ON THE ROLE OF BOGOLYUBOV'S INTEGRAL MANIFOLD METHOD IN BIFURCATION ANALYSIS OF MULTIDIMENSIONAL NONLINEAR MODELS

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It is shown that the results by Bogolyubov on the reduction of a multidimensional dynamical system on its integral manifold are the base of the center manifold theory which, together with normal forms, is a powerful tool of the bifurcation analysis of multidimensional nonlinear models.

1. In 1945, the monograph by M.M. Bogolyubov "On Some Statistical Methods in Mathematical Physics" [1] was published. This monograph contains, along with the results obtained by M.M. Bogolyubov in the stochastic theory of perturbations, the theory of random processes, *etc.*, a strictly mathematically substantiated (on both finite and infinite time intervals) method of averaging in the nonlinear mechanics, as well as the main aspects of the integral manifold theory constructed by M.M. Bogolyubov.

The fundamental meaning of results presented in this monograph follows from the fact that it was included into the list of the best mathematical works published over the world for 1900–1950. The list composed by about 50 famous mathematicians of the contemporaneity was published in 1994 [2].

In the foreword to his monograph, M.M. Bogolyubov wrote, in particular:

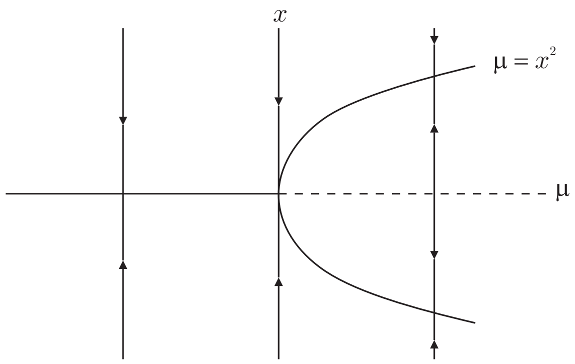
"It is worth noting that the construction of at least a local theory of the existence of integral manifolds, which would generalize the Poincaré local theory, can be of independent interest irrespective of the problem of substantiation of the principle of averaging. Indeed, a qualitative study of solutions is significantly simplified, if these solutions lie on a manifold with a less dimension than that of the initial phase space, especially if such a manifold turns out to be one-dimensional or two-dimensional."

Thus, as early as in the 1940s, M.M. Bogolyubov indicated the role of integral manifolds of minimal dimensionalities in relation to their use, in particular, in a qualitative bifurcation analysis of multidimensional nonlinear models. The point is that simple bifurcations are manifested on manifolds of minimal dimensionalities (on which the reduction of the input multidimensional system is realized), if we take them as new phase spaces. These bifurcations are well studied in the qualitative theory of ordinary differential equations and are widely applied in physics, the mechanics of solids, thermodynamics, statistical physics, the theory of lasers, *etc.*

For example, the physicians use widely bifurcations, at which the stable mode bifurcates into a stable mode. As an example of such a bifurcation, I mention a simple pitchfork bifurcation of codimension 1 which is modeled by the equation

$$\dot{x} = \mu x - x^3, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}. \tag{1}$$

The analysis of this equation leads to the following conclusion (Figure). For any value of the parameter μ, Eq. (1) has the stationary point $x = 0$. The eigenvalue for this point coincides with the parameter μ. Therefore, the point $x = 0$ is stable at $\mu < 0$ and unstable at $\mu > 0$. But, for $\mu > 0$, there exist else two stationary points: $x_1 = \sqrt{\mu}$ and $x_2 = -\sqrt{\mu}$ are two branches of a parabola: $\mu = x^2$, into which the point $x = 0$ bifurcates. Both these points have the eigenvalue equal to -2μ. Thus, both branches of the parabola are stable.



The representation of any characteristic property of solutions as a function of the bifurcation parameter forms a bifurcation diagram in the extended phase space. In this case, it has the form of a pitchfork.

This bifurcation underlies some theories which explain the spontaneous symmetry breaking. Namely, the stable fixed point $x = 0$ at $\mu < 0$ corresponds to a symmetric state, for example, in the absence of magnetization in a ferromagnetic, whereas the fixed points $x_{1,2} = \pm\sqrt{\mu}$ generated at $\mu > 0$ correspond to the state with the spontaneous symmetry breaking.

In the mechanics of solids, this bifurcation is used in the qualitative analysis of the dynamics of the "modified" Duffing equation $\ddot{x} + \alpha\dot{x} - x + x^3 = 0$ (with the negative coefficient of stiffness) which is a one-mode model of the motion of a beam in a stationary rigid body without external force [10].

In thermodynamics, the pitchfork bifurcation was used by Landau in his scenario of phase transitions of the second kind in ferromagnetics [3].

In statistical physics, it is frequently assumed for phenomenological reasons that the temporal course of the order parameter q is governed by the equation

$$\dot{q} = -\frac{\partial F(q,T)}{\partial q}, \tag{2}$$

where T is the temperature. In the case where the free energy $F(q,T)$ is set by the Ginzburg–Landau potential

$$F(q,T) = F(0,T) + \frac{\mu}{2}q^2 + \frac{1}{4}q^4,$$

Eq. (2) yields

$$\dot{q} = -\mu q - q^3.$$

Equations of type (1) have a striking analogy with the equation for a laser [4]

$$\dot{B} = GB - C(B^+B)B + \tilde{F} \tag{3}$$

which differs from (1) by that Eq. (1) has no fluctuation terms $\tilde{F}$, and the variable x is real, whereas B in (3) is a complex-valued amplitude. As was mentioned in [4], Eq. (3) is, in turn, typical of the equations describing the effects of self-organization.

2. M.M. Bogolyubov constructed the theory of integral manifolds by the example of a system in the standard form

$$\dot{x} = \varepsilon X(t,x), \qquad \varepsilon > 0, \tag{4}$$

where x and X are n-vectors. This system of equations is involved in numerous problems of the nonlinear theory of oscillations.

As for the right-hand part of system (4), it is assumed that the vector-function $X(t,x)$ belongs to the class C^1 in some D_ρ-neighborhood of a one-parameter family of the averaged system corresponding to (4).

By the example of a system of form (4), M.M. Bogolyubov introduced, for the first time, the definition of integral manifold [1, p. 25].

Definition. *Let every t in the interval $(-\infty, \infty)$ correspond to some set S_t of points x which can be represented analytically in a parametric form by an equation of the form $x = f(t, u_1, \ldots, u_S)$, where f satisfies the Lipschitz condition relative to the parameters $u_1, \ldots, u_S$ in the whole domain of their variation.*

Then we say that S_t is an integral manifold of system (4), if, for every solution $x = x(t)$ of this system, the relation $x(t) \in S_t$ valid at some time moment $t = t_0$ yields its validity for any real $t > t_0$.

M.M. Bogolyubov proved a theorem [1, Theorem III] which establishes the conditions for system (4), under which there exists a one-parameter integral manifold possessing the property to attract, in the course of the time, "close" integral curves in some neighborhood of a one-parameter family of periodic solutions of the t-averaged system corresponding to (4). In this case, the dynamics of the input system is controlled by the dynamics of an equation which is a result of the reduction of the input system onto its one-parameter integral manifold.

The basic idea of the proof of the theorem consists in the following. In some D_ρ-neighborhood of a one-parameter family of periodic solutions of the averaged system corresponding to (4), let us introduce the change $x \to \{\varphi, h\}$ (φ is the angular variable, and h is an $(n-1)$-vector directed along a normal to φ) which reduces the input system (4) to the form

$$\begin{aligned} \dot{\varphi} &= \omega + P(t,\varphi,h,\varepsilon), \\ \dot{h} &= Hh + Q(t,\varphi,h,\varepsilon) \end{aligned} \tag{5}$$

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with the separated variables φ and h, where H is a constant matrix, whose all eigenvalues have negative real parts.

By the obtained system, the mapping S is constructed on some class $C_\rho(\eta)$ of ρ-bounded and η-Lipschitz vector-functions $h = f(t, \varphi, \varepsilon)$.

The conditions imposed on the input system ensure the fulfilment of conditions for the contraction mapping principle for S. It is proved that the fixed point of a contraction mapping S sets the required integral manifold. It is proved also that the dynamics of the input system is controlled by the dynamics defined by the scalar equation

$$\dot{\varphi} = \omega + P(t, \varphi, f(t, \varphi, \varepsilon), \varepsilon) \tag{6}$$

which is a result of the reduction of system (5) onto its integral manifold set by the vector-function $h = f(t, \varphi, \varepsilon)$.

3. By advancing the idea of the development of the theory of local integral manifolds, M.M. Bogolyubov emphasized the expediency of the construction of local integral manifolds with minimum dimensionalities (one or two).

In 1954, by the example of an n-dimensional system of nonlinear differential equations of the form

$$\dot{x} = X(x) + \varepsilon X(t, x), \qquad \varepsilon > 0, \tag{7}$$

M.M. Bogolyubov posed the problem on the existence of two-dimensional local integral manifolds in a neighborhood of a static (and also periodic) solution of the nonperturbed system

$$\dot{x} = X(x) \tag{8}$$

corresponding to (7). The results obtained in this direction were published in [5,6].

In 1964, the article by V.A. Pliss [7] which is considered to be a pioneer study on the center manifold theory was published in the journal Izv. Akad. Nauk SSSR. This article cited work [6].

The necessity of the analysis of bifurcation problems, which appear more and more often in the qualitative studies of applied problems, caused the further intense development of the method of center manifolds [8–12].

For the sake of simplicity, we present the essence of the center manifold method according to [8].

Let

$$\dot{x} = F(x, c), \quad x \in \mathbb{R}^n, \quad c \in \mathbb{R}^c, \quad F(x, c) \in \mathbb{R}^n \tag{9}$$

be a k-parameter family of dynamical systems.

Let $x = x^0$ be a stationary point of system (9) at $c = c^0$. Then the linearization of (9) in a neighborhood of this point gives the relation $\dot{x}_i = F_{i,j}(x^0, c^0)\delta x_j +$ terms of higher orders.

The linear vector space V of displacements δx from x^0 can be divided into three linear vector subspaces:

$$V = V_S + V_C + V_U. \tag{10}$$

Here, the stable subspace V_S is formed by eigenvectors of the matrix $F_{i,j}$, whose corresponding eigenvalues have negative real parts; the unstable subspace V_U is formed by eigenvectors of the matrix $F_{i,j}$, whose corresponding eigenvalues have positive real parts; and the center subspace V_C is formed by eigenvectors of the matrix $F_{i,j}$, whose corresponding eigenvalues have zero real parts or are zero. Just the last subspace is critical, because it is related to bifurcations of system (9), and it can be locally extended to a manifold called the center (local) manifold.

We say that some manifold M is the center manifold of a stationary point of the given dynamical system, if it is invariant relative to this system and includes its stationary point, and if its tangent space at this point is a critical eigensubspace.

If δx is an arbitrary displacement from x^0, then, in a neighborhood of $(x^0; c^0)$, the equations describing the dynamical system can be represented in a simpler form as

$$\delta x = \delta v_S + \delta v_C + \delta v_U, \tag{11}$$

$$\begin{aligned} \delta\dot{v}_S &= G_S \delta v_S, \\ \delta\dot{v}_C &= G_C(\delta v_C, c), \\ \delta\dot{v}_U &= G_U \delta v_U, \end{aligned} \tag{10'}$$

where G_S is an $(s \times s)$-matrix ($s = \dim V_S$); G_U is a $(u \times u)$-matrix ($u = \dim V_u$), and the operator $G_C(\delta v_C, c)$ is nonlinear.

It is seen from Eqs. (10′) that the equation describing the dynamical system can be linearized far from the center manifold. The terms of higher orders must be preserved only on the center manifold with dimensionality $c < n$. All bifurcations of the dynamical system in a neighborhood of c^0 are determined by the operator $G_C(\delta v_C, c)$. The indicated decrease in the dimensionality simplifies significantly the study of bifurcations related to dynamical systems (9). Thus, it is possible to state that the study of bifurcations of an input dynamical system is reduced to the study of only such bifurcations which can appear on the center manifold [8].

It is easy to see that the last assertion is an analog of the thought of M.M. Bogolyubov on the application of the reduction of the initial problem concerning the qualitative study of a multidimensional nonlinear system onto its integral manifold.

4. In 1994, the article by D.V. Anosov "On the contribution of N.N. Bogolyubov to the theory of dynamical systems" was published in the journal Uspekhi Mat. Nauk [13], in which the fundamental role of ideas by M.M. Bogolyubov in the construction of the center manifold theory was indicated. In particular, D.V. Anosov wrote: "N.N. Bogolyubov emphasized the pragmatic meaning of integral (invariant) manifolds: they allow one as if to divide the dimension of the initial problem. This idea is the main one also in the center manifold theory".

"...As for a center manifold, the first, devoted to it, article by V.A. Pliss contains the direct reference onto N.N. Bogolyubov's works in connection with the use of an analogous integral equation."

Thus, I may conclude that the integral manifold theory of Bogolyubov is a basis of the center manifold theory which, together with normal forms, is considered a powerful tool of the bifurcation analysis of multidimensional dynamical systems with simple dynamics. In turn, such dynamical systems model, in many cases, the applied problems such as those, for example, related to neuron networks, whose partial cases are dynamical systems on lattices and cell automata [14, p. 15].

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ПРО РОЛЬ МЕТОДУ ІНТЕГРАЛЬНИХ МНОГОВИДІВ БОГОЛЮБОВА В БІФУРКАЦІЙНОМУ АНАЛІЗІ БАГАТОВИМІРНИХ НЕЛІНІЙНИХ МОДЕЛЕЙ

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Р е з ю м е

Показано, що результати Боголюбова щодо редукції багатовимірної динамічної системи на її інтегральному многовиді становлять фундамент теорії центрального многовиду, яка разом з нормальними формами є потужним апаратом біфуркаційного аналізу багатовимірних нелінійних моделей.