APPROACHES TO DERIVATION OF QUANTUM KINETIC EQUATIONS

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We discuss possible approaches to the problem of the rigorous derivation of quantum kinetic equations from the underlying many-particle dynamics.

For the description of a many-particle evolution, we construct solutions of the Cauchy problems of the BBGKY hierarchy and the dual BBGKY hierarchy in suitable Banach spaces.

In the framework of the conventional approach to the description of a kinetic evolution, the mean-field asymptotics of the quantum BBGKY hierarchy solution is constructed. We develop also alternative approaches. One method is based on the construction of the asymptotics of a solution of the initial-value problem of the quantum dual BBGKY hierarchy. One more approach is based on the generalized quantum kinetic equation that is a consequence of the equivalence of the Cauchy problems of such evolution equation and the BBGKY hierarchy with initial data determined by the one-particle density operator.

1. Introduction

We develop a formalism suggested by Bogolyubov [1, 2] for the description of the evolution of infinitely many particles. The evolution equations of quantum many-particle systems arise in many problems of modern statistical mechanics [5]. In the theory of such equations during the last decade, many new results have been obtained, in particular concerning the fundamental problem of the rigorous derivation of quantum kinetic equations and, among them, the kinetic equations describing the Bose condensate [3, 10–13].

A description of quantum many-particle systems is formulated in terms of two sets of objects: observables and states. The mean value functional defines a duality between observables and states. As a consequence, there exist two approaches to the description of the evolution. Usually, the evolution of many-particle systems is described in the framework of the evolution of states by the BBGKY hierarchy for marginal density operators [1, 2, 4–6]. An equivalent approach to the description of the evolution of many-particle systems is given by the dual BBGKY hierarchy [5, 17] in the framework of the evolution of marginal observables.

The aim of this work is to consider links between the infinite-particle dynamics and quantum kinetic equations.

A conventional approach to the problem of the rigorous derivation of kinetic equations from the underlying many-particle dynamics consists in the construction of a suitable scaling limit [9], for instance, the Boltzmann–Grad limit or the mean-field limit [5, 15] of a solution of the initial-value problem of the BBGKY hierarchy. As a result, the solution limit is governed by the limit hierarchy preserving the chaos property, and the one-particle density operator satisfies the kinetic equation [10–13]. Here, we formulate new methods of solving the mentioned problem which are based on the description of a many-particle evolution by the dual BBGKY hierarchy.

We outline the structure of the paper and the main results.

In Section 2, we introduce some preliminary definitions and construct a solution of the Cauchy problem to the dual BBGKY hierarchy for marginal observables and the canonical BBGKY hierarchy for marginal density operators of quantum many-particle systems. We formulate also one more approach to the description of the quantum many-particle dynamics which is based on an equivalence of the Cauchy problem of the BBGKY hierarchy with initial data determined by the one-particle density operator and the corresponding initial value-problem for a generalized quantum kinetic equation.

In Section 3, the results obtained in the previous section are used to analyze the mean-field asymptotics of constructed solutions, in particular to derive a nonlinear Schrödinger equation and its generalizations. We formulate also new methods of the derivation of quantum kinetic equations from the underlying many-particle dynamics. One method is based on the study of the scaling limits of a solution of the initial-value problem of the dual BBGKY hierarchy. Another method is based on a generalized quantum kinetic equation.

Finally in Section 4, we conclude with some observations and perspectives for the future research.

2. Dynamics of Quantum Many-Particle Systems

We study possible approaches to the description of the evolution of quantum many-particle systems, namely the Heisenberg and Schrödinger pictures of the evolution. We introduce hierarchies of evolution equations for marginal observables and states and construct a solution of the Cauchy problems of these hierarchies in suitable Banach spaces. We develop also one more approach based on the generalized quantum kinetic equation that is a consequence of the equivalence of the Cauchy problems of such evolution equation and the BBGKY hierarchy for a certain class of initial data.

2.1. Dual BBGKY hierarchy

We will consider a quantum system of a non-fixed (i.e. arbitrary but finite [14]) number of identical (spinless) particles obeying the Maxwell–Boltzmann statistics in the space $\mathbb{R}^\nu$. We will use units where $h = 2\pi\hbar = 1$ is the Planck constant, and $m = 1$ is the mass of particles. The Hamiltonian of such a system $H = \bigoplus_{n=0}^{\infty} H_n$ is a self-adjoint operator with the domain $\mathcal{D}(H) = \{\psi = \oplus_{n=0}^{\infty}\psi_n \in \mathcal{F}_\mathcal{H} \mid \psi_n \in \mathcal{D}(H_n) \in \mathcal{H}_n, \sum_n \|H_n\psi_n\|^2 < \infty\} \subset \mathcal{F}_\mathcal{H}$, where $\mathcal{F}_\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}^{\otimes n}$ is the Fock space over the Hilbert space $\mathcal{H}$ $(\mathcal{H}^0 = \mathbb{C})$. Assume $\mathcal{H} = L^2(\mathbb{R}^\nu)$ (the coordinate representation); then an element $\psi \in \mathcal{F}_\mathcal{H} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^{\nu n})$ is a sequence of functions $\psi = \big(\psi_0, \psi_1(q_1), \ldots, \psi_n(q_1, \ldots, q_n), \ldots\big)$ such that $\|\psi\|^2 = |\psi_0|^2 + \sum_{n=1}^{\infty} \int dq_1 \ldots dq_n |\psi_n(q_1, \ldots, q_n)|^2 < +\infty$. On the subspace of infinitely differentiable functions with compact supports $\psi_n \in L_0^2(\mathbb{R}^{\nu n}) \subset L^2(\mathbb{R}^{\nu n})$, the n-particle Hamiltonian H_n acts according to the formula $(H_0 = 0)$

$$H_n\psi_n = \sum_{i=1}^{n} K(i)\psi_n + \epsilon \sum_{i<j=1}^{n} \Phi(i,j)\psi_n, \tag{1}$$

where $K(i)\psi_n = -\frac{1}{2}\Delta_{q_i}\psi_n$ is the operator of kinetic energy, $\Phi(i,j)\psi_n = \Phi(|q_i - q_j|)\psi_n$ is the operator of a two-body interaction potential satisfying the Kato conditions, and $\epsilon > 0$ is a scaling parameter.

Let a sequence $g = \big(I, g_1, \ldots, g_n, \ldots\big)$ be an infinite sequence of self-adjoint bounded operators g_n defined on the Fock space $\mathcal{F}_\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}^{\otimes n}$ over the Hilbert space $\mathcal{H}$ $(\mathcal{H}^0 = \mathbb{C}$, and I is a unit operator). An operator g_n defined in the n-particle Hilbert space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ will be denoted by $g_n(1, \ldots, n)$. For a system of identical particles obeying the Maxwell–Boltzmann statistics, one has $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$ for any permutation of indices $\{i_1, \ldots, i_n\} \in \{1, \ldots, n\}$.

Let the space $\mathfrak{L}(\mathcal{F}_\mathcal{H})$ be the space of sequences $g = \big(I, g_1, \ldots, g_n, \ldots\big)$ of bounded operators g_n (I is a unit operator) defined on the Hilbert space $\mathcal{H}_n$ and satisfying the symmetry property $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$, if $\{i_1, \ldots, i_n\} \in \{1, \ldots, n\}$, with an operator norm. We will also consider a more general space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ with a norm

$$\|g\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})} = \max_{n \geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)},$$

where $0 < \gamma < 1$ and $\|.\|_{\mathfrak{L}(\mathcal{H}_n)}$ is an operator norm. An observable of the many-particle quantum system is a sequence of self-adjoint operators from $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$.

On the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, we consider the initial-value problem of the dual BBGKY hierarchy.

The evolution of marginal observables is described by the initial-value problem for the following hierarchy of evolution equations:

$$\frac{\partial}{\partial t}G_s(t,Y) = \big(\sum_{i=1}^{s}\mathcal{N}_0(i) + \epsilon \sum_{i<j=1}^{s}\mathcal{N}_{\text{int}}(i,j)\big)G_s(t,Y) +$$

$$+\epsilon \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{N}_{\text{int}}(j_1, j_2) G_{s-1}(t, Y\backslash\{j_1\}), \tag{2}$$

$$G_s(t)\mid_{t=0} = G_s(0), \quad s \geq 1. \tag{3}$$

In Eqs. (2), we use the notation $Y \equiv (1, \ldots, s)$. The operators $\mathcal{N}_0, \mathcal{N}_{\text{int}}$ are consequently defined on $\mathcal{D}(\mathcal{N}_0) \subset \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ as follows:

$$\mathcal{N}_0(j)g = -i\big[g, K(i)\big], \tag{4}$$

$$\mathcal{N}_{\text{int}}(i,j)g = -i\big[g, \Phi(i,j)\big]. \tag{5}$$

Here, $[\,\cdot\,,\cdot\,]$ is a commutator of operators. We refer to the evolution equations (2) as the quantum dual BBGKY hierarchy, since the canonical BBGKY hierarchy [5] for marginal density operators $F(t)$ is the dual hierarchy of evolution equations with respect to the following bilinear form [17, 20]:

$$\big\langle G(t)\big|F(0)\big\rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s}\, G_s(t)F_s(0). \tag{6}$$

If $\mathcal{H} = L^2(\mathbb{R}^\nu)$, the evolution equations (2) in terms of the kernels of operators $G_s(t)$, $s \geq 1$, are given in the form of the equations

$$i\frac{\partial}{\partial t}G_s(t, q_1, \ldots, q_s; q'_1, \ldots, q'_s) =$$

$$= \Big(-\frac{1}{2}\sum_{i=1}^{s}(-\Delta_{q_i}+\Delta_{q'_i}) + +\epsilon\sum_{1=i<j}^{s}\big(\Phi(q'_i-q'_j)-$$

$$-\Phi(q_i-q_j)\big)\Big)G_s(t,q_1,\dots,q_s;q'_1,\dots,q'_s)+$$

$$+\epsilon\sum_{1=i\neq j}^{s}(\Phi(q'_i-q'_j)-\Phi(q_i-q_j))\times$$

$$\times G_{s-1}\big(t,q_1,\dots,q^j,\dots,q_s;q'_1,\dots,q^j,\dots,q'_s\big),$$

where $(q_1,\dots,q^j,\dots,q_s)\equiv(q_1,\dots,q_{j-1},q_{j+1},\dots,q_s)$.

To construct a solution of the abstract initial-value problem (2), (3), we introduce some necessary facts.

If $g\in\mathfrak{L}(\mathcal{F}_\mathcal{H})$, we define the group $\mathcal{G}(t)=\oplus_{n=0}^{\infty}\mathcal{G}_n(t)$ of operators

$$\mathcal{G}_n(t)g_n = e^{itH_n}\,g_n\,e^{-itH_n}. \tag{7}$$

This group of operators is defined by a solution of the initial-value problem of the Heisenberg equation for observables of quantum many-particle systems.

On the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, the one-parameter mapping $\mathbb{R}^1\ni t\mapsto\mathcal{G}(t)g$ defines an isometric $*$-weak continuous group of operators, i.e. it is a C_0^*-group. The infinitesimal generator $\mathcal{N}=\bigoplus_{n=0}^{\infty}\mathcal{N}_n$ of the group of operators (7) is a closed operator for the $*$-weak topology. On its domain of the definition $\mathcal{D}(\mathcal{N})\subset\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, which is everywhere dense for the $*$-weak topology, $\mathcal{N}$ is defined in the sense of the $*$-weak convergence of the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ as follows:

$$\mathrm{w}^*\!-\lim_{t\to0}\frac{1}{t}\big(\mathcal{G}(t)g-g\big)=i(Hg-gH)\equiv\mathcal{N}g. \tag{8}$$

Here, $H=\bigoplus_{n=0}^{\infty}H_n$ is Hamiltonian (1) of the many-particle system, and the operator: $\mathcal{N}g=-i(gH-Hg)$ is defined on the domain $\mathcal{D}(H)\subset\mathcal{F}_\mathcal{H}$. We remark that operator (8) is the generator of the Heisenberg equation.

We define the n-th order ($n\geq1$) cumulant of the groups of operators (7) as

$$\mathfrak{A}_n(t)\equiv\mathfrak{A}_n(t,X)= \tag{9}$$

$$=\sum_{\mathrm{P}:\,X=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}\mathcal{G}_{|X_i|}(t),$$

where $\sum_\mathrm{P}$ is the sum over all possible partitions P of the set $X\equiv(1,\dots,n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i\subset X$.

We formulate some properties of cumulants (9) of groups of operators (7) [21]. If $n=1$, for $g_1\in\mathcal{D}(\mathcal{N}_1)\subset\mathfrak{L}(\mathcal{H}_1)$, the generator of the first-order cumulant in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_1)$ is given by operator (8), i.e.

$$\mathrm{w}^*\!-\lim_{t\to0}\big(\frac{1}{t}\big(\mathfrak{A}_1(t,1)-I\big)g_1-\big(\mathcal{N}g\big)_1\big)=0,$$

where the operator $\mathcal{N}$ is defined by (8) or (4). In the case $n=2$, we have, in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_2)$,

$$\mathrm{w}^*\!-\lim_{t\to0}\big(\frac{1}{t}\,\mathfrak{A}_2(t,1,2)g_2-\epsilon\big(\mathcal{N}_{\mathrm{int}}(1,2)\big)g_2\big)=0.$$

Let $n>2$. As a consequence that we consider a system of particles interacting by a two-body potential (1), we have

$$\mathrm{w}^*\!-\lim_{t\to0}\frac{1}{t}\mathfrak{A}_n(t)g_n=0.$$

We introduce also some abridged notations: $Y\equiv(1,\dots,s)$, $X\equiv Y\backslash\{j_1,\dots,j_{s-n}\}$, the set $(Y\backslash X)_1$ consists of one element of $Y\backslash X=(j_1,\dots,j_{s-n})$, i.e. the set $\{j_1,\dots,j_{s-n}\}$ is a connected subset of the partition P ($|\mathrm{P}|=1$, $|\mathrm{P}|$ denotes the number of partitions). We will also denote the set $(Y\backslash X)_1$ by the symbol $\{j_1,\dots,j_{s-n}\}_1$.

On the space $\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ for the abstract initial-value problem (2),(3), the following statement is valid.

A solution of the initial-value problem to the quantum dual BBGKY hierarchy (2),(3) is determined by the expansion ($s\geq1$)

$$G_s(t,Y)=\sum_{n=0}^{s}\frac{1}{(s-n)!}\times \tag{10}$$

$$\times\sum_{j_1\neq\dots\neq j_{s-n}=1}^{s}\mathfrak{A}_{1+n}\big(t,(Y\backslash X)_1,X\big)\,G_{s-n}(0,Y\backslash X),$$

where the operator $\mathfrak{A}_{1+n}\big(t,(Y\backslash X)_1,X\big)$ *is the* $(1+n)$*-th order cumulant (9) defined by the formula*

$$\mathfrak{A}_{1+n}\big(t,(Y\backslash X)_1,X\big)=$$

$$=\sum_{\mathrm{P}:\,\{(Y\backslash X)_1,X\}=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}\mathcal{G}_{|X_i|}(t,X_i).$$

If $G(0)\in\mathcal{D}(\mathcal{N})\subset\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$*, it is a classical solution, and, for arbitrary initial data* $G(0)\in\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$*, it is a generalized (weak) solution.*

Thus, solutions of the first two equations of hierarchy (2) are given by the expansions

$$G_1(t,1)=\mathfrak{A}_1(t,1)G_1(0,1),$$

$$G_2(t,1,2)=\mathfrak{A}_1\big(t,\{1,2\}_1\big)G_2(0,1,2)+$$

$$+\mathfrak{A}_2(t,1,2)\big(G_1(0,1)+G_1(0,2)\big),$$

where the first-order cumulant $\mathfrak{A}_1\big(t,\{1,2\}_1\big) = \mathcal{G}_2(t,1,2)$ is defined by group (7).

2.2. BBGKY hierarchy

The sequence $F=(I,F_1,\dots,F_n,\dots)$ defined on the Fock space $\mathcal{F}_\mathcal{H}$ of self-adjoint positive density operators F_n (I is an identity operator) describes the state of a quantum system of a non-fixed number of particles. The marginal density operators F_n, $n\geq 1$, whose kernels are known as marginal or n-particle density matrices defined on the n-particle Hilbert space $\mathcal{H}_n=\mathcal{H}^{\otimes n}$, are denoted by $F_n(1,\dots,n)$. For a system of identical particles described by the Maxwell–Boltzmann statistics, one has $F_n(1,\dots,n)=F_n(i_1,\dots,i_n)$ if $\{i_1,\dots,i_n\}\in\{1,\dots,n\}$.

We will consider states of a system that belong to the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})=\bigoplus_{n=0}^\infty \alpha^n\mathfrak{L}^1(\mathcal{H}_n)$ of sequences $f=\big(I,f_1,\dots,f_n,\dots\big)$ of trace class operators $f_n=f_n(1,\dots,n)\in\mathfrak{L}^1(\mathcal{H}_n)$ satisfying the above-mentioned symmetry condition, equipped with the trace norm

$$\|f\|_{\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})}=\sum_{n=0}^\infty \alpha^n\,\mathrm{Tr}_{1,\dots,\mathrm{n}}|f_n(1,\dots,n)|,$$

where $\mathrm{Tr}_{1,\dots,\mathrm{n}}$ are the partial traces over $1,\dots,n$ particles, and $\alpha>1$ is a real number. By $\mathfrak{L}^1_{\alpha,0}$, we denote the everywhere dense set in $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$ of finite sequences of degenerate operators with infinitely differentiable kernels and compact supports.

On the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, we consider the following initial-value problem of the quantum BBGKY hierarchy (*the quantum Bogolyubov chain of equations*):

$$\frac{\partial}{\partial t}F_s(t)=-\big(\sum_{i=1}^s\mathcal{N}_0(i)+\epsilon\sum_{i<j=1}^s\mathcal{N}_{\mathrm{int}}(i,j)\big)F_s(t)+$$

$$+\sum_{i=1}^s\mathrm{Tr}_{s+1}\big(-\mathcal{N}_{\mathrm{int}}(i,s+1)\big)F_{s+1}(t), \tag{11}$$

$$F_s(t)\mid_{t=0}=F_s(0),\quad s\geq 1. \tag{12}$$

If $f\in\mathfrak{L}^1_0(\mathcal{F}_\mathcal{H})\subset\mathcal{D}(\mathcal{N})\subset\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, the operators $\mathcal{N}_0$, $\mathcal{N}_{\mathrm{int}}$ are consequently defined by (4), (5). We remark that hierarchy (11) is the dual hierarchy of equations to hierarchy (2).

In terms of the kernels $F_s(t,q_1,\dots,q_s;q'_1,\dots,q'_s)$ of s-particle density operators $F_s(t)$, i.e. marginal or s-particle density matrices, Eqs. (11) take the canonical form of the quantum BBGKY hierarchy [2]

$$i\frac{\partial}{\partial t}F_s(t,q_1,\dots,q_s;q'_1,\dots,q'_s)=$$

$$=\Big(-\frac{1}{2}\sum_{i=1}^s(\Delta_{q_i}-\Delta_{q'_i})+\sum_{i<j=1}^s\big(\Phi(q_i-q_j)-$$

$$-\Phi(q'_i-q'_j)\big)\Big)F_s(t,q_1,\dots,q_s;q'_1,\dots,q'_s)+$$

$$+\sum_{i=1}^s\int dq_{s+1}\big(\Phi(q_i-q_{s+1})-$$

$$-\Phi(q'_i-q_{s+1})\big)F_{s+1}(t,q_1,\dots,q_s,q_{s+1};q'_1,\dots,q'_s,q_{s+1}).$$

To construct a solution of the initial-value problem (11)-(12), we introduce some preliminary facts.

On the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, we define the following group $\mathcal{G}(-t)=\oplus_{n=0}^\infty\mathcal{G}_n(-t)$ of operators:

$$\mathcal{G}_n(-t)f_n := e^{-itH_n}\,f_n\,e^{itH_n}. \tag{13}$$

On the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, mapping (13): $t\rightarrow\mathcal{G}(-t)f$ is an isometric strongly continuous group which preserves the positivity and the self-adjointness of operators. A solution of the initial-value problem of the von Neumann equation for a statistical operator is defined by this group.

If $f\in\mathfrak{L}^1_{\alpha,0}(\mathcal{F}_\mathcal{H})\subset\mathcal{D}(\mathcal{N})$ in the sense of the norm convergence of the space $\mathfrak{L}^1(\mathcal{F}_\mathcal{H})$, there exists a limit, by which the infinitesimal generator $-\mathcal{N}=\oplus_{n=0}^\infty(-\mathcal{N}_n)$ of the group of operators (13) is determined as

$$\lim_{t\rightarrow 0}\frac{1}{t}\big(\mathcal{G}(-t)f-f\big)=-i(Hf-fH):=-\mathcal{N}f, \tag{14}$$

where $H=\bigoplus_{n=0}^\infty H_n$ is Hamiltonian (1) and the operator $-i(Hf-fH)$ is defined on the domain $\mathcal{D}(H)\subset\mathcal{F}_\mathcal{H}$. We note that operator (14) is the generator of the von Neumann evolution equation.

Let $X\equiv(1,\dots,n)$. The n-th order cumulant [18, 19] of the groups of operators (13) is defined as ($n\geq 1$)

$$\mathfrak{A}_n(-t)\equiv\mathfrak{A}_n(-t,X)= \tag{15}$$

$$= \sum_{\mathrm{P}:\, X=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i\subset \mathrm{P}} \mathcal{G}_{|X_i|}(-t),$$

where $\sum_{\mathrm{P}}$ is the sum over all possible partitions P of the set $\{1,\ldots,n\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset X$.

If $n=1$, for $f_1 \in \mathfrak{L}^1_0(\mathcal{H}_1) \subset \mathcal{D}(\mathcal{N}_1) \subset \mathfrak{L}^1(\mathcal{H}_1)$ in the sense of the norm convergence in $\mathfrak{L}^1(\mathcal{H}_1)$, the generator of the first-order cumulant is given by operator (14), i.e.

$$\lim_{t\to 0} \big\| \frac{1}{t}\big(\mathfrak{A}_1(-t,1)-I\big)f_1 - \big(-\mathcal{N}f\big)_1\big\|_{\mathfrak{L}^1(\mathcal{H}_1)} = 0.$$

In the case $n=2$ for cumulant (15), we have

$$\lim_{t\to 0} \big\| \frac{1}{t}\mathfrak{A}_2(-t)f_2 - \epsilon\big(-\mathcal{N}_{\mathrm{int}}(1,2)\big)f_2\big\|_{\mathfrak{L}^1(\mathcal{H}_2)} = 0.$$

Let $n>2$. As a consequence that we consider a system of particles interacting by a two-body potential, it holds

$$\lim_{t\to 0} \big\| \frac{1}{t}\mathfrak{A}_n(-t)f_n\big\|_{\mathfrak{L}^1(\mathcal{H}_n)} = 0.$$

We introduce the following notations: $Y_{\mathrm{P}} \equiv (X_1,\ldots,X_{|\mathrm{P}|})$ is a set, whose elements are $|\mathrm{P}|$ mutually disjoint subsets $X_i \subset Y \equiv (1,\ldots,s)$ of the partition $\mathrm{P}: Y = \bigcup_{i=1}^{|\mathrm{P}|} X_i$. Since $Y_{\mathrm{P}} = (X_1,\ldots,X_{|\mathrm{P}|})$, Y_1 is the set consisting of one element $Y=(1,\ldots,s)$ of the partition P ($|\mathrm{P}|=1$). To underline that the set $(1,\ldots,s)$ is a connected subset (the cluster of s elements) of a partition P ($|\mathrm{P}|=1$), we will also denote the set Y_1 by the symbol $\{1,\ldots,s\}_1$.

On the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$ for the abstract initial-value problem (11), (12), the following statement is valid [19].

If $F(0) \in \mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$ and $\alpha > e$, then, for $t\in\mathbb{R}^1$, there exists a unique solution of the initial-value problem (11), (12) given by the expansion ($s\geq 1$)

$$F_s(t,Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{s+1,\ldots,s+n}\, \mathfrak{A}_{1+n}(-t) F_{s+n}(0,X), \quad (16)$$

where

$$\mathfrak{A}_{1+n}(-t) \equiv \mathfrak{A}_{1+n}(-t, Y_1, s+1,\ldots,s+n) =$$

$$= \sum_{\mathrm{P}:\{Y_1, X\setminus Y\}=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i\subset\mathrm{P}} \mathcal{G}_{|X_i|}(-t)$$

is the $(1+n)$-th order cumulant (15) of the groups of operators (13), $\sum_{\mathrm{P}}$ is the sum over all possible partitions P of the set $\{Y_1, s+1,\ldots,s+n\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset \{Y_1, X\setminus Y\}$.

For initial data $F(0) \in \mathfrak{L}^1_{\alpha,0}(\mathcal{F}_\mathcal{H})$, it is a strong solution, and, for arbitrary initial data of the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, it is a weak solution.

The condition $\alpha > e$ guarantees the convergence of series (16) and implies that the mean value of a number of particles is finite. This fact follows if we renormalize sequence (16) in such a way: $\widetilde{F}_s(t) = \langle N\rangle^s F_s(t)$. For arbitrary $F(0) \in \mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, the mean value (6) of the number of particles

$$\langle N\rangle(t) = \mathrm{Tr}_1 F_1(t,1) \quad (17)$$

in state (16) is finite. In fact,

$$|\langle N\rangle(t)| \leq c_\alpha \|F(0)\|_{\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})} < \infty,$$

where $c_\alpha = e^2(1-\frac{e}{\alpha})^{-1}$ is a constant. To describe the evolution of an infinite-particle system, we have to construct a solution of the initial-value problem (11), (12) in more general spaces than $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$. This problem will be discussed in Conclusion.

We remark that, for classical systems of particles, the first few terms of the cumulant expansion (16) for the BBGKY hierarchy were obtained in [7],[8]. The methods used by Green and Cohen were based on the analogy with the Ursell–Mayer cluster expansions for equilibrium states.

A solution of the initial-value problem (11), (12) is usually represented as the perturbation (iteration) series [4, 11, 12]. On the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, expansion (16) is equivalent to the iteration series.

Indeed, if an interaction potential is a bounded operator, then, for $f_s \in \mathfrak{L}^1(\mathcal{H}_s)$, an analog of the Duhamel formula for group (13) holds

$$\big(\mathcal{G}_s(-t,1,\ldots,s) - \prod_{l=1}^{s}\mathcal{G}_1(-t,l)\big)f_s = \quad (18)$$

$$= \epsilon \int_0^t d\tau \prod_{l=1}^{s} \mathcal{G}_1(-t+\tau,l)\big(-\sum_{i<j=1}^{s}\mathcal{N}_{\mathrm{int}}(i,j)\big)\mathcal{G}_s(-\tau)f_s.$$

Then, according to the unitary property of group (13) on the space $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, the solution expansion (16) reduces to the iteration series of BBGKY hierarchy (11)

$$F_s(t,1,\ldots,s) =$$

$$= \sum_{n=0}^{\infty} \epsilon^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \mathrm{Tr}_{s+1,\ldots,s+n}\mathcal{G}_s(-t+t_1)\times$$

$$\times \sum_{i_1=1}^{s} \big(-\mathcal{N}_{\rm int}(i_1, s+1)\big)\mathcal{G}_{s+1}(-t_1+t_2)\ldots$$

$$\ldots \mathcal{G}_{s+n-1}(-t_{n-1}+t_n) \sum_{i_n=1}^{s+n-1} \big(-\mathcal{N}_{\rm int}(i_n, s+n)\big)\times$$

$$\times \mathcal{G}_{s+n}(-t_n) F_{s+n}(0,1,\ldots,s+n). \qquad (19)$$

If $F(0) \in \mathfrak{L}^1_0(\mathcal{F}_\mathcal{H})$, this series exists and converges for a finite time interval [4, 11].

As was mentioned above, functional (6) of mean values defines a duality between marginal observables and marginal states. If $G(t) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $F(0) \in \mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, then functional (6) exists, provided that $\alpha = \gamma^{-1} > e$, and the following estimate holds:

$$\big|\big\langle G(0)\big|F(t)\big\rangle\big| = \big|\big\langle G(t)\big|F(0)\big\rangle\big| \le$$

$$\le e^2(1-\gamma e)^{-1}\big\|G(0)\big\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}\big\|F(0)\big\|_{\mathfrak{L}^1_{\gamma^{-1}}(\mathcal{F}_\mathcal{H})}.$$

2.3. Generalized quantum kinetic equation

We consider one more approach to the description of the evolution of states of quantum many-particle systems. Let the initial data be completely characterized by the one-particle density operator $F_1(0)$, for example, the initial data satisfying the chaos property (Maxwell–Boltzmann statistics)

$$F^{(c)}(0) = \big(I, F_1(0,1), \ldots, \prod\nolimits_{i=1}^{s} F_1(0,i), \ldots\big).$$

In that case, the initial-value problem of BBGKY hierarchy (11), (12) is not a completely well-defined Cauchy problem, because the generic initial data are not independent for every density operator $F_s(t)$, $s \ge 1$, of the hierarchy of equations (11). Thus, this naturally yields the opportunity of reformulating such initial-value problem as a new Cauchy problem for the one-particle density operator, i.e. $F_1(t)$, with independent initial data $F_1(0)$ and the explicitly defined functionals $F_s\big(t, 1, \ldots, s \mid F_1(t)\big)$, $s \ge 2$, of the solution $F_1(t)$ of this Cauchy problem instead other s-particle density operators $F_s(t)$, $s \ge 2$ [5, 16].

Consequently, for an initial state satisfying the chaos property, i.e. $F^{(c)}(0)$, the state of a many-particle system described by the sequence $F(t) = (I, F_1(t,1), \ldots, F_s(t,1,\ldots,s), \ldots)$ of the s-particle density operators (16) can be described by the sequence

$$F\big(t \mid F_1(t)\big) = \big(I, F_1(t,1), F_2\big(t,1,2 \mid F_1(t)\big), \ldots$$

$$\ldots, F_s\big(t, 1, \ldots, s \mid F_1(t)\big), \ldots\big)$$

of the functionals stated above.

At first, we define the sequence $F\big(t \mid F_1(t)\big)$ of functionals. The functionals $F_s\big(t, 1, \ldots, s \mid F_1(t)\big)$, $s \ge 2$, are represented by the expansions over products of the one-particle density operator $F_1(t)$ (for particles obeying the Maxwell–Boltzmann statistics)

$$F_s\big(t, 1, \ldots, s \mid F_1(t)\big) = \qquad (20)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{s+1,\ldots,s+n}\, \mathfrak{V}_{1+n}(t) \prod_{i=1}^{s+n} F_1(t,i),$$

where the evolution operators $\mathfrak{V}_{1+n}(t) \equiv \mathfrak{V}_{1+n}(t, \{1,\ldots,s\}_1, s+1, \ldots, s+n)$, $n \ge 0$, are defined from the condition that expansion (20) of the functional $F_s\big(t \mid F_1(t)\big)$ must be equal term-by-term to expansion (16) of the s-particle density operator $F_s(t)$.

The low-order evolution operators $\mathfrak{V}_{1+n}(t)$, $n \ge 0$, have the form

$$\mathfrak{V}_1(t, Y_1) = \widehat{\mathfrak{A}}_1(t, Y_1), \qquad (21)$$

$$\mathfrak{V}_2(t, Y_1, s+1) = \qquad (22)$$

$$= \widehat{\mathfrak{A}}_2(t, Y_1, s+1) - \widehat{\mathfrak{A}}_1(t, Y_1) \sum_{j=1}^{s} \widehat{\mathfrak{A}}_2(t, j, s+1),$$

where $\widehat{\mathfrak{A}}_n(t)$ is the n-th order cumulant (semiinvariants) of scattering operators

$$\widehat{\mathcal{G}}_n(t, 1, \ldots, n) := \mathcal{G}_n(-t, 1, \ldots, n) \prod_{i=1}^{n} \mathcal{G}_1(t, i), \qquad (23)$$

$\widehat{\mathcal{G}}_1(t) = I$ is the identity operator.

In terms of scattering operators (23), evolution operators (21),(22) get the form

$$\mathfrak{V}_1(t, Y_1) = \widehat{\mathcal{G}}_s(t, Y),$$

$$\mathfrak{V}_2(t, Y_1, s+1) = \widehat{\mathcal{G}}_{s+1}(t, Y, s+1) -$$

$$-\widehat{\mathcal{G}}_s(t, Y) \sum_{j=1}^{s} \widehat{\mathcal{G}}_2(t, j, s+1) + (s-1)\, \widehat{\mathcal{G}}_s(t, Y).$$

For $F_1(0) \in \mathfrak{L}^1(\mathcal{H})$, the sequence $F\big(t \mid F_1(t)\big)$ of functionals (20) exists, and series (20) converges under the condition that $\|F_1(0)\| < e^{-1}$, i.e. if the mean value of particles is finite [16].

We remark that expansions (20) are an nonequilibrium analog of expansions in powers of the density of the equilibrium marginal density operators [1, 7, 8].

We now formulate the evolution equation for the one-particle density operator $F_1(t)$, i.e. for the first element of the sequence $F\big(t \mid F_1(t)\big)$. If $\|F_1(0)\| < e^{-1}$, it represents by series (16) convergent in the norm of the space $\mathfrak{L}^1(\mathcal{H})$

$$F_1(t,1) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{2,\ldots,1+\mathrm{n}}\, \mathfrak{A}_{1+n}(-t) \prod_{i=1}^{n+1} F_1(0,i), \qquad (24)$$

where $\mathfrak{A}_{1+n}(-t)$ is the $(1+n)$-th order cumulant (15) of groups of operators (13). Let $F_1(0) \in \mathfrak{L}^1_0(\mathcal{H})$. Then, by differentiating series (24) with respect to the time variable in the sense of norm convergence of the space $\mathfrak{L}^1(\mathcal{H}_1)$, according to properties of cumulants (15), we find that the one-particle density operator $F_1(t)$ is governed by the initial-value problem of the following nonlinear evolution equation (*the generalized quantum kinetic equation*)

$$\frac{\partial}{\partial t} F_1(t,1) = -\mathcal{N}_1(1) F_1(t,1) + \qquad (25)$$

$$+ \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{2,3,\ldots,\mathrm{n}+2}\big(-\mathcal{N}_{\mathrm{int}}(1,2)\big) \mathfrak{V}_{1+n}(t) \prod_{i=1}^{n+2} F_1(t,i),$$

$$F_1(t,1)|_{t=0} = F_1(0,1). \qquad (26)$$

In the kinetic equation (25), the evolution operators $\mathfrak{V}_{1+n}(t) \equiv \mathfrak{V}_{1+n}(t, \{1,2\}_1, 3, \ldots, 2+n)$, $n \geq 0$, are defined as above.

For initial-value problem (25), (26), the following statement holds [16].

If $F_1(0) \in \mathfrak{L}^1_0(\mathcal{H})$ is a non-negative density operator, then, provided $\|F_1(0)\| < e^{-1}$, there exist a unique strong global in time solution of the initial-value problem (25), (26) which is a non-negative density operator represented by series (24) convergent in the norm of the space $\mathfrak{L}^1(\mathcal{H})$ and a weak one for arbitrary initial data $F_1(0) \in \mathfrak{L}^1(\mathcal{H})$.

As a result, the following principle of equivalence of the initial-value problems (11), (12) and (25), (26) is true.

If the initial data are completely defined by the trace class operators $F_1(0)$, then the Cauchy problem (11), (12) is equivalent to the initial-value problem (25), (26) for the generalized kinetic equation and functionals $F_s\big(t, 1, \ldots, s \mid F_1(t)\big)$, $s \geq 2$, defined by expansions (20) under the condition that $\|F_1(0)\| < e^{-1}$.

We note that this statement is valid also for more general initial data than $F^{(c)}(0)$, namely the initial data determined by the one-particle density operator $F_1(0)$ and operators describing initial correlations. In this case, the initial correlations are a part of the coefficients of Eq. (25) and functionals (20).

Thus, if the initial data are completely defined by the one-particle density operator, then all possible states of infinite-particle systems at an arbitrary moment of time can be described within the framework of the one-particle density operator without any approximations.

We remark that functionals (20) are formally concerned with the corresponding functionals of the Bogolyubov method of the derivation of kinetic equations [1]. Indeed, functionals (20) and the corresponding Bogolyubov functionals coincide if the principle of weakening of correlations for functionals (20) holds. The proof of this assertion is similar to the proof [5] of an equivalence of the BBGKY hierarchy solution (16) and iteration series (19).

3. Derivation of a Nonlinear Schrödinger Equation

We consider the problem of the rigorous derivation of quantum kinetic equations from an underlying many-particle dynamics by the example of the mean-field asymptotics of the above-constructed solutions of quantum evolution equations. We formulate new approaches to the derivation of a nonlinear Schrödinger equation (subsections 3.2 and 3.3).

3.1. Mean-field limit of the BBGKY hierarchy solution

We present the main steps of the construction of the mean-field asymptotics of solution (16) of the initial-value problem (11), (12). For that, we introduce some preliminary facts on the asymptotic perturbation of cumulants.

If $f_s \in \mathfrak{L}^1(\mathcal{H}_s)$, then, for an arbitrary finite time interval, there exists the following limit of the strongly continuous group (13):

$$\lim_{\epsilon \to 0} \big\|\big(\mathcal{G}_s(-t) - \prod_{j=1}^{s} \mathcal{G}_1(-t,j)\big) f_s\big\|_{\mathfrak{L}^1(\mathcal{H}_s)} = 0. \qquad (27)$$

According to an analog of the Duhamel formula (18) and (27) for the second-order cumulant $\mathfrak{A}_2(-t, Y_1, s+1)$, we

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have

$$\lim_{\epsilon\rightarrow 0}\big\|\big(\frac{1}{\epsilon}\,\mathfrak{A}_2(-t,Y_1,s+1)-\int\limits_0^t dt_1\prod_{j=1}^{s+1}\mathcal{G}_1(-t+t_1,j)\times$$

$$\times\big(-\sum_{i=1}^{s}\mathcal{N}_{\mathrm{int}}(i,s+1)\big)\prod_{l=1}^{s+1}\mathcal{G}_1(-t_1,l)\big)f_{s+1}\big\|_{\mathfrak{L}^1(\mathcal{H}_{s+1})}=0.$$

In general case, the following equality holds:

$$\lim_{\epsilon\rightarrow 0}\big\|\big(\frac{1}{\epsilon^n}\,\mathfrak{A}_{1+n}(-t)-\int\limits_0^t dt_1\ldots\int\limits_0^{t_{n-1}}dt_n\prod_{j=1}^{s}\mathcal{G}_1(-t+$$

$$+t_1,j)\sum_{i_1=1}^{s}\big(-\mathcal{N}_{\mathrm{int}}(i_1,s+1)\big)\prod_{j_1=1}^{s+1}\mathcal{G}_1(-t_1+t_2,j_1)\ldots$$

$$\prod_{j_{n-1}=1}^{s+n-1}\mathcal{G}_1(-t_{n-1}+t_n,j_{n-1})\sum_{i_n=1}^{s+n-1}\big(-\mathcal{N}_{\mathrm{int}}(i_n,s+n)\big)\times$$

$$\times\prod_{j_n=1}^{s+n}\mathcal{G}_1(-t_n,j_n)\big)f_{s+n}\big\|_{\mathfrak{L}^1(\mathcal{H}_{s+n})}=0. \qquad (28)$$

Thus, if, for the initial data $F_s(0)\in\mathfrak{L}^1(\mathcal{H}_s)$, there exists the limit $f_s(0)\in\mathfrak{L}^1(\mathcal{H}_s)$, i.e.,

$$\lim_{\epsilon\rightarrow 0}\big\|\epsilon^s F_s(0)-f_s(0)\big\|_{\mathfrak{L}^1(\mathcal{H}_s)}=0,$$

then, according to (28) for an arbitrary finite time interval, there exists the mean-field limit of solution (16) of the BBGKY hierarchy

$$\lim_{\epsilon\rightarrow 0}\big\|\epsilon^s F_s(t)-f_s(t)\big\|_{\mathfrak{L}^1(\mathcal{H}_s)}=0,$$

where $f_s(t)$ is given by the series

$$f_s(t,1,\ldots,s)= \qquad (29)$$

$$=\sum_{n=0}^{\infty}\int\limits_0^t dt_1\ldots\int\limits_0^{t_{n-1}}dt_n\mathrm{Tr}_{\mathrm{s+1,\ldots,s+n}}\prod_{j=1}^{s}\mathcal{G}_1(-t+t_1,j)\times$$

$$\times\sum_{i_1=1}^{s}\big(-\mathcal{N}_{\mathrm{int}}(i_1,s+1)\big)\prod_{j_1=1}^{s+1}\mathcal{G}_1(-t_1+t_2,j_1)\ldots$$

$$\ldots\prod_{j_{n-1}=1}^{s+n-1}\mathcal{G}_1(-t_{n-1}+t_n,j_{n-1})\sum_{i_n=1}^{s+n-1}\big(-\mathcal{N}_{\mathrm{int}}(i_n,s+n)\big)\times$$

$$\times\prod_{j_n=1}^{s+n}\mathcal{G}_1(-t_n,j_n)f_{s+n}(0),$$

which converges for a bounded interaction potential for a finite time interval [11].

If $f(0)\in\mathfrak{L}^1_0(\mathcal{F}_{\mathcal{H}})$, the sequence $f(t)=(I,f_1(t),\ldots,f_s(t),\ldots)$ of limit marginal density operators (29) is a strong solution of the Cauchy problem of the *Vlasov hierarchy*

$$\frac{\partial}{\partial t}f_s(t)=\sum_{i=1}^{s}\big(-\mathcal{N}_0(i)\big)f_s(t)+ \qquad (30)$$

$$+\sum_{i=1}^{s}\mathrm{Tr}_{\mathrm{s+1}}\big(-\mathcal{N}_{\mathrm{int}}(i,s+1)\big)f_{s+1}(t),$$

$$f_s(t)|_{t=0}=f_s(0),\qquad s\geq 1. \qquad (31)$$

We observe that, if the initial data satisfy the chaos property (for particles obeying the Maxwell–Boltzmann statistics)

$$f_s(t,1,\ldots,s)|_{t=0}=\prod_{j=1}^{s}f_1(0,j),\quad s\geq 2,$$

then solution (29) of the initial-value problem of the Vlasov hierarchy (30), (31) possesses the same property

$$f_s(t,1,\ldots,s)=\prod_{j=1}^{s}f_1(t,j),\quad s\geq 2. \qquad (32)$$

To establish equality (32), we introduce marginal correlation density operators [20]

$$G_s(t,1,\ldots,s)= \qquad (33)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{\mathrm{s+1,\ldots,s+n}}\,\mathfrak{A}_{s+n}(-t)\prod_{i=1}^{s+n}G_1(0,i),$$

where $\mathfrak{A}_{s+n}(-t)\equiv\mathfrak{A}_{s+n}(-t,1,\ldots,s+n)$ is the $(s+n)$-th order cumulant (15) of the groups of operators (13), and $G_1(0)=F_1(0)$. In the same way as (28) for arbitrary $t\in\mathbb{R}$, we have the equality

$$\lim_{\epsilon\rightarrow 0}\big\|\frac{1}{\epsilon^n}\,\mathfrak{A}_{s+n}(-t,1,\ldots,s+n)f_{s+n}\big\|_{\mathfrak{L}^1(\mathcal{H}_{s+n})}=0. \qquad (34)$$

Let

$$\lim_{\epsilon\to 0}\big\|\epsilon\, G_1(0)-f_1(0)\big\|_{\mathfrak{L}^1(\mathcal{H}_1)}=0$$

hold. Then, according to (34) for the correlation density operators (33), we obtain

$$\lim_{\epsilon\to 0}\big\|\epsilon^s\, G_s(t)\big\|_{\mathfrak{L}^1(\mathcal{H}_s)}=0. \tag{35}$$

In view of the fact that the marginal density operators (16) are expressed in terms of the correlation density operators (33) by the cluster expansions

$$F_s(t,Y)=\prod_{i=1}^{s}F_1(t,i)+$$

$$+\sum_{\substack{\mathrm{P}\,:\,\{Y\}=\bigcup_i X_i,\\ |\mathrm{P}|\neq s}}\ \prod_{X_i\subset \mathrm{P}}G_{|X_i|}(t,X_i),\quad s\geq 2,$$

and taking equality (35) into account, the following statement is valid.

If there exists the mean-field limit of the initial data $F_s(0)\in\mathfrak{L}^1(\mathcal{H}_s)$

$$\lim_{\epsilon\to 0}\big\|\epsilon^s F_s(0,1,\ldots,s)-\prod_{j=1}^{s}f_1(0,j)\big\|_{\mathfrak{L}^1(\mathcal{H}_s)}=0,$$

then, for a finite time interval for solution (16) of the BBGKY hierarchy, the limit

$$\lim_{\epsilon\to 0}\big\|\epsilon^s F_s(t,1,\ldots,s)-\prod_{j=1}^{s}f_1(t,j)\big\|_{\mathfrak{L}^1(\mathcal{H}_s)}=0$$

holds, where $f_1(t)$ *is the solution of the Cauchy problem of the quantum Vlasov equation*

$$\frac{\partial}{\partial t}f_1(t,1)=\big(-\mathcal{N}_0(1)\big)f_1(t,1)+ \tag{36}$$

$$+\mathrm{Tr}_2\big(-\mathcal{N}_{\mathrm{int}}(1,2)\big)f_1(t,1)f_1(t,2),$$

$$f_1(t)|_{t=0}=f_1(0). \tag{37}$$

Thus, in consequence of the chaos property (32), we derive the quantum Vlasov kinetic equation (36).

For a system in the pure state, i.e. $f_1(t)=|\psi_t\rangle\langle\psi_t|$ ($P_{\psi_t}\equiv|\psi_t\rangle\langle\psi_t|$ is a one-dimensional projector onto a unit vector $|\psi_t\rangle$) or in terms of the kernel $f_1(t,q,q')=\psi(t,q)\psi(t,q')$ of the marginal one-particle density operator $f_1(t)$, the Vlasov kinetic equation (36) is transformed to the *Hartree equation*

$$i\frac{\partial}{\partial t}\psi(t,q)=-\frac{1}{2}\Delta_q\psi(t,q)+$$

$$+\int dq'\Phi(q-q')|\psi(t,q')|^2\psi(t,q). \tag{38}$$

If the kernel of the interaction potential $\Phi(q)=\delta(q)$ is the Dirac measure, then, from (38), we derive the cubic *nonlinear Schrödinger equation*

$$i\frac{\partial}{\partial t}\psi(t,q)=-\frac{1}{2}\Delta_q\psi(t,q)+|\psi(t,q)|^2\psi(t,q).$$

Thus, the following statement holds:

$$\lim_{\epsilon\to 0}\big\|\,\epsilon^s F_s(t)-|\psi_t\rangle\langle\psi_t|^{\otimes s}\,\big\|_{\mathfrak{L}^1(\mathcal{H}_s)}=0,$$

where $|\psi_t\rangle$ is the solution of the cubic nonlinear Schrödinger equation.

In the case of representation (19) of the solution of the Cauchy problem (11), (12) of the BBGKY hierarchy by the iteration series, the last statement is proved in works [10–13].

3.2. Mean-field limit of generalized kinetic equation

We construct the mean-field limit of a solution of the initial-value problem of the generalized kinetic equation (25).

If there exists the limit $f_1(0)\in\mathfrak{L}^1(\mathcal{H}_1)$ of initial data (26),

$$\lim_{\epsilon\to 0}\big\|\epsilon\, F_1(0)-f_1(0)\big\|_{\mathfrak{L}^1(\mathcal{H}_1)}=0,$$

then, according to (27) and (28) for an arbitrary finite time interval, there exists the limit of solution (24) of the generalized kinetic equation (25)

$$\lim_{\epsilon\to 0}\big\|\epsilon\, F_1(t)-f_1(t)\big\|_{\mathfrak{L}^1(\mathcal{H}_1)}=0, \tag{39}$$

where $f_1(t)$ is a strong solution of the Cauchy problem (36), (37) of the quantum Vlasov equation represented in the form of the expansion

$$f_1(t,1)=$$

$$=\sum_{n=0}^{\infty}\int_0^t dt_1\ldots\int_0^{t_{n-1}}dt_n\mathrm{Tr}_{\mathrm{s+1,\ldots,s+n}}\prod_{j=1}^{s}\mathcal{G}_1(-t+t_1,j)\times$$

$$\times \sum_{i_1=1}^{s} \big(-\mathcal{N}_{\mathrm{int}}(i_1, s+1)\big) \prod_{j_1=1}^{s+1} \mathcal{G}_1(-t_1+t_2, j_1) \ldots$$

$$\prod_{j_{n-1}=1}^{s+n-1} \mathcal{G}_1(-t_{n-1}+t_n, j_{n-1}) \sum_{i_n=1}^{s+n-1} \big(-\mathcal{N}_{\mathrm{int}}(i_n, s+n)\big)\times$$

$$\times \prod_{j_n=1}^{s+n} \mathcal{G}_1(-t_n, j_n) \prod_{i=1}^{s+n} f_1(0,i), \qquad (40)$$

and the operator $\mathcal{N}_{\mathrm{int}}$ is defined by formula (5). For bounded interaction potentials, series (40) converges for a finite time interval.

If $f_s \in \mathfrak{L}^1(\mathcal{H}_s)$ and the interaction potential is a bounded operator, then, for scattering operators (23), an analog of the Duhamel formula holds:

$$\big(\widehat{\mathcal{G}}_s(t,1,\ldots,s) - I\big) f_s =$$

$$= \epsilon \int_0^t d\tau \prod_{l=1}^{s} \mathcal{G}_1(\tau, l)\big(-\sum_{i<j=1}^{s} \mathcal{N}_{\mathrm{int}}(i,j)\big)\mathcal{G}_s(-\tau) f_s. \qquad (41)$$

Then, according to definition (21) of the evolution operators $\mathfrak{V}_{1+n}(t, \{1,\ldots,s\}_1, s+1,\ldots,s+n)$, $n \geq 0$, expansion (20) and equality (41) yield

$$\lim_{\epsilon\to 0} \big\| \big(\mathfrak{V}_1(t, \{1,\ldots,s\}_1) - I\big) f_s \big\|_{\mathfrak{L}^1(\mathcal{H}_s)} = 0.$$

Correspondingly for $n \geq 1$, it holds:

$$\lim_{\epsilon\to 0} \big\| \mathfrak{V}_{1+n}(t) f_{s+n} \big\|_{\mathfrak{L}^1(\mathcal{H}_{s+n})} = 0.$$

Since a solution of the initial-value problem (25), (26) of the generalized kinetic equation converges to a solution of the initial-value problem (36), (37) of the quantum Vlasov kinetic equation as (39), functionals (20) satisfy the relation

$$\lim_{\epsilon\to 0} \big\| \epsilon^s F_s\big(t,1,\ldots,s \mid F_1(t)\big) - \prod_{j=1}^{s} f_1(t,j) \big\|_{\mathfrak{L}^1(\mathcal{H}_s)} = 0,$$

where $f_1(t)$ is defined by series (40) which converges for a finite time interval.

The last equality means that the chaos property (32) preserves in time in the mean-field scaling limit.

Thus, we conclude that the results of the previous subsection concerning the derivation of the Hartree equation and the nonlinear Schrödinger equation take place also in the case of the generalized quantum kinetic equation (25).

3.3. Mean-field limit of a dual BBGKY hierarchy solution

Consider the mean-field limit of a solution of the initial-value problem of the dual BBGKY hierarchy (2).

For an arbitrary finite time interval, there exists the following limit of the $*$-weak continuous group of operators (7) in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_s)$:

$$\mathrm{w}^*\!-\lim_{t\to 0} \big(\mathcal{G}_s(t) g_s - \prod_{j=1}^{s} \mathcal{G}_1(-t,j) g_s\big) = 0. \qquad (42)$$

According to an analog of the Duhamel formula (18) and (42) for the second-order cumulant $\mathfrak{A}_2(t,1,2)$ in the same sense as above, it holds:

$$\mathrm{w}^*\!-\lim_{\epsilon\to 0} \big(\frac{1}{\epsilon}\, \mathfrak{A}_2(t,1,2) g_2 -$$

$$- \int_0^t dt_1 \prod_{j=1}^{2} \mathcal{G}_1(t-t_1, j) \mathcal{N}_{\mathrm{int}}(1,2) \prod_{l=1}^{2} \mathcal{G}_1(t_1, l) g_2\big) = 0. \qquad (43)$$

Thus, if, for initial data $G_s(0) \in \mathfrak{L}(\mathcal{H}_s)$, there exists the limit $g_s(0) \in \mathfrak{L}(\mathcal{H}_s)$, i.e. the relation

$$\mathrm{w}^*\!-\lim_{\epsilon\to 0} \big(\epsilon^{-s} G_s(0) - g_s(0)\big) = 0 \qquad (44)$$

holds, then, according to (42) and (43), for an arbitrary finite time interval, there exists the mean-field limit of solution (10) of the dual BBGKY hierarchy (2) in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_s)$:

$$\mathrm{w}^*\!-\lim_{\epsilon\to 0} \big(\epsilon^{-s} G_s(t) - g_s(t)\big) = 0. \qquad (45)$$

The limit operator $g_s(t)$ in (45) is given by the expansion

$$g_s(t,Y) = \sum_{n=0}^{s-1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \mathcal{G}_s^0(t-t_1)\times$$

$$\times \sum_{i_{k_1} \neq i_{k_2}=1}^{s} \mathcal{N}_{\mathrm{int}}(i_{k_1}, i_{k_2})\, \mathcal{G}_{s-1}^0(t_1-t_2) \ldots$$

$$\ldots \mathcal{G}_{s-n+1}^0(t_{n-1}-t_n) \sum_{i_{k_n} \neq i_{k_{n+1}}=1}^{s} \mathcal{N}_{\mathrm{int}}(i_{k_n}, i_{k_{n+1}})\times$$

$$\times\mathcal{G}^0_{s-n}(t_n)\, g_{s-n}(0,Y\backslash\{i_{k_1},\ldots,i_{k_n}\}), \tag{46}$$

where $\mathcal{G}^0_{s-n+1}(t_{n-1}-t_n)\equiv\mathcal{G}^0_{s-n+1}(t_{n-1}-t_n,Y\backslash\{i_{k_1},\ldots,i_{k_{n-1}}\})=\prod_{j\in Y\backslash\{i_{k_1},\ldots,i_{k_{n-1}}\}}\mathcal{G}_1(t_{n-1}-t_n,j)$ is the group of operators (7) of noninteracting particles.

If $g(0)\in\mathfrak{L}(\mathcal{F}_\mathcal{H})$, the sequence $g(t)=(g_0,g_1(t),\ldots,g_s(t),\ldots)$ of the limit marginal observables (46) is a generalized solution of the initial-value problem of the *dual Vlasov hierarchy*

$$\frac{\partial}{\partial t}g_s(t,Y)=\sum_{i=1}^s\mathcal{N}_0(i)\, g_s(t,Y)+$$

$$+\sum_{j_1\neq j_2=1}^s\mathcal{N}_{\rm int}(j_1,j_2)\, g_{s-1}(t,Y\backslash\{j_1\}), \tag{47}$$

$$g_s(t)\mid_{t=0}=g_s(0),\qquad s\geq 1. \tag{48}$$

Consider the mean-field limit of the additive-type observables, i.e.

$$G^{(1)}(0)=(0,G_1^{(1)}(0,1),0,\ldots).$$

In that case, solution (10) of the dual BBGKY hierarchy (2) has the form

$$G_s^{(1)}(t,Y)=\mathfrak{A}_s(t,Y)\sum_{j=1}^s G_1^{(1)}(0,j). \tag{49}$$

If, for the additive-type observables $G^{(1)}(0)$, condition (44) holds, i.e.

$$\mathrm{w}^*-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{-1}G_1^{(1)}(0)-g_1^{(1)}(0)\big)=0,$$

then, according to statement (45), for (49), we have

$$\mathrm{w}^*-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{-s}G_s^{(1)}(t)-g_s^{(1)}(t)\big)=0,$$

where

$$g_1^{(1)}(t,1)=\mathcal{G}_1(t,1)g_1^{(1)}(0,1),$$

$$g_2^{(1)}(t,1,2)=$$

$$=\int_0^t dt_1\prod_{j=1}^2\mathcal{G}_1(t-t_1,j)\mathcal{N}_{\rm int}(1,2)\sum_{l=1}^2\mathcal{G}_1(t_1,l)g_1^{(1)}(0,l)$$

or, as a special case of (46), the limit operator $g_s^{(1)}(t)$ is defined by the expansion

$$g_s^{(1)}(t,Y)=\int_0^t dt_1\ldots\int_0^{t_{s-2}}dt_{s-1}\,\mathcal{G}^0_s(t-t_1)\times$$

$$\times\sum_{i_{k_1}\neq i_{k_2}=1}^s\mathcal{N}_{\rm int}(i_{k_1},i_{k_2})\,\mathcal{G}^0_{s-1}(t_1-t_2)\ldots$$

$$\ldots\mathcal{G}^0_2(t_{s-2}-t_{s-1})\sum_{i_{k_{s-1}}\neq i_{k_s}=1}^s\mathcal{N}_{\rm int}(i_{k_{s-1}},i_{k_s})\times$$

$$\times\mathcal{G}^0_1(t_{s-1})\,g_1^{(1)}(0,Y\backslash\{i_{k_1},\ldots,i_{k_{s-1}}\}). \tag{50}$$

Let the initial state satisfy the chaos property (32)

$$f_s^{(c)}(0,1,\ldots,s)=\prod_{j=1}^s f_1(0,j),\quad s\geq 2.$$

Then, if $g(t)\in\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $f_1(0)\in\mathfrak{L}^1(\mathcal{H}_1)$, the mean value functional

$$\big\langle g(t)\big|f(0)\big\rangle=\sum_{s=0}^\infty\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\,g_s(t,1,\ldots,s)\prod_{i=1}^s f_1(0,i)$$

exists, provided that $\|f_1(0)\|_{\mathfrak{L}^1(\mathcal{H}_1)}<\gamma$.

In consequence of the equality

$$\big\langle g^{(1)}(t)\big|f^{(c)}(0)\big\rangle=\sum_{s=0}^\infty\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\,g_s^{(1)}(t)\prod_{i=1}^s f_1(0,i)$$

$$=\mathrm{Tr}_1\,g_1^{(1)}(0)f_1(t,1),$$

where $g_s^{(1)}(t)$ is given by (50) and $f_1(t,1)$ is solution (40) of the quantum Vlasov equation (36), we find that the initial-value problem (47), (48) for additive-type observables and the initial state $f^{(c)}(0)$ describes the evolution of quantum many-particle systems as by the Vlasov kinetic equation.

Correspondingly, the chaos property (32) in the Heisenberg picture of evolution of quantum many-particle systems is fulfilled, which follows from the equality ($k\geq 2$)

$$\big\langle g^{(k)}(t)\big|f^{(c)}(0)\big\rangle=\sum_{s=0}^\infty\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\,g_s^{(k)}(t)\prod_{i=1}^s f_1(0,i)=$$

$$= \frac{1}{k!}\mathrm{Tr}_{1,\ldots,k}\, g_k^{(k)}(0)\prod_{i=1}^{k} f_1(t,i),$$

where $f_1(t,1)$ is given by expansion (40).

Thus, if the initial state is a pure state, i.e. $f_s(0) = |\psi_0\rangle\langle\psi_0|^{\otimes s}$, we conclude that, in the Heisenberg picture of evolution, the initial-value problem (47), (48) describes the evolution of quantum many-particle systems which is governed by the Hartree equation (38) in the Schrödinger picture of evolution or it is governed by the cubic nonlinear Schrödinger equation, if the interaction potential $\Phi(q) = \delta(q)$ is the Dirac measure.

4. Conclusion

The concept of cumulants (9) of the groups of operators (7) of the Heisenberg equations or cumulants (15) of the groups of operators (13) of the von Neumann equations forms the basis of the groups of operators for quantum evolution equations, as well as the quantum dual BBGKY hierarchy and the BBGKY hierarchy for marginal density operators [18, 19, 21].

As was mentioned above, for the initial data $F(0) \in \mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$, the average number (17) of particles is finite. In order to describe the evolution of infinitely many particles [5], we have to construct solutions for initial marginal density operators belonging to more general Banach spaces than $\mathfrak{L}^1_\alpha(\mathcal{F}_\mathcal{H})$. For example, it can be the space of sequences of bounded operators containing the equilibrium states [4]. In that case, every term of solution expansions for the BBGKY hierarchy (11) and correspondingly for the generalized kinetic equation (25) and functionals (20) contains the divergent traces [5, 8]. In the case of the dual BBGKY hierarchy (2), the problem consists in the definition of mean value functional (6) which contains the divergent traces [17, 21]. The analysis of such a question for quantum systems remains an open problem.

We formulate two new approaches to the rigorous derivation of kinetic equations from the underlying many-particle dynamics. These approaches enable one to describe the kinetic evolution if the chaos property (32) is not fulfilled initially, i.e. in the presence of initial correlations. Such Cauchy problem takes place in the case of a kinetic evolution of the Bose condensate [2, 3]. As a result, we can formulate the kinetic equations both for a Bose gas and a Bose condensate, i.e. the nonlinear Schrödinger equation and the Gross–Pitaevskii equation [11, 12], respectively.

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ПІДХОДИ ДО ВИВОДУ КВАНТОВИХ КІНЕТИЧНИХ РІВНЯНЬ

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Р е з ю м е

У роботі розглянуто можливі підходи до проблеми строгого виводу квантових кінетичних рівнянь із багаточастинкової динаміки.

Для опису багаточастинкової еволюції побудовано розв'язки задач Коші для ієрархії ББГКІ та дуальної ієрархії рівнянь ББГКІ у відповідних банахових просторах.

На основі традиційного підходу до опису кінетичної еволюції побудовано границю середнього поля для розв'язку квантової ієрархії ББГКІ. Розвинуто також альтернативні підходи. Один з них полягає в побудові асимптотики розв'язку початкової задачі для квантової дуальної ієрархії ББГКІ. Ще один ґрунтується на узагальненому квантовому кінетичному рівнянні, що є наслідком еквівалентності задач Коші для цього еволюційного рівняння та ієрархії ББГКІ у випадку початкових умов, які визначаються одночастинковим статистичним оператором.