SPECTRA OF GAUDIN QUANTUM INTEGRABLE MODELS AND A DISTRIBUTION OF ZEROS OF POLYNOMIALS

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The spectra of Gaudin quantum integrable models is defined by solving the Richardson algebraic equations. Under the narrow-band assumption, the solution of the Richardson equations is presented in terms of a distribution of zeros of the scaled Laguerre polynomial in a rational case and the scaled Jacobi polynomial in a general case (rational, trigonometric, and hyperbolic). The asymptotic limit of the distribution of zeros for Laguerre and Jacobi polynomials is studied, and the spectral density for Gaudin models is calculated.

1. Introduction

In 1957, N.N. Bogolyubov with students [1] proved the integrability of the BCS Hamiltonian in the thermodynamic limit. In 1963, R.W. Richardson [2] proved the integrability of the BCS pairing Hamiltonian for a finite number of particles. Later on, M.Gaudin [3, 4] developed the appropriate mathematical theory and proposed a lot of integrable models which are of interest for a number of physical problems.

In this paper, we discuss a spectrum of integrable Gaudin models which is defined by the Richardson equations

$$2\sum_{\beta=1,\beta\neq\alpha}^{M}\frac{1}{\omega_\beta-\omega_\alpha}=\sum_{l=1}^{N}\frac{1}{u_l-\omega_\alpha}-\frac{1}{G}.$$

Here, the variables u_l, $l=1,\dots,N$, are eigenvalues of the Hamiltonian without interaction, the variables ω_α, $\alpha=1,\dots,M$, are eigenvalues of the Hamiltonian with interaction, and G is an interaction constant of the Hamiltonian. (For details see, e.g., [5]).

A structure of the paper is as follows. In Section 1, we obtain a solution of the Richardson equations under "the narrow-band assumption" in the rational case in terms of a distribution of zeros of the scaled Laguerre polynomial. Section 2 presents a solution of the Richardson equations under "the narrow-band assumption" in the general (rational, trigonometric, and hyperbolic) case in terms of a distribution of zeros of the scaled Jacobi polynomial.

2. Solution of the Richardson Equations and Zeros of the Laguerre Polynomials

2.1. *The Richardson equations and the Laguerre polynomials*

Let us study a solution of the Richardson equations in the rational case under the assumption

$$u_l=0,\quad l=1,2,\dots,N,$$

which we call "the narrow-band assumption".

Theorem 2.1. *If, in the Richardson equations for the rational case,*

$$2\sum_{\beta=1,\beta\neq\alpha}^{M}\frac{1}{\omega_\beta-\omega_\alpha}=\sum_{l=1}^{N}\frac{1}{u_l-\omega_\alpha}-\frac{1}{G},$$

the conditions

$$u_l=0,\quad l=1,2,\dots,N,$$

are satisfied, then the Richardson equations have the solution

$$\omega_\alpha,\quad \alpha=1,2,\dots,M,$$

where ω_α are zeros of the generalized Laguerre polynomial

$$L_M^{AM}\left(M\frac{\omega_\alpha}{B}\right).$$

Here,

$$A=-\frac{N+1}{M},\quad B=MG=-\frac{1}{A}(N+1)G.$$

Proof. Let the Richardson equations in the rational case,

$$2\sum_{\beta=1,\beta\neq\alpha}^{M}\frac{1}{\omega_\alpha-\omega_\beta}+\sum_{l=1}^{N}\frac{1}{u_l-\omega_\alpha}-\frac{1}{G}=0,$$

satisfy the conditions

$$u_l=0,\quad l=1,2,\dots,N.$$

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Then these equations attain the form

$$2\sum_{\beta=1,\beta\neq\alpha}^{M}\frac{1}{\omega_\alpha-\omega_\beta}-\frac{N}{\omega_\alpha}-\frac{1}{G}=0.$$

According to last equalities, the polynomial

$$f(z)=\prod_{\beta=1}^{M}(z-\omega_\beta)$$

is a solution of the differential equation

$$z\frac{d^2f}{dz^2}-\left(\frac{z}{G}+N\right)\frac{df}{dz}+\frac{M}{G}f=0.$$

A unique polynomial solution of this equation is the scaled Laguerre polynomial

$$y(z)=L_M^{-(N+1)}\left(\frac{z}{G}\right).$$

It is convenient to introduce new parameters A, B, and g in the following way:

$$A=-\frac{N+1}{M},\quad B=MG=-\frac{1}{A}(N+1)G=-\frac{g}{A}.$$

Then the polynomial

$$p_M(z)=L_M^{AM}\left(M\frac{z}{B}\right)$$

satisfies the differential equation

$$z\frac{d^2p_M}{dz^2}-\left(\frac{M}{B}z-AM-1\right)\frac{dp_M}{dz}+\frac{M^2}{B}p_M=0.$$

□

In what follows, we will study a distribution of zeros of the polynomial

$$p_M(z)=C\prod_{j=1}^{M}(z-a_j)$$

at the limit $M\to\infty$ with parameters A, B, and g being constants. The parameter B defines just a scale of the variable x, and there is no problem to take it into account. Therefore, we set further $B=1$ in order to simplify the calculations.

2.2. Basic facts on the Laguerre polynomials

We recall that the Laguerre polynomial of the n-th order,

$$L_n^{(\alpha)}(z)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-z)^k}{k!},$$

is a polynomial solution of the second-order linear differential equation

$$zy''(z)+(\alpha+1-z)y'(z)+ny(z)=0.$$

Since $L_n^{(\alpha)}(z)$ is also a polynomial of the order n in the variable α, it is an analytical function in this variable. Let us recall some facts on zeros of the Laguerre polynomials (G. Szegö [6]). If $-1<\alpha$, then all n zeros of $L_n^{(\alpha)}(z)$ are simple and located on the real half-line $[0,+\infty)$. If $\alpha=-k\in\{-1,-2,\ldots,-n\}$, then we have a zero of the $k-$th order at $z=0$ and the other $(n-k)$ simple zeros which are located on the half-line $(0,+\infty)$. If $\alpha<-n$, then all n zeros are simple; if n is even, these zeros are all complex pairwise conjugated, if n is odd, these zeros are all complex pairwise conjugated but one. At $\alpha=-n-1$, the Laguerre polynomial is proportional to a partial sum for the function e^z,

$$L_n^{(-n-1)}(z)=(-1)^n\sum_{k=0}^{n}\frac{z^k}{k!},$$

and can be used for the polynomial approximations for the exponential function e^z :

$$e^z=\lim_{n\to\infty}(-1)^nL_n^{-n-1}(z).$$

Let us scale the independent variable, $z\to nz$, and introduce two power series in the scaled variable,

$$T_n(z)=(-1)^nL_n^{-n-1}(nz),$$

$$E_n(z)=e^{nz}-(-1)^nL_n^{-n-1}(nz).$$

Let $Z_n=Z(T_n)$ and $W_n=W(E_n)$ denote the sets of zeros of the polynomial $T_n(z)$ and the remainder term $E_n(z)$, respectively.

Theorem 2.2. *(G. Szegö [7]) As $n\to\infty$, the zeros Z_n of the polynomial $T_n(z)$ cluster on the curve $S_0\cap B$, and the zeros W_n of the transformed remainder term $E_n(z)$ cluster on the curve $S_0\backslash B$, where the curve*

$$S_0=\{z:|ze^{1-z}|=1\}$$

is called the Szegö curve and

$$B=\{z:|z|<1\}$$

is an open disc.

For recent publications on the Szegö curve, see [8].

2.3. The limit zero distribution of scaled Laguerre polynomials

Here, we study the zero distribution of scaled Laguerre polynomials.

Theorem 2.3. *Let us consider a sequence of scaled Laguerre polynomials,*

$$p_n(z) = L_n^{(\alpha_n)}(nz),$$

and a sequence of their zero measures,

$$\mu_n = \frac{1}{n} \sum_{p_n(z)=0} \delta_z.$$

Then, as $n \to \infty$ and under the condition

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = A$$

there exists a limit measure

$$\mu = \lim_{n\to\infty} \mu_n$$

and a support of this measure Γ which is a simple analytic arc, symmetric with respect to $\mathbb{R}$. These quantities are described in terms of the parameters

$$\xi_\pm = A + 2 \pm 2\sqrt{A+1},$$

and the polynomial

$$D(z) = (z - \xi_+)(z - \xi_-).$$

More precisely,

1) if $A > 0$, then $\xi_-, \xi_+ \in \mathbb{R}$ and $\Gamma = [\xi_-, \xi_+] \subset \mathbb{R}$;

2) if $A < -1$, then $\xi_- = \bar{\xi}_+$, and Γ is given by the equation

$$\mathrm{Re} \int_{\xi_-}^{z} \frac{\sqrt{D(t)}}{t} dt = 0$$

which, when computed explicitly, yields

$$\left| \frac{z^A e^{\sqrt{D(z)}}}{4(A+1)} \frac{[z - \sqrt{D(z)} - (A+2)]^{A+2}}{[A(A+\sqrt{D(z)}) - z(A+2)]^A} \right| = 1;$$

3) if $A = -1$, then $\xi_- = \xi_+ = 1$, and Γ is the Szegö curve defined by the equation

$$\left| z e^{1-z} \right| = 1, \quad |z| \le 1,$$

which is a closed curve around the origin passing through 1 and a point in $(-\infty, 0)$.

In all cases above, the zero distribution measure μ is absolutely continuous with respect to the linear Lebesgue measure on Γ and

$$d\mu(z) = \frac{1}{\pi i} \frac{\sqrt{D(z)}}{z} dz, \quad z \in \Gamma;$$

4) if $A \in (-1, 0)$ and there exists the limit

$$\lim_{n\to\infty} [\mathrm{dist}(\alpha_n, \mathbb{Z})]^{1/n} = 0,$$

then the zeros of $L_n^{\alpha_n}(nz)$ accumulate on $\Gamma = \{0\} \cup [\xi_-, \xi_+]$, and the asymptotic zero distribution measure is

$$d\mu(x) = -A\delta_0 + \frac{\sqrt{(x-\xi_-)(\xi_+ - x)}}{2\pi x} \chi_{[\xi_-, \xi_+]} dx;$$

5) if $A \in (-1, 0)$ and there exists the limit

$$\lim_{n\to\infty} [\mathrm{dist}(\alpha_n, \mathbb{Z})]^{1/n} = e^{-r}, \quad 0 \le r < \infty,$$

then the zeros of $L_n^{\alpha_n}(nz)$ accumulate on $\Gamma = \Gamma_r \cup [\xi_-, \xi_+]$ and the asymptotic zero distribution measure is

$$d\mu_r = d\nu_r(s) + \frac{\sqrt{(x-\xi_-)(\xi_+ - x)}}{2\pi x} \chi_{[\xi_-, \xi_+]} dx,$$

$$d\nu_r(s) = \frac{1}{2\pi i} \frac{\sqrt{D(s)}}{s} ds, \quad s \in \Gamma_r,$$

$$\mathrm{Re} \int_{\xi_-}^{z} \frac{\sqrt{D(s)}}{s} ds = r, \quad z \in \Gamma_r.$$

The case $r = 0$ represents the typical one in the sense that if a sequence $\{\alpha_n\}$ is chosen randomly, then, with probability one, the following equality is valid:

$$\lim_{n\to\infty} [\mathrm{dist}(\alpha_n, \mathbb{Z})]^{1/n} = 1.$$

So, the zeros cluster on $\Gamma_0 \cup [\xi_-, \xi_+] inthetypicalcase$. The case $0 < r < \infty$ is more special, since the members of the sequence $\{\alpha_n\}$ should be very close to integers.

In this theorem, results 1)–3) are due to A. Martinez-Finkelshtein, P. Martinez-Gonzalez, R. Orive [9] and results 4), 5) are due to A.B.J. Kuijlaars, K.T-R. McLaughlin [10,11].

Since these results were obtained by different techniques, it is reasonable to present their proof in a single manner [12].

The proof of the theorem is the result of the following lemmas.

Lemma 2.3.1. *The Cauchy transform*

$$\hat{\mu}(z) = \int \frac{d\mu(t)}{z-t}$$

of the limit zero distribution measure $\mu(z)$ for the Laguerre polynomials $p_n(z) = L_n^{(\alpha_n)}(nz)$ looks as

$$\hat{\mu}(z) = \frac{1}{2} - \frac{A}{2z} - \frac{\sqrt{(z-A)^2 - 4z}}{2z} =$$

$$= \frac{1}{2} - \frac{A}{2z} - \frac{\sqrt{(z-z_+)(z-z_-)}}{2z},$$

$$R(z) = \sqrt{(z-A)^2 - 4z} = \sqrt{(z-z_+)(z-z_-)},$$

$$z_\pm = A + 2 \pm 2\sqrt{A+1}.$$

Proof. We derive an expression for the Cauchy transform of the limit zero distribution $\mu(z)$ directly from the differential equation for the Laguerre polynomials.

The polynomial

$$p_n(z) = L_n^{(An)}(nz) = C \prod_{j=1}^{n} (z - a_j)$$

is a solution of the differential equation

$$\frac{z}{n} p_n''(z) + \left(A - z + \frac{1}{n} \right) p_n'(z) + n p_n(z) = 0.$$

The zero distribution measure is

$$\mu_{p_n}(z) = \frac{1}{n} \sum_{p_n(z)=0} \delta(z),$$

and the Cauchy transform for the zero distribution measure is

$$\hat{\mu}_{p_n}(z) = \int \frac{d\mu_{p_n}(t)}{z-t}.$$

The expression of the Cauchy transform for the zero distribution measure in terms of the logarithmic derivative of the polynomial is

$$\frac{1}{n}\frac{p_n'(z)}{p_n(z)} = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{z-a_j} = \int \frac{d\mu_{p_n}(t)}{z-t} = \hat{\mu}_{p_n}(z).$$

The differential equation for the Cauchy transform of the zero distribution measure looks as

$$z\left(\frac{\hat{\mu}_{p_n}'(z)}{n} + \hat{\mu}_{p_n}^2(z) \right) + \left(A - z + \frac{1}{n} \right) \hat{\mu}_{p_n}(z) + 1 = 0.$$

Now let us consider the limit $n \to \infty$ under assumption that A is constant, $|\hat{\mu}_{p_n}(z)|$, $|\hat{\mu}_{p_n}'(z)|$ are uniformly bounded. Then, for the quantity

$$\lim_{n\to\infty} \hat{\mu}_{p_n}(z) = \hat{\mu}(z),$$

we obtain the algebraic quadratic equation

$$z\hat{\mu}^2(z) + (A - z)\hat{\mu}(z) + 1 = 0.$$

Therefore,

$$\hat{\mu}(z) = \frac{z - A \pm \sqrt{(z-z_+)(z-z_-)}}{2z},$$

$$z_\pm = A + 2 \pm 2\sqrt{A+1}.$$

Since

$$\lim_{z\to\infty} \frac{\sqrt{(z-z_+)(z-z_-)}}{z} = 1, \quad \lim_{z\to\infty} z\hat{\mu}(z) = 1,$$

the expression for $\hat{\mu}(z)$ has to have negative sign against the square root.

□

Lemma 2.3.2. *The limit zero distribution measure looks as*

$$d\mu(t) = \frac{1}{\pi i} \frac{D(t)}{t} dt, \quad t \in \Gamma.$$

Proof. The limit zero distribution measure $\mu(t)$ defines, by means of the Cauchy-type integral, a piecewise analytic function

$$\tilde{\mu}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\mu(t)}{t-z} dt, \quad z \in \mathbb{C}\backslash\Gamma.$$

This function $\tilde{\mu}(z)$ is analytic on the complex plane everywhere except for the integration contour,

$$\tilde{\mu}(z) \in A(\mathbb{C}\backslash\Gamma).$$

In terms of the piecewise analytic function $\tilde{\mu}(z)$, the limit zero distribution measure $\mu(t)$ is defined by means of the Sokhotskii–Plemelj formulae,

$$\mu(t) = \tilde{\mu}^+(z) - \tilde{\mu}^-(z),$$

where $\tilde{\mu}^{+}(z)$ and $\tilde{\mu}^{-}(z)$ are values of the function $\tilde{\mu}(z)$ on two sides of the integration contour Γ.

In the case under consideration,

$$\tilde{\mu}(z) = -\frac{1}{\pi i}\left(\frac{1}{2} - \frac{A}{2z} - \frac{\sqrt{D(z)}}{2z}\right),$$

$$D(z) = (z-\xi_-)(z-\xi_+), \quad \xi_\pm = A+2 \pm 2\sqrt{A+1}.$$

If the contour Γ consists of a single arc or a finite number of separate arcs, then

$$d\mu(t) = \frac{1}{\pi i}\frac{D(t)}{t}dt, \quad t \in \Gamma.$$

□

Lemma 2.3.3. *If $A < -1$, then $\xi_- = \bar{\xi}_+$ and Γ is given by the equation*

$$\mathrm{Re}\int\limits_{\xi_-}^{z} \frac{\sqrt{D(t)}}{t}dt = 0$$

which, when computed explicitly, yields

$$\Big|\frac{z^A e^{\sqrt{D(z)}}}{4(A+1)}\frac{[z-\sqrt{D(z)}-(A+2)]^{A+2}}{[A(A+\sqrt{D(z)})-z(A+2)]^A}\Big| = 1.$$

Proof. The integral is equal

$$\int\limits_{\xi_-}^{z}\frac{\sqrt{D(t)}}{t}dt = \ln\left\{\frac{[A^2-(A+2)z+A\sqrt{D(z)}]^A}{(-z)^A\, e^{\sqrt{D(z)}}}\times\right.$$

$$\left.\times\left(\frac{\xi_-}{\xi_+}\right)^A\frac{4(A+1)}{[z-(A+2)-\sqrt{D(z)}]^{A+2}}\right\},$$

where

$$D(z) = (z-\xi_-)(z-\xi_+).$$

The curve is defined by the equation

$$\mathrm{Re}\int\limits_{\xi_-}^{z} \frac{\sqrt{D(t)}}{t}dt = 0$$

which is equivalent to the equation

$$\Big|\frac{[A^2-(A+2)z+A\sqrt{D(z)}]^A}{(-z)^A e^{\sqrt{D(z)}}}\left(\frac{\xi_-}{\xi_+}\right)^A\times$$

$$\times\frac{4(A+1)}{[z-(A+2)-\sqrt{D(z)}]^{A+2}}\Big| = 1.$$

With regard for the equalities

$$|(\xi_-/\xi_+)| = 1,$$

we obtain finally

$$\Big|\frac{[A(A+\sqrt{D(z)})-z(A+2)]^A}{[z-\sqrt{D(z)}-(A+2)]^{A+2}}\;\frac{4(A+1)}{z^A e^{\sqrt{D(z)}}}\Big| = 1.$$

□

Lemma 2.3.4. *If $A=-1$, then $\xi_- = \xi_+ = 1$ and Γ is the Szegö curve defined by the equation*

$$\left|ze^{1-z}\right| = 1, \quad |z| \le 1,$$

which is a closed curve around the origin passing through 1 and a point in $(-\infty, 0)$.

Proof. The Szegö curve is the curve of the previous lemma, taken at the limit $A \to -1$.

□

The results of consideration in the interval $A \in (-1,0)$ depend essentially on the way, which the sequence $\{\alpha_n\}$ tends to an integer with.

Lemma 2.3.5. *If $A \in (-1,0)$ and there exists the limit*

$$\lim_{n\to\infty}[\mathrm{dist}(\alpha_n,\mathbb{Z})]^{1/n} = 0,$$

then the zeros of $L_n^{\alpha_n}(nz)$ accumulate on $\Gamma = \{0\} \cup [\xi_-,\xi_+]$, and the asymptotic zero distribution measure is

$$d\mu(x) = -A\delta_0 + \frac{\sqrt{(x-\xi_-)(\xi_+-x)}}{2\pi x}\chi_{[\xi_-,\xi_+]}dx.$$

If $A \in (-1,0)$ and there exists the limit

$$\lim_{n\to\infty}[\mathrm{dist}(\alpha_n,\mathbb{Z})]^{1/n} = e^{-r}, \quad 0 \le r < \infty,$$

then the zeros of $L_n^{\alpha_n}(nz)$ accumulate on $\Gamma = \Gamma_r \cup [\xi_-,\xi_+]$, and the asymptotic zero distribution measure is

$$d\mu_r = d\nu_r(s) + \frac{\sqrt{(x-\xi_-)(\xi_+-x)}}{2\pi x}\chi_{[\xi_-,\xi_+]}dx,$$

$$d\nu_r(s) = \frac{1}{2\pi i}\frac{\sqrt{D(s)}}{s}ds, \quad s \in \Gamma_r,$$

$$\mathrm{Re}\int\limits_{\xi_-}^{z}\frac{\sqrt{D(s)}}{s}ds=r,\quad z\in\Gamma_r.$$

The case $r=0$ is the typical one. If a sequence $\{\alpha_n\}$ is chosen randomly, then, with probability one, the equality

$$\lim_{n\to\infty}[\mathrm{dist}(\alpha_n,\mathbb{Z})]^{1/n}=1$$

is valid. In this case, the zeros cluster on $\Gamma_0\cup[\xi_-,\xi_+]$.

The case $0<r<\infty$ is much more special. In this case, the sequence $\{\alpha_n\}$ should approximate some integer very closely.

Proof. Multiple zeros of the polynomial $y(z)=L_n^{(\alpha)}(z)$ occur only at the point $z=0$. Indeed, if $z_0\neq 0$ at some point, we assume $y(z_0)=y'(z_0)=0$. Then, according to the differential equation

$$zy''(z)+(\alpha+1-z)y'(z)+ny(z)=0,$$

we have $y^{(k)}(z)=0$, $k=0,1,2,\dots$, and, therefore, $y(z)=0$.

If $\alpha=-k\in\{-1,-2,\dots,-n\}$, then the polynomial $L_n^{(\alpha)}(z)$ has a zero of the order k at the point $z=0$. All the other $(n-k)$ zeros are simple and are located on the half-line $(0,+\infty)$. It is obvious due to the relation

$$L_n^{(-k)}(z)=(-z)^k\frac{(n-k)!}{n!}L_{n-k}^{(k)}(z).$$

If the real parameter α tends from above to the integer $-k$, then k simple zeros of the $L_n^{(\alpha)}(z)$ approach the point $z=0$ in the directions $(+1)^{1/k}$. If the real parameter α decreases further below the integer $-k$, then k simple zeros of the $L_n^{(\alpha)}(z)$ emerge from the point $z=0$ in the directions $(-1)^{1/k}$. It is similar to the behavior of the hypergeometric function (see [13]).

Now let us consider a sequence of rational numbers $\{\alpha_n\}$. If the sequence $\{\alpha_n\}$ converges to an integer in such a way that there exists the limit

$$\lim_{n\to\infty}[\mathrm{dist}(\alpha_n,\mathbb{Z})]^{1/n}=0,$$

then zeros of $L_n^{\alpha_n}(nz)$ accumulate on $\Gamma=\{0\}\cup[\xi_-,\xi_+]$, and the asymptotic zero distribution measure is

$$d\mu(x)=-A\delta_0+\frac{\sqrt{(x-\xi_-)(\xi_+-x)}}{2\pi x}\chi_{[\xi_-,\xi_+]}dx.$$

If the sequence $\{\alpha_n\}$ converges to an integer in the way

$$\lim_{n\to\infty}[\mathrm{dist}(\alpha_n,\mathbb{Z})]^{1/n}=e^{-r},\quad 0\le r<\infty,$$

then, using the asymptotics for the scaled Laguerre polynomial $L_n^{\alpha_n}(nz)$, we prove easily that its zeros accumulate on the curve

$$\Gamma=\Gamma_r\cup[\xi_-,\xi_+],\quad \mathrm{Re}\int\limits_{\xi_-}^{z}\frac{\sqrt{D(s)}}{s}ds=r,\quad z\in\Gamma_r,$$

and the appropriate asymptotic zero distribution measure is

$$d\mu_r=d\nu_r(s)+\frac{\sqrt{(x-\xi_-)(\xi_+-x)}}{2\pi x}\chi_{[\xi_-,\xi_+]}dx,$$

$$d\nu_r(s)=\frac{1}{2\pi i}\frac{\sqrt{D(s)}}{s}ds,\quad s\in\Gamma_r.$$

We skip details of these calculations.

□

2.4. Asymptotic solution of the Richardson equations in rational case

Applying the results of the Theorem for the zero distribution of the scaled Laguerre polynomials to solutions of the Richardson equations in rational case, we should take into account that, first of all, $N\ge M>0$ and, therefore, $A\le -1$. This means that the zero distribution measure of the Richardson equations is of the form

$$d\mu(z)=\frac{1}{\pi i}\frac{\sqrt{D(z)}}{z}dz,\quad z\in\Gamma,$$

and is defined on the curve Γ which is given by the equation

$$\mathrm{Re}\int\limits_{\xi_-}^{z}\frac{\sqrt{D(t)}}{t}dt=0,$$

or, which is the same, by the equation

$$\left|\frac{z^Ae^{\sqrt{D(z)}}}{4(A+1)}\frac{[z-\sqrt{D(z)}-(A+2)]^{A+2}}{[A(A+\sqrt{D(z)})-z(A+2)]^A}\right|=1,$$

where

$$D(z)=(z-\xi_+)(z-\xi_-),\quad \xi_\pm=A+2\pm2\sqrt{A+1}.$$

3. Solution of the Richardson Equations and Zeros of the Jacobi Polynomials

3.1. The Richardson equations and the Jacobi polynomials

Let us consider the Richardson equations in the general case,

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} q\coth[q(\omega_\alpha-\omega_\beta)] = \sum_{j=1}^{N} q\coth[(\omega_\alpha-u_j)] + \frac{1}{G},$$

where $q=0$ corresponds to rational case, $q=i$ corresponds to trigonometric case, and $q=1$ corresponds to hyperbolic case. We study a solution of the Richardson equations under "the narrow-band assumption",

$$u_l = \text{const.},\quad l=1,2,\ldots,N.$$

Theorem 3.1. *If, in the Richardson equations*

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} q\coth[q(\omega_\alpha-\omega_\beta)] = \sum_{j=1}^{N} q\coth[(\omega_\alpha-u_j)] + \frac{1}{G},$$

the conditions

$$u_j = (1/2q)\ln 2,\quad j=1,2,\ldots,N,$$

are satisfied, then the Richardson equations have the solution

$$\omega_\alpha = (1/2q)\ln(x_\alpha+1),\quad \alpha=1,2,\ldots,M,$$

where x_α are zeros of a Jacobi polynomial,

$$P_M^{-(N+1),-(L+1)}(x_\alpha)=0.$$

Here,

$$L=\frac{1}{2qG}-\frac{N}{2}+M-1.$$

Proof. For the Richardson equations in general case,

$$2\sum_{\beta=1,\beta\neq\alpha}^{M} q\coth q(\omega_\alpha-\omega_\beta)+$$

$$+\sum_{l=1}^{N} q\coth(u_l-\omega_\alpha)-\frac{1}{G}=0,$$

let us introduce new variables

$$x_\alpha=\exp(2q\omega_\alpha)-1,\quad \alpha=1,2,\ldots,M,$$

$$\zeta_l=\exp(2qu_l),\quad l=1,2,\ldots,N,$$

and present these equations as

$$2\sum_{\beta=1,\beta\neq\alpha}^{M}\frac{1}{x_\alpha-x_\beta}+\sum_{l=1}^{N}\frac{1}{\zeta_l-x_\alpha-1}-\frac{L}{x_\alpha+1}=0,$$

where

$$L=\frac{1}{2qG}-\frac{N}{2}+M-1.$$

If we assume

$$\zeta_l=2,\quad l=1,2,\ldots,N,$$

then the equations attain the form

$$2\sum_{\beta=1,\beta\neq\alpha}^{M}\frac{1}{x_\alpha-x_\beta}-\frac{N}{x_\alpha-1}-\frac{L}{x_\alpha+1}=0.$$

As a result of the last equalities, the polynomial

$$f(x)=\prod_{\beta=1}^{M}(x-x_\beta),\quad x_\beta=\exp(2q\omega_\beta)-1$$

satisfies the differential equation

$$(1-x^2)\frac{d^2f}{dx^2}+[(N-L)+(N+L)x]\frac{df}{dx}+$$

$$+M(M-N-L-1)f=0.$$

The Jacobi polynomial

$$y(x)=P_n^{(a,b)}(x)$$

satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+[(b-a)-(a+b+2)x]\frac{dy}{dx}+$$

$$+n(n+a+b+1)y=0.$$

Comparing these equations, we get

$$a=-(N+1),\quad b=-(L+1),\quad n=M,$$

and, therefore,

$$f(x)=P_M^{-(N+1),-(L+1)}(x).$$

□

3.2. Basic facts on the Jacobi polynomials

The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is a polynomial of the n-th order in the variable z,

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k},$$

which is a solution of the second-order linear differential equation,

$$(1-z^2)y''(z) + [\beta - \alpha - (\alpha+\beta+2)z]y'(z)+$$

$$+n(n+\alpha+\beta+1)y(z) = 0,$$

and $P_n^{(\alpha,\beta)}(z)$ is also a polynomial in the variables α, β.

Now we recall some facts on zeros of the Jacobi polynomials [6]. Let α and β be arbitrary real numbers, and $n \geq 1$. The polynomial $P_n^{(\alpha,\beta)}(z)$ can have a multiple zero at $z = 1$ if $\alpha = -1, -2, \cdots, -n$, at $z = -1$ if $\beta = -1, -2, \cdots, -n$ or at $z = \infty$ (which means a degree reduction) if $n + \alpha + \beta = -1, -2, \cdots, -n$. We exclude these zeros from the further consideration. All other zeros are different with $+1, -1, \infty$ and with themselves. Let us define values $N_1(\alpha,\beta)$, $N_2(\alpha,\beta)$ and $N_3(\alpha,\beta)$ as the numbers of zeros of the Jacobi polynomials on the segments of the real line $(-1,+1)$, $(-\infty,-1)$, and $(+1,+\infty)$, respectively. Then

$$N_1(\alpha,\beta) =$$

$$= \begin{cases} 2\left[\frac{X_1(\alpha,\beta)+1}{2}\right] \text{if} (-1)^n \binom{n+\alpha}{n}\binom{n+\beta}{n} > 0, \\ 2\left[\frac{X_1(\alpha,\beta)}{2}\right] + 1 \text{if} (-1)^n \binom{n+\alpha}{n}\binom{n+\beta}{n} < 0, \end{cases}$$

$$N_2(\alpha,\beta) =$$

$$= \begin{cases} 2\left[\frac{X_2(\alpha,\beta)+1}{2}\right] \text{if} \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n} > 0, \\ 2\left[\frac{X_2(\alpha,\beta)}{2}\right] + 1 \text{if} \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n} < 0, \end{cases}$$

$$N_3(\alpha,\beta) =$$

$$= \begin{cases} 2\left[\frac{X_3(\alpha,\beta)+1}{2}\right] \text{if} \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n} > 0, \\ 2\left[\frac{X_3(\alpha,\beta)}{2}\right] + 1 \text{if} \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n} < 0. \end{cases}$$

Here,

$$X_1(\alpha,\beta) = E\left\{\frac{1}{2}(|2n+\alpha+\beta+1| - |\alpha| - |\beta| + 1)\right\},$$

$$X_2(\alpha,\beta) = E\left\{\frac{1}{2}(-|2n+\alpha+\beta+1| + |\alpha| - |\beta| + 1)\right\},$$

$$X_3(\alpha,\beta) = E\left\{\frac{1}{2}(-|2n+\alpha+\beta+1| - |\alpha| + |\beta| + 1)\right\},$$

and $E\{u\}$ is the Klein symbol,

$$E\{u\} = \begin{cases} 0 & \text{if } u \leq 0, \\ [u] & \text{if } u > 0 \text{ and } u \text{ is noninteger}, \\ u-1 & \text{if } u = 1, 2, \cdots. \end{cases}$$

Therefore, the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ has $N_1(\alpha,\beta) + N_2(\alpha,\beta) + N_3(\alpha,\beta)$ real zeros, all other zeros are complex-valued.

3.3. The limit zero distribution of the scaled Jacobi polynomials

Here, we study the zero distribution of the scaled Jacobi polynomials.

Theorem 3.2. *Let us consider, at the limit $n \to \infty$, a sequence of Jacobi polynomials,*

$$p_n(z) = P_n^{\alpha_n,\beta_n}(z),$$

and a sequence of their zero measures,

$$\mu_n = \frac{1}{n} \sum_{p_n(z)=0} \delta_z,$$

under condition

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = A, \quad \lim_{n\to\infty} \frac{\beta_n}{n} = B.$$

Then there exists a weak limit measure

$$\mu = \lim_{n\to\infty} \mu_n,$$

and the support of this measure is a simple analytic arc Γ symmetric with respect to $\mathbb{R}$.

1) If the parameters A, B satisfy one of the three conditions, $A > 0$, $B > 0$, or $A > 0$, $A + B < -2$, or $B > 0$, $A + B < -2$, then $\Gamma = [\zeta_-, \zeta_+] \subset \mathbb{R}$, where

$$\zeta_\pm = \frac{B^2 - A^2 \pm 4\sqrt{(A+1)(B+1)(A+B+1)}}{(A+B+2)^2};$$

2) If the parameters A, B satisfy one of the three conditions, $A < -1,\ B < -1$ or $A < -1,\ A + B > -1$, or $B < -1,\ A + B > -1$, then Γ is a set of $z \in \mathbb{C}$, defined by the equation

$$\mathrm{Re} \int\limits_{\zeta_-}^{z} \frac{R(t)}{t^2 - 1} dt = 0, \quad \mathrm{Re}\, z > \mathrm{Re}\, \zeta_\pm,$$

$$R(z) = \sqrt{(z - \zeta_-)(z - \zeta_+)}.$$

In both cases, the measure of zeros is

$$d\mu(z) = \frac{A + B + 2}{2\pi i} \frac{R_+(z)}{1 - z^2}\, dz, \quad z \in \Gamma.$$

3) If the parameters A, B satisfy one of the three conditions, $-1 < A < 0$, or $-1 < B < 0$ or $-2 < A + B < -1$, and the condition

$$\lim_{n \to \infty} [\mathrm{dist}(\alpha_n, \mathbb{Z})]^{1/n} = e^{-r}, \quad 0 \le r \le +\infty,$$

holds for some $0 \le r \le +\infty$, then the support of the measure is

$$[\zeta_-, \zeta_+] \cup \Gamma_r,$$

where Γ_r is a set of $z \in \mathbb{C}$ defined by the equation

$$\mathrm{Re} \int\limits_{\zeta_+}^{z} \frac{R(t)}{t^2 - 1} dt = \frac{r}{A + B + 2}, \quad \mathrm{Re}\, z > \mathrm{Re}\, \zeta_+,$$

$$R(z) = \sqrt{(z - \zeta_-)(z - \zeta_+)}.$$

For each $r \in [0, \infty)$, the measure is

$$d\mu_r(z) = \frac{A + B + 2}{2\pi i} \frac{R_+(z)}{1 - z^2}\, dz,$$

and, for $r = \infty$, the measure is

$$d\mu_\infty(z) = -A\delta_1 + \frac{A + B + 2}{2\pi i} \frac{R_+(z)}{1 - z^2}\, \chi_{[\zeta_-, \zeta_+]}\, dz.$$

This theorem is due to A. Martinez-Finkelshtein, R. Orive [14], A.B.J. Kuijlaars, A. Martinez-Finkelshtein [15], and A.B.J. Kuijlaars, A. Martinez-Finkelshtein, R. Orive [16]. Like the previous section, the proof of the theorem can be presented as a result of similar lemmas.

Lemma 3.2.1. *The Cauchy transform*

$$\hat{\mu}(z) = \int \frac{d\mu(t)}{z - t}$$

of the limit zero distribution measure $\mu(z)$ for the Jacobi polynomials $p_n(z) = P_n^{(\alpha_n, \beta_n)}(nz)$ looks as

$$\hat{\mu}(z) = -\frac{A}{2(z-1)} - \frac{B}{2(z+1)} +$$

$$+ \frac{A + B + 2}{2(z^2 - 1)} \sqrt{(z - z_+)(z - z_-)},$$

$$z_\pm = \frac{1}{(A + B + 2)^2} \times$$

$$\times \left[B^2 - A^2 \pm 4\sqrt{(A+1)(B+1)(A+B+1)} \right].$$

Proof. We derive an expression for the Cauchy transform of the limit zero distribution $\mu(z)$ directly from the differential equation for the Jacobi polynomials.

The polynomial

$$p_n(z) = P_n^{(\alpha_n, \beta_n)}(nz) = C \prod_{j=1}^{n} (z - a_j)$$

is a solution of the differential equation

$$p_n''(z) + \left(\frac{\alpha_n + 1}{z - 1} + \frac{\beta_n + 1}{z + 1} \right) p_n'(z) - \frac{\lambda_n}{z^2 - 1} p_n(z) = 0,$$

where $\lambda_n = n(n + \alpha_n + \beta_n + 1)$. The zero distribution measure is

$$\mu_{p_n}(z) = \frac{1}{n} \sum_{p_n(z) = 0} \delta(z),$$

and its Cauchy transform is

$$\hat{\mu}_{p_n}(z) = \int \frac{d\mu_{p_n}(t)}{z - t}.$$

The expression of the Cauchy transform for the zero distribution measure in terms of the logarithmic derivative of the polynomial is

$$\frac{1}{n} \frac{p_n'(z)}{p_n(z)} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - a_j} = \int \frac{d\mu_{p_n}(t)}{z - t} = \hat{\mu}_{p_n}(z).$$

The differential equation for the Cauchy transform of the zero distribution measure looks as

$$\frac{\hat{\mu}_{p_n}'(z)}{n} + \hat{\mu}_{p_n}^2(z) + \left(\frac{\alpha_n + 1}{z - 1} + \frac{\beta_n + 1}{z + 1} \right) \frac{\hat{\mu}_{p_n}(z)}{n} -$$

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$$-\frac{\lambda_n}{n^2}\frac{1}{z^2-1}=0.$$

Now let us consider the limit $n \to \infty$ under the assumptions: $|\hat{\mu}_{p_n}(z)|$, $|\hat{\mu}'_{p_n}(z)|$ are uniformly bounded, $\frac{\alpha_n}{n}=A$, $\frac{\beta_n}{n}=B$,

$$\frac{\lambda_n}{n^2}=\frac{n+\alpha_n+\beta_n+1}{n}=A+B+1+\frac{1}{n},$$

where A and B are constants. Then, for the quantity

$$\lim_{n\to\infty}\hat{\mu}_{p_n}(z)=\hat{\mu}(z),$$

we obtain the algebraic quadratic equation

$$\hat{\mu}^2(z)+\left(\frac{A}{z-1}+\frac{B}{z+1}\right)\hat{\mu}(z)-\frac{A+B+1}{z^2-1}=0.$$

Therefore,

$$\hat{\mu}(z)=-\frac{A}{2(z-1)}-\frac{B}{2(z+1)}\pm$$

$$\pm\frac{A+B+2}{2(z^2-1)}\sqrt{(z-z_+)(z-z_-)},$$

$$z_\pm=\frac{1}{(A+B+2)^2}\Big[B^2-A^2\pm$$

$$\pm4\sqrt{(A+1)(B+1)(A+B+1)}\Big].$$

Since

$$\lim_{z\to\infty}\frac{\sqrt{(z-z_+)(z-z_-)}}{z}=1,\quad \lim_{z\to\infty}z\hat{\mu}(z)=1,$$

the expression for $\hat{\mu}(z)$ has to have positive sign against the square root.

$\square$

The proof of the other lemmas is similar to that for rational case, and we skip them out.

3.4. Asymptotic solution of the Richardson equations in general case

Applying the results of the theorem for the asymptotic zero distribution of the scaled Jacobi polynomials $P_n^{(An,Bn)}(z)$ to solutions of the Richardson equations in general case, we should remember that, in fact, we consider the scaled Jacobi polynomials of the form

$$P_M^{-(N+1),-(L+1)}(z),$$

where

$$L=\frac{1}{2qG}-\frac{N}{2}+M-1,$$

under the asymptotic condition

$$M,N,G^{-1}\to\infty,\quad \lim_{M\to\infty}\frac{N}{M}=A,$$

$$\lim_{M\to\infty}\frac{L}{M}=\frac{1}{2qg}-\frac{A}{2}+1=B,\quad \lim_{M\to\infty}\frac{1}{GM}=g.$$

For rational ($q=0$) and hyperbolic ($q=1$) cases, the indices A and B are real. For trigonometric case ($q=i$), the index A is real, but the index B is complex-valued; this case deserves a special consideration, and we do not study it here.

Since $N \geq M > 0$, $A \leq -1$. The sign of B in rational and hyperbolic cases may be arbitrary.

4. Discussion

The constructed analytical solutions of the Richardson equations are in good agreement with the results of computer calculations obtained earlier [17]. Although our assumption on the parameters u_l (all u_l are equal) and the assumption in [17] (all u_l are distributed uniformly along a given interval of the real line) are different, the curves of the support for spectral measures are very similar in both cases. In Fig. 2 of work [17], the reader can even see the Szegö curve.

Applications of solutions of the Richardson equations to various integrable quantum systems deserve a special discussion and will be postponed to the other publication.

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СПЕКТРИ КВАНТОВИХ ІНТЕГРОВНИХ МОДЕЛЕЙ ГОДЕНА ТА РОЗПОДІЛ НУЛІВ ПОЛІНОМІВ

Є.Д. Білоколос

Р е з ю м е

Спектри квантових інтегровних моделей Годена визначаються розв'язком алгебраїчних рівнянь Річардсона. При припущенні вузької зони розв'язок рівнянь Річардсона представлено в термінах розподілу нулів масштабованого полінома Лагерра в раціональному випадку і масштабованого полінома Якобі в загальному випадку (раціональному, тригонометричному і гіперболічному). Досліджено асимптотичну межу поділу нулів поліномів Лагерра і Якобі та розраховано спектральну густину для моделей Годена.