This may be connected with the consideration of stationary waves and the corresponding assumption that all harmonics move with the same velocity, whereas in this work, the time evolution of the zeroth and first harmonics is governed by interdependent equations of motion, and the velocity of the zeroth harmonic is not assumed to be equal to that of the first harmonic.

5. Conclusion

Besides the first harmonic, an essential role in the formation of an additional instability region is played by the zeroth one. Therefore, the long-term evolution of the considered instability can result in the formation of structures that are intermediate between the soliton-like envelope of fast oscillations described by the NSE and the isolated waves without filling, which are characteristic at small depths.

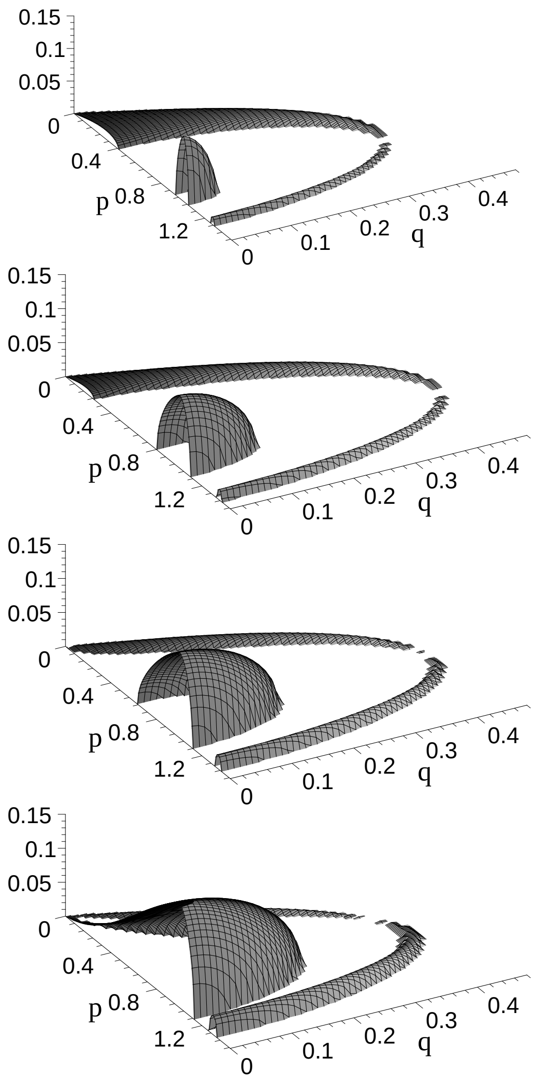

Fig. 5. Imaginary part of the normalized frequency $\widehat{\Omega}$ (see Eq. (33)) as a function of two components of the vector $\boldsymbol{\varkappa}/k_0 = (p, q)$ for $k_0h = 10$, 2, 1.363, and 1 (from top to bottom). $k_0A_0 = 0.2$

APPENDIX

Here, the amplitudes of the zeroth, B, and second, A_2, harmonics expressed in terms of the first harmonic amplitude A [4, 12] are given.

$$A_2(x,t) = \frac{3-\sigma^2}{8\sigma^3}k_0A^2(x,t), \quad \sigma = \tanh k_0h \tag{46}$$

$$B(x,t) = \frac{\sigma^2-1}{4\sigma}k_0|A|^2(x,t) - \frac{\sigma k_0}{\omega_0^2}\frac{\partial\Psi}{\partial t},$$

$$\frac{\partial\Psi}{\partial t} = -c_g\frac{\partial\Psi}{\partial x},$$

$$\frac{\partial\Psi}{\partial x} = \frac{\omega_0^2}{4\sigma^2}\frac{\left(2\frac{\omega_0}{k_0} + \left(1-\sigma^2\right)c_g\right)}{c_g^2 - gh}|A|^2(x,t), \tag{47}$$

$$q = \frac{\omega_0k_0^2}{16\sigma^2}\left\{-\frac{9\sigma^4 - 10\sigma^2 + 9}{\sigma^2} + 2(1-\sigma^2)^2 + \right.$$
$$\left. + \frac{1}{gh - c_g^2}\left(2\frac{\omega_0}{k_0} + (1-\sigma^2)c_g\right)^2\right\}. \tag{48}$$

1. V.P. Lukomsky and I.S. Gandzha, Ukr. J. Phys. **54**, 207 (2009).
2. G.G. Stokes, Trans. Cambridge Philos. Soc. **8**, 441 (1847).
3. V.E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).
4. H. Hasimoto and H. Ono, J. Phys. Soc. Jpn. **33**, 805 (1972).
5. V.E. Zakharov, Sov. Phys. JETP **24**, 740 (1967).
6. T.B. Benjamin, Proc. Roy. Soc. (London) A **299**, 59 (1967); J.E. Feir, Proc. Roy. Soc. (London) A **299**, 54 (1967).
7. G.B.J. Whitham, J. Fluid Mech. **27**, 399 (1967).
8. A.R. Its and V.P. Kotljarov, Dopovidi Akad. Nauk UkrRSR, ser. A **11**, 965 (1976); E.R. Tracy and H.H. Chen, Phys. Rev. A **37**, 815 (1988).
9. H. Berland, A.L. Islas, and C.M. Schober, J. Comp. Phys. **225**, 284 (2007); C.M. Schober, Eur. J. Mech. B/Fluids **25**, 602 (2006); A. Slunyaew, Eur. J. Mech. B/Fluids **25**, 621 (2006); A. Osborne, M. Onorato, and M. Serio, Phys. Lett. A **275**, 386 (2000); D. Cai and D.W. McLaughlin, J. Math. Phys, **41**, 4125 (2000); K. Dysthe and K. Trulsen, Phys. Scripta T **82**, 48 (1999); N.N. Akhmediev, V.M. Eleonskii, and N.E. Kulagin, Theor. Math. Phys. (USSR) **72**, 809 (1987).
10. M.J. Ablowitz, J. Hammack, D. Henderson, and C.M. Schober, Phys. Rev. Lett. **84**, 887 (2000); Physica **152-153**, 416 (2001).
11. J.W. McLean, J. Fluid Mech. **114**, 331 (1982).
12. A.V. Slunyaev, JETP **101**, 926 (2005); Yu.V. Sedletsky, JETP **97**, 180 (2003); U. Brinch-Nielsen and I.G. Jonsson, Wave Motion **8**, 455 (1986).
13. V.E. Zakharov and V.G. Kharitonov, J. Appl. Mech. Tech. Phys. **11**, 747 (1970).
14. D.J. Benney and G.J. Roskes, Stud. Appl. Math. **48**, 377 (1969).
15. A. Davey and K. Stewartson, Proc. Roy. Soc. (London) A **338**, 101 (1974).
16. H. Tomita, J. Oceanogr. Soc. Japan **42**, 153 (1986).
17. W.D. Hayes, Proc. Roy. Soc. (London) A **332**, 199 (1973).
18. T.J. Bridges, Phil. Trans. R. Soc. **354**, 533 (1996).
19. Yu.V. Sedletsky, Phys. Lett. A **243/4**, 293 (2005).
20. Yu.V. Sedletsky, J. Phys. A: Math. Gen. **39**, L529 (2006).
21. Yu.V. Sedletsky, JETP Letters **86**, 502 (2007).

22. Yu.V. Sedletsky, J. Phys. A: Math. Theor. **41**, 035502 (2008).
23. V.P. Krasitskii, JETP **71**, 921 (1990).
24. V.P. Krasitskii, J. Fluid Mech. **272**, 1 (1994).
25. V.P. Krasitskii, Morsk. Gidrofiz. Zh. **6**, 17 (1993).
26. V.E. Zakharov, A.I. Dyachenko, and A.O. Prokofiev, Eur. J. Mech. B **25**, 677 (2006).
27. M. Stiasnie and L. Shemer, J. Fluid Mech. **143**, 47 (1984).
28. M.M. Zaslavskii and V.P. Krasitskii, Izv. Atmos. Ocean Phys. **37**, 105 (2001).
29. P.A.E.M. Janssen and M. Onorato, J. Phys. Oceanogr. **37**, 2389 (2007).
30. D.R. Crawford, B.M. Lake, P.G. Saffman, and H.C. Yuen, J. Fluid Mech. **105**, 177 (1981).
31. H.C. Yuen and B.M. Lake, Nonlinear Dynamics of Deep-Water Gravity Waves (Academic Press, New York, 1982).
32. V.P. Krasitskii and V.A. Kalmykov, Izv. Atmos. Ocean Phys. **29**, 222 (1993).
33. M. Onorato, A.R. Osborne, and M. Serio, Phys. Rev. Lett. **96**, 014503 (2006); P.K. Shukla, I. Kourakis, B. Eliasson, M. Marklund, L. Steflo, Phys. Rev. Lett. **97**, 094501 (2006); Suma Debsarma, K.P. Das, Phys. Fluids **19**, 097101 (2007); K.W. Chow, K.K.Y. Wong, and K. Lam, Phys. Lett. A **372**, 4596 (2008).
34. O.M. Phillips, J. Fluid Mech. **9**, 193 (1960); Proc. Roy. Soc.(London) **299**, 104 (1967).
35. D.K. Rollins and B. Shivamoggi, Phys. Scripta **57**, 28 (1998).

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НЕСТАБІЛЬНІСТЬ ОСНОВНОЇ ГАРМОНІКИ СТОКСОВИХ ХВИЛЬ ДО ДВОВИМІРНИХ ЗБУРЕНЬ В ПРИСУТНОСТІ НУЛЬОВОЇ ГАРМОНІКИ

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Р е з ю м е

Раніше отримані в одновимірному випадку (Письма в ЖЭТФ **86**, 574 (2007)) результати про додаткову, недовгохвильову область нестабільності слабонелінійних хвиль на поверхні шару рідини обмеженої глибини h поширено на випадок двовимірних збурень. Опис проведено у рамках гамільтонівського підходу Захарова для зв'язаної системи фур'є-амплітуд першої гармоніки з хвильовим вектором k_0 і неосцилюючої компоненти хвилі (нульової гармоніки). При аналізі лінійної (не)стабільності слабконелінійного стоксового розв'язку такої системи дисперсійне рівняння 4-го порядку (отримане раніше Захаровим і вивчене аналітично в області малих хвильових векторів хвилі збурювання) дає, поряд з раніше відомою, додаткову область нестабільності при незанадто великих кутах θ між основною і збурюючою хвилями, на відміну від звичайно одержуваного в довгохвильовій області квадратного рівняння, що дає тільки звичайну нестабільність Бенджамена–Фейра. У той час як область звичайної нестабільності звужується зі зменшенням глибини й зникає при $k_0h = 1,363$ для $\theta = 0$, а при $k_0h = 0,38$ для всіх θ, додаткова область нестабільності розширюється із зменшенням глибини.