HIDDEN SYMMETRIES AND CRITICAL DIMENSIONS IN THE THEORY OF MODULATED STRUCTURES

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PACS 64.60.-i

Some aspects of the theory of the critical phenomena in systems with spontaneous symmetry breaking are considered. The applicability range of the mean field approximation for the systems with modulated structures is discussed. Connection between symmetries of a corresponding model and the existence of exact solutions is showed. The role of symmetries in the theory of dynamic long-range ordering is discussed.

1. Introduction

The importance of N.N. Bogolubov's contribution to the condensed matter physics and the theory of phase transitions can scarcely be overestimated [1]. Spontaneous symmetry breaking is one of the most important ideas of the modern physics. It arises in the various fields of physics and gives the possibility to describe the phenomena which, at first sight, have nothing in common from the general point of view [2]. In particular, the notion of spontaneous symmetry breaking is one of the backgrounds of the Landau theory of phase transitions of the second kind (the other background is the mean-field theory). The ability of the Landau theory to describe phase transitions in different physical systems in a general way has a close connection with the generality of the spontaneous symmetry breaking principle.

2. Hidden Symmetries and the Critical Dimensions

It is well known that the Landau theory has a limited range of applications. But the symmetry background of the Landau theory remains valid in the area, where the mean-field approximation fails. The reason for the mean-field approximation failure is an increasing of the critical order parameter fluctuation. The calculation of critical indices now needs very difficult methods based on the renormgroup technique. But the symmetry principles again turn out useful. On the one hand, the information about the group structure of a system under consideration allows one to obtain useful data avoiding the direct calculation (conservation laws, *etc.*). On the other hand, the group methods are used to determine the validity range for the Landau theory. The important criterion of the validity of the Landau theory is the dimension of the space. If the dimension of a physical space is larger than a certain value called the critical dimension (CD), then the Landau theory is valid.

In the φ^4 model which describes the phase transition (PT) at the usual critical point (CP), the CD is equal to 4. However, in the models which describe the PT in the system with both the multicritical and Lifshits points, the CD differs from 4 and depends on the power of a nonlinearity of the model and the order of the Lifshits point.

To describe the critical phenomena in the systems with the Lifshits point, one need to take the higher gradients of order parameter (OP) into account [3]. The CD of such models is $d_c = 4\,(p+1)$, where $\frac{p}{2}$ is the order of the higher gradient. Describing the critical phenomena in the systems with multicritical points require to consider the terms with higher nonlinearities of the OP in the effective Hamiltonian. In this case, the CD has the form $d_c = 4\,(N+1)\,/\,(N-1)$, where N is the nonlinearity power of the model. The generalizations of these cases are the systems with joint multicritical and Lifshits point.

The thermodynamic potential of the system in the vicinity of the aforementioned critical point may be writ-

ISSN 2071-0194. Ukr. J. Phys. 2009. Vol. 54, No. 8-9

ten as [4,5]

$$\Phi = \int^{dm} x_i \times$$

$$\times d^{d-m x_c \left\{ \frac{r}{2}\varphi^2 + \frac{\gamma}{2}\left(\Delta^{\frac{1}{2}}{}_i\,\varphi\right)^2 + \frac{\delta}{2}\left(\Delta^{\frac{1}{2}}{}_c\,\varphi\right)^2 + \frac{\beta}{2}\left(\Delta^{\frac{p}{2}}{}_i\,\varphi\right)^2 + u\varphi^{N+1}\right\}}, \tag{1}$$

where φ is the one-component OP, d is the dimension of the physical space, r, γ, δ, and β are the material parameters. We assume that the physical space can be divided into two subspaces with the dimensions $d-m$ and m. In the first case, denoted by c, there are no wave vectors of modulation. In the second one, denoted by i, the wave vectors of modulation are present. We assume that d and m can be considered as continuous variables, and, of course, $d > m$. The quantities Δ_c and Δ_i are the Laplacian operators in subspaces c and i, respectively. The operators $\Delta^l = \Delta(\Delta^{l-1})$. If l is a non-integer number, then Δ^l should be understood as the pseudodifferential operators defined with the help of integral Fourier transforms. In the CP, $r = \gamma = 0$.

We will find the CD of model (1) from the condition of stability of a fixed point of the renormgroup transformation for Hamiltonian (1). This condition looks as follows [4, 5]:

$$d > d_c = m\left(1 - \frac{1}{p}\right) + 2\frac{N+1}{N-1}. \tag{2}$$

If the dimension of the physical space is more than d_c, then the Landau theory is valid, otherwise it is invalid due to the anomalous increase of the OP fluctuation in the vicinity of the CP.

The space with the dimension which coincides with that of the CD has some interesting properties. In such a space, the model which describes the PT is renormalizable and allows the variational scale symmetry. The variational scale invariance of the model is the important property which can be useful in the analysis of the corresponding variational equations. The symmetry of the case under consideration has a connection with the conformal symmetry of field theory.

3. Exact Solvability in the System with Spontaneous Symmetry Breaking

It has been pointed out many times that the nature of peculiarities of physical magnitudes at the critical points is probably caused by some hidden symmetries of the exact solutions which are lost in the approximations (such as the renormgroup one and others) and generally are not related to the invariance spontaneously broken under the phase transition.

Namely, the infinite-parametric conformal symmetry (the exclusive property of the two-dimensional space) provides the exact solvability of both the problem of the thermodynamics of plane Ising-like models and the problem of its N-point correlators (at the critical temperature $T = T_c$). Then, the upper boundary of the band of the existence of a superstrong wave collapse in the nonlinear Schrödinger equation coincides with the condition of the conformal symmetry and the condition of existence of the exact algebraic soliton solutions of the corresponding differential equations. The list of such examples can be continued.

The conformal symmetry is also of a special significance for the Ginzburg–Landau–Wilson models of phase transitions with arbitrary nonlinearity (it is a special case of model (1)):

$$\Phi = \int^{d} d_r \left[(\nabla\varphi)^2 - \lambda\varphi^2 + \frac{\mu}{N+1}\varphi^{N+1}\right], \tag{3}$$

where φ(r) is the order parameter.

Using (2), one can easily obtain the condition for the critical dimension d_c in such models:

$$N = \frac{d_c + 2}{d_c - 2}. \tag{4}$$

Both of the properties of model (3) (the presence of critical dimension and renormalizability) providing the efficiency of its renormgroup description are caused just by the conformal invariance of the variational differential equation for Φ, taking place only at the critical point $\lambda = 0$ and only under condition (4).

Such a symmetric description is especially efficient in the models with one-dimensional gradients (of an arbitrary order). This can be easily seen from the following example. Let us consider the differential equation

$$\varphi'' - \lambda\varphi + \varphi^3 = 0. \tag{5}$$

Its general solution can be represented in the form

$$\varphi(x) = \sqrt{\lambda}\frac{\sqrt{2C^2}}{\sqrt{2C^2 - 1}} Cn\left(\frac{x\sqrt{\lambda} + C_1}{\sqrt{2C^2 - 1}}, C_1\right), \tag{6}$$

where C, C_1 are the integration constants. The differential equation (5) for the simplest Ginzburg–Landau–Wilson model (4), as is seen from (6), has the poles not only on the x-plane, but on the OC-axis too. Passing to the critical point ($\lambda \to 0$), one will restore the scale symmetry of differential equation (5). At the point λ=0, the general solution of differential equation (5) has the form

$$\varphi(x) = \tilde{C}Cn\left(\tilde{C}x + C_1, \frac{1}{\sqrt{2}}\right), \qquad (7)$$

(in addition, there is also the exceptional solution $\varphi(x) = (\pm i\sqrt{})/(x+D))$. Solution (7) cannot be obtained from solution (6) (by the usual limit transition $\lambda \to 0$). The restoration of scale symmetry of the differential equation (5) requires the combined transition: $\lambda \to 0$, $C^2 \to 1/2$.

Hidden symmetries of the field theory equations have a closed connection with the existence of a soliton-like solution at the critical point. We will illustrate this, by using the model

$$\hat{\Phi} = \hat{\Phi}_0 \int [(\Delta f)^2 + \hat{g}r^\alpha (f\boldsymbol{\nabla} f)^2 + \frac{r^{2\alpha}}{3} f^6] d^d\mathbf{r}. \qquad (8)$$

The Euler–Poisson equation for functional (8) is invariant under the transformation

$$\mathbf{r}^* = \lambda\mathbf{r}, \quad f^*(\mathbf{r}^*) = \lambda^{-\delta} f(\mathbf{r}), \qquad (9)$$

where $\delta = (\alpha+2)/2$.

On the assumption of $\alpha = d-6$, (9) becomes a variational equation. After the change of the variable $t = \ln r$, $f(r) = r^{-1}\varphi(t)$.

In the case $d = 6$, the variational equation for functional (8) takes the form

$$\varphi^{(IV)} - \hat{g}(\varphi^2\varphi'' + \varphi\varphi^2) - 10\varphi'' + 9\varphi + 2\hat{g}\varphi^3 + \varphi^5 = 0. \quad (10)$$

The hidden symmetry of (9) leads to the existence of a soliton-like solution [6,7]

$$\varphi(x) = a_c \cdot cn(bx, 1) = \frac{\hat{a}}{ch(wx)} = a_d \cdot dn(b_d x, 1), \qquad (11)$$

where $a_c, b_c, \hat{a}, a_d$, and b_d are some coefficients depending on parameters of the initial equation.

The solutions of type (11) play an important role in the theory of modulated structures.

4. Symmetries and the Dynamic Ordering

Dynamic long-range ordering is a generalization of the long-range ordering to the case of nonequilibrium time-dependent systems. The analogy between dynamic (superplasticity, superradiation, *etc.*) and time-independent (superconductivity, superfluidity, ferroelectricity) collective effects arises not only in a quantitative description [8], but also in fundamental symmetry aspects based on the Higgs and Goldstone theorems. In contrast to the time-independent case, the description of dynamical effects requires to use non-Abelian groups of symmetries.

The investigation of the field approximation of a superplastic transition shows that the dynamical equations for corresponding fields have some properties of the modulated structure theory equations [9]. The existence of soliton-like solutions in the Maxwell–Bloch model of superradiation is related to the hidden symmetry of the equation [10]

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + Ke^y + F(x) = 0. \qquad (12)$$

This symmetry arises in the case of $F = 2(f^2 + f')$, (such a type of the connection between F and f arises in the supersymmetry theories). It is easy to find the exact soliton solution of (12), by using this symmetry.

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Received 01.07.09

ПРИХОВАНІ СИМЕТРІЇ ТА КРИТИЧНІ РОЗМІРНОСТІ В ТЕОРІЇ МОДУЛЬОВАНИХ СТРУКТУР

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Р е з ю м е

Розглянуто деякі аспекти теорії критичних явищ в системах зі спонтанно порушеною симетрією. Обговорено застосовність наближення середнього поля для опису систем із модульованим впорядкуванням. Досліджено зв'язок між симетріями відповідних моделей та існуванням точних розв'язків. Обговорено роль симетрій в теорії динамічного дальнього впорядкування.