FROM BBGKY HIERARCHY TO NON-MARKOVIAN EVOLUTION EQUATIONS

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The problem of description of the evolution of the microscopic phase density and its generalizations is discussed. With this purpose, the sequence of marginal microscopic phase densities is introduced, and the appropriate BBGKY hierarchy for these microscopic distributions and their average values is formulated. The microscopic derivation of the generalized evolution equation for the average value of the microscopic phase density is given, and the non-Markovian generalization of the Fokker–Planck collision integral is proposed.

1. Introduction

Recently, much interest has been generated to the studies of physical systems with statistical properties which cannot be described within the concept of the Markovian random processes. This concerns the anomalous transport in turbulent plasma, strange diffusion of magnetic field lines in fusion plasma, the Brownian motion of macroparticles in complex fluids, *etc.* (see, for example, [1–7] and references cited therein). One of the challenging and most important, at the same time, problems is to describe the kinetic properties of such systems and to understand the role of non-Markovian effects of the particle and energy transport in such systems. The key point of such calculations is the formulation of the consistent kinetic-type equations with regard to the time-nonlocality of collision integrals and the renormalization of a free-streaming particle propagator due to the fluctuation influence on particle trajectories. In [8–10], this problem was solved on the basis of the Green function method similar to that proposed in [11]. The main disadvantage of such a treatment is the uncertainty introduced by the specific splitting of higher correlations made on the basis of some physical arguments.

The aim of the present paper is to derive the non-Markovian kinetic-type equations in a more consistent way on the basis of a rigorous solution of the appropriate BBGKY hierarchy (Bogolyubov–Born–Green–Kirkwood–Yvon) for the generalized microscopic phase densities and their average values.

The paper is organized in the following order.

In Section 2, we introduce the definitions used for the description of the evolution of the microscopic phase density and its generalization. In Section 3, we derive the evolution equations for the average values of microscopic phase densities and construct a solution of the initial-value problem of the obtained hierarchy of equations. In Section 4 with the use of the results obtained above, we develop a new approach to the description of the evolution of the average microscopic phase density; namely, we formulate a generalized non-Markovian equation for this quantity and introduce a non-Markovian generalization of the Fokker–Planck equation.

2. Microscopic Evolution of Many-particle Systems

Let us introduce the definitions which are necessary for the description of the evolution of the observables of many-particle systems such as the microscopic phase density and its generalization.

2.1. Approaches to the description of the evolution and average values of observables

We consider the system of a non-fixed (i.e. arbitrary but finite) number of identical particles with unit mass $m = 1$ in the space $\mathbb{R}^3$ (in the terminology of statistical mechanics, it is known as *nonequilibrium grand canonical ensemble* [15]). Every particle is characterized by the phase space coordinates $x_i \equiv (q_i, p_i)$, i.e. by a position in the space $q_i \in \mathbb{R}^3$ and a momentum $p_i \in \mathbb{R}^3$. A description of many-particle systems can be formulated in terms of two sets of objects: by the sequence of observables $A = (A_0, A_1(x_1), \ldots, A_n(x_1, \ldots, x_n), \ldots)$ and by the sequence of states $D = (1, D_1(x_1), \ldots, D_n(x_1, \ldots, x_n), \ldots)$. The average values of observables (mean values or expectation values of observables) determine a duality between observables and states. As a consequence, there exist two approaches to the description of the many-particle system evolution, namely those concerning the evolution of observables or the evolution of states:

$$\langle A\rangle(t) = \big(1, D(0)\big)^{-1}\big(A(t), D(0)\big) =$$

$$= \big(1, D(0)\big)^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\dots dx_n A_n(t)D_n(0) =$$

$$= \big(1, D(0)\big)^{-1}\big(A(0), D(t)\big) =$$

$$= \big(1, D(0)\big)^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\dots dx_n A_n(0)D_n(t). \qquad (1)$$

Here, $\big(1, D(0)\big) = \sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\dots dx_n D_n(0)$ is a normalizing factor (*grand canonical partition function*). The sequence $D(t) = (1, D_1(t, x_1), \dots, D_n(t, x_1, \dots, x_n), \dots)$ of probability densities of the distribution functions $D_n(t)$ is a solution of the initial-value problem of the Liouville equation, and the sequence of observables $A(t) = (A_0, A_1(t, x_1), \dots, A_n(t, x_1, \dots, x_n), \dots)$ is a solution of the initial-value problem of the Liouville equation for observables. If $A(0)$ is the sequence of continuous functions and $D(0)$ is the sequence of integrable functions, then functional (1) exists.

An equivalent approach to the description of the evolution of many-particle systems, that enables to describe systems in the thermodynamic limit, is given by the sequences of s-particle (marginal) distribution functions $F(t) = \big(1, F_1(t, x_1), \dots, F_s(t, x_1, \dots, x_s), \dots\big)$ and s-particle (marginal) observables $G(t) = \big(G_0, G_1(t, x_1), \dots, G_s(t, x_1, \dots, x_s), \dots\big)$. The sequence $F(t)$ is a solution of the initial-value problem of the Bogolyubov chain of equations (BBGKY hierarchy), and $G(t)$ is a solution of the initial-value problem of the dual Bogolyubov chain of equations (dual BBGKY hierarchy). In that case, the average values of observables at a time moment $t \in \mathbb{R}$ are determined by the functional

$$\langle A\rangle(t) = \big(G(0), F(t)\big) =$$

$$= \sum_{s=0}^{\infty}\frac{1}{s!}\int dx_1\dots dx_s G_s(0)F_s(t) =$$

$$= \big(G(t), F(0)\big) = \sum_{s=0}^{\infty}\frac{1}{s!}\int dx_1\dots dx_s G_s(t)F_s(0). \qquad (2)$$

Thus, the sequence of marginal observables $G(t)$ in terms of the sequence $A(t)$ is defined by the formula

$$G_s(t, x_1, \dots, x_s) =$$

$$= \sum_{n=0}^{s}\frac{(-1)^n}{n!}\sum_{j_1\neq\dots\neq j_n=1}^{s} A_{s-n}\big(t, Y\backslash\{x_{j_1}, \dots, x_{j_n}\}\big), \qquad (3)$$

where $Y \equiv (x_1, \dots, x_s)$, $s \geq 1$, and the sequence $F(t)$ of marginal distribution functions is defined in terms of the sequence $D(t)$ as follows (*nonequilibrium grand canonical ensemble*)

$$F_s(t, x_1, \dots, x_s) =$$

$$= \big(1, D(0)\big)^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_{s+1}\dots dx_{s+n} D_{s+n}(t). \qquad (4)$$

We remark that, in the case of a system with a fixed number N of particles (*nonequilibrium canonical ensemble*), the observables and states are the one-component sequences, respectively, $A^{(N)} = (0, \dots, 0, A_N, 0, \dots)$, $D^{(N)} = (0, \dots, 0, D_N, 0, \dots)$. Therefore, the formula for average values (1) reduces to the expression

$$\langle A^{(N)}\rangle = \big(1, D^{(N)}\big)^{-1}\int dx_1\dots dx_N A_N D_N,$$

where $\big(1, D^{(N)}\big) = \int dx_1\dots dx_N D_N$ is a normalizing factor (*canonical partition function*) [13].

2.2. Evolution of microscopic phase density

We introduce the observables known as the microscopic phase densities of the system with a non-fixed number of identical particles with unit mass $m = 1$ in the space $\mathbb{R}^3$. Let $N(t) \equiv \big(N^{(1)}(t), \dots, N^{(k)}(t), \dots\big)$, where $N^{(k)}(t) = \big(0, \dots, 0, N_k^{(k)}(t), \dots, N_n^{(k)}(t), \dots\big)$, $k \geq 1$, is the sequence of microscopic phase densities of the k-ary type

$$N_n^{(k)}(t) \equiv N_n^{(k)}(t, \xi_1, \dots, \xi_k; x_1, \dots, x_n) =$$

$$= \sum_{i_1\neq\dots\neq i_k=1}^{n}\prod_{l=1}^{k}\delta(\xi_l - X_{i_l}(t, x_1, \dots, x_n)), \qquad (5)$$

where δ is the Dirac δ-function, $\xi_1, \dots, \xi_k$ are the macroscopic variables $\xi_i = (v_i, r_i) \in \mathbb{R}^3 \times \mathbb{R}^3$. The set of functions $\big\{X_i(t, x_1, \dots, x_n)\big\}_{i=1}^{n}$, $n \geq k \geq 1$, are the solution

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of the Cauchy problem of the Hamilton equations for n particles with the Hamiltonian

$$H_n = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i<j=1}^{n} \Phi(q_i - q_j), \tag{6}$$

where $\Phi(q_i - q_j)$ is the pairwise interaction potential, and with the initial data $x_1, \ldots, x_n$.

For example, if $k = 1$, i.e. in the case of an additive-type observable, we have the microscopic phase density [8]

$$N_n^{(1)}(t, \xi_1; x_1, \ldots, x_n) = \sum_{i=1}^{n} \delta(\xi_1 - X_i(t, x_1, \ldots, x_n)).$$

Microscopic phase densities (5) are the solutions of a sequence of the Cauchy problems of the Liouville equations for observables

$$\frac{\partial}{\partial t} N_n^{(k)}(t) = \Bigg(\sum_{i=1}^{n} \langle p_i, \frac{\partial}{\partial q_i} \rangle -$$

$$- \sum_{i \neq j=1}^{n} \langle \frac{\partial}{\partial q_i} \Phi(q_i - q_j), \frac{\partial}{\partial p_i} \rangle \Bigg) N_n^{(k)}(t), \tag{7}$$

with the initial data $(1 \leq k \leq n)$

$$N_n^{(k)}(t)|_{t=0} = \sum_{i_1 \neq \ldots \neq i_k = 1}^{n} \prod_{l=1}^{k} \delta(\xi_l - x_{i_l}), \tag{8}$$

where the brackets $\langle \cdot, \cdot \rangle$ is a scalar product of vectors.

We note that solution (5) of Cauchy problem (7)-(8) defines the one-parametric group of operators $\mathbb{R}^1 \ni t \mapsto S_n(t) N_n(0)$, i.e.

$$N_n^{(k)}(t, \xi_1, \ldots, \xi_k; x_1, \ldots, x_n) = S_n(t) N_n^{(k)}(0), \tag{9}$$

where $N_n^{(k)}(0)$ is the microscopic phase density (8).

In terms of the variables $\xi_1, \ldots, \xi_k$, the sequence of Liouville equations (7) for microscopic phase densities (5) is represented as the Bogolyubov system of equations with respect to the arity index $k \geq 1$, while it is a sequence of equations with respect to the index of the number of particles $n \geq k$. Indeed, we have

$$\frac{\partial}{\partial t} N_n^{(k)}(t) = \Bigg(- \sum_{i=1}^{k} \Big\langle v_i, \frac{\partial}{\partial r_i} \Big\rangle +$$

$$+ \sum_{i \neq j=1}^{k} \Big\langle \frac{\partial}{\partial r_i} \Phi(r_i - r_j), \frac{\partial}{\partial v_i} \Big\rangle \Bigg) N_n^{(k)}(t) +$$

$$+ \sum_{i=1}^{k} \int d\xi_{k+1} \Big\langle \frac{\partial}{\partial r_i} \Phi(r_i - r_{k+1}), \frac{\partial}{\partial v_i} \Big\rangle N_n^{(k+1)}(t). \tag{10}$$

Here, $k < n$, and if $k = n$, it is the Liouville equation. Using such a representation of (7), we can directly derive the evolution equations for average values (1) of microscopic phase densities (5).

We observe that, in the case of a system with a fixed number N of particles, equations (10) are the hierarchy-type system of equations with respect to the index $k \geq 1$, and the Liouville equation with respect to a number of particles.

The fact that the microscopic phase densities (5) are exactly governed by the BBGKY-type hierarchy of equations (10) is closely connected with the structure of the hierarchy. Indeed, it is known [20] that the BBGKY hierarchy for the states (see (22) below) has the explicit solution - the marginal pure state which is the sequence of functions of the phase space variables similar to microscopic phase densities (5) as functions with respect to the "macroscopic variables".

2.3. Evolution of marginal microscopic phase densities

We introduce the sequence of marginal observables $G(t) \equiv \big(G^{(1)}(t), \ldots, G^{(k)}(t), \ldots\big)$ of k-ary type $G^{(k)}(t) = \big(0, \ldots, 0, G_k^{(k)}(t), \ldots, G_s^{(k)}(t), \ldots\big)$ defined by (3) through microscopic phase densities (5).

For example, according to (3) at the initial time moment $t = 0$, the sequence of marginal additive-type microscopic phase densities has the form

$$G^{(1)}(0) = \big(0, \ldots, \delta(\xi_1 - x_1), 0, \ldots\big).$$

Correspondingly, the sequence of marginal observables of the k-ary type microscopic phase densities (5) is given as follows:

$$G^{(k)}(0) = \big(0, \ldots, 0, \sum_{i_1 \neq \ldots \neq i_k = 1}^{k} \prod_{l=1}^{k} \delta(\xi_l - x_{i_l}), 0, \ldots\big). \tag{11}$$

Let $Y \equiv (x_1, \ldots, x_s)$ and $(x_1, \ldots, \stackrel{j}{\bigvee}, \ldots, x_s) \equiv (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_s) = Y \setminus x_j$.

The marginal microscopic phase densities $G_s^{(k)}(t) \equiv G_s^{(k)}(t, \xi_1, \ldots, \xi_k; x_1, \ldots, x_s)$ of every k-ary type are governed by the initial-value problem of the dual BBGKY hierarchy [16]

$$\frac{\partial}{\partial t} G_s^{(k)}(t) = \Bigg(\sum_{i=1}^{s} \Big\langle p_i, \frac{\partial}{\partial q_i} \Big\rangle -$$

$$-\sum_{i\neq j=1}^{s}\Big\langle\frac{\partial}{\partial q_i}\Phi(q_i-q_j),\frac{\partial}{\partial p_i}\Big\rangle\Big)G_s^{(k)}(t)-$$

$$-\sum_{i\neq j=1}^{s}\Big\langle\frac{\partial}{\partial q_i}\Phi(q_i-q_j),\frac{\partial}{\partial p_i}\Big\rangle G_{s-1}^{(k)}(t,Y\setminus x_j) \qquad (12)$$

with the initial data

$$G_s^{(k)}(t)\mid_{t=0}=G_s^{(k)}(0),\quad s\geq k\geq 1. \qquad (13)$$

As a case in point, we adduce the first equation of hierarchy (12)

$$\frac{\partial}{\partial t}G_k^{(k)}(t)=\Big(\sum_{i=1}^{k}\Big\langle p_i,\frac{\partial}{\partial q_i}\Big\rangle-$$

$$-\sum_{i\neq j=1}^{k}\Big\langle\frac{\partial}{\partial q_i}\Phi(q_i-q_j),\frac{\partial}{\partial p_i}\Big\rangle\Big)G_k^{(k)}(t).$$

For the marginal additive-type microscopic phase density, the first two equations have the form

$$\frac{\partial}{\partial t}G_1^{(1)}(t,\xi_1;x_1)=\Big\langle p_1,\frac{\partial}{\partial q_1}\Big\rangle G_1^{(1)}(t,\xi_1;x_1)$$

and

$$\frac{\partial}{\partial t}G_2^{(1)}(t,\xi_1;x_1,x_2)=\Big(\sum_{i=1}^{2}\Big\langle p_i,\frac{\partial}{\partial q_i}\Big\rangle-$$

$$-\sum_{i\neq j=1}^{2}\Big\langle\frac{\partial}{\partial q_i}\Phi(q_i-q_j),\frac{\partial}{\partial p_i}\Big\rangle\Big)G_2^{(1)}(t,\xi_1;x_1,x_2)-$$

$$-\sum_{i\neq j=1}^{2}\Big\langle\frac{\partial}{\partial q_i}\Phi(q_i-q_j),\frac{\partial}{\partial p_i}\Big\rangle G_1^{(1)}(t,\xi_1;x_i).$$

To determine a solution of the dual BBGKY hierarchy (12), we introduce some preliminaries.

We define the *nth*-order cumulant of the groups of operators (9) on the continuous functions ($n\geq 1$) as

$$\mathfrak{A}_n(t)\equiv\mathfrak{A}_n(t,X)=$$

$$=\sum_{\mathrm{P}:\,X=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}S_{|X_i|}(t), \qquad (14)$$

where $\sum_{\mathrm{P}}$ is the sum over all possible partitions P of the set $X\equiv(x_1,\ldots,x_n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i\subset X$, the operator $S_{|X_i|}(t)$ being defined by formula (9).

The simplest examples of cumulants (14) have the form

$$\mathfrak{A}_1(t,x_1)=S_1(t,x_1),$$

$$\mathfrak{A}_2(t,x_1,x_2)=S_2(t,x_1,x_2)-S_1(t,x_1)S_1(t,x_2),$$

$$\mathfrak{A}_3(t,x_1,x_2,x_3)=S_3(t,x_1,x_2,x_3)-$$

$$-S_1(t,x_1)S_2(t,x_2,x_3)-S_1(t,x_2)S_2(t,x_1,x_3)-$$

$$-S_1(t,x_3)S_2(t,x_1,x_2)+2!S_1(t,x_1)S_1(t,x_2)S_1(t,x_3).$$

The generator of the first-order cumulant is defined on a continuously differentiable function $g_1=g_1(x_1)$ by the operator

$$\lim_{t\to 0}\frac{1}{t}(\mathfrak{A}_1(t,x_1)-I)g_1=\Big\langle p_1,\frac{\partial}{\partial q_1}\Big\rangle g_1.$$

In the case $n=2$, we have

$$\lim_{t\to 0}\frac{1}{t}\,\mathfrak{A}_2(t,x_1,x_2)g_2=-\sum_{i\neq j=1}^{2}\Big\langle\frac{\partial}{\partial q_i}\Phi(q_i-q_j),\frac{\partial}{\partial p_i}\Big\rangle g_2.$$

If $n>2$, as a consequence of the fact that we consider a system of particles interacting by a two-body potential (6), the limit

$$\lim_{t\to 0}\frac{1}{t}\mathfrak{A}_n(t)g_n=0$$

holds. We introduce also the following notations: $Y=(x_1,\ldots,x_s)$, $X=Y\setminus\{x_{j_1},\ldots,x_{j_{s-n}}\}$.

For continuous functions in the capacity of initial data, a solution of Cauchy problem (12)–(13) is defined by the following one-parametric group of operators $\mathbb{R}^1\ni t\mapsto U^+(t)G(0)$

$$G_s^{(k)}(t,Y)=(U^+(t)G^{(k)}(0))_s(Y)=$$

$$=\sum_{n=0}^{s}\frac{1}{n!}\sum_{j_1\neq\ldots\neq j_n=1}^{s}\mathfrak{A}_{1+n}(t)G_{s-n}^{(k)}(0,Y\setminus\{x_{j_1},\ldots,x_{j_n}\}), \qquad (15)$$

where the evolution operator

$$\mathfrak{A}_{1+n}(t) = \mathfrak{A}_{1+n}(t, (Y \setminus X)_1, X) =$$

$$= \sum_{\mathrm{P}:\{(Y\setminus X)_1, X\}=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} S_{|X_i|}(t, X_i)$$

is the $(1+n)th$-order cumulant (14) of groups $S_{|X_i|}(t)$ of operators (9), and $\sum_{\mathrm{P}}$ is the sum over all possible partitions P of the set $\{(Y \setminus X)_1, X\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset \{(Y \setminus X))_1, X\}$. The set $(Y \setminus X)_1$ consists of one element of $Y\setminus X$, i.e. the set $Y\setminus X = \{x_{j_1}, \ldots, x_{j_{s-n}}\}$ is a connected subset of the partition P ($|\mathrm{P}| = 1$).

Solution (15) is an expansion over particle clusters, whose evolutions are governed by the corresponding-order cumulant (semiinvariant) of the evolution operators of finitely many particles.

Consider the solution expansion (15) in the case of initial marginal microscopic phase densities of the k-ary type. For the additive-type microscopic phase density (11), we derive

$$G_s^{(1)}(t, Y) = \mathfrak{A}_s(t, Y) \sum_{j=1}^{s} \delta(\xi_1 - x_j), \qquad (16)$$

where $Y \equiv (x_1, \ldots, x_s)$.

Then, in terms of variables $\xi_1, \ldots, \xi_k$, the first equation of hierarchy (12) for the additive-type microscopic phase density (16) takes the form

$$\frac{\partial}{\partial t} G_s^{(1)}(t, \xi_1; x_1, \ldots, x_s) =$$

$$= -\Big\langle v_1, \frac{\partial}{\partial r_1}\Big\rangle G_s^{(1)}(t, \xi_1; x_1, \ldots, x_s) +$$

$$+ \int d\xi_2 \Big\langle \frac{\partial}{\partial r_1}\Phi(r_1 - r_2), \frac{\partial}{\partial v_1}\Big\rangle G_s^{(2)}(t, \xi_1, \xi_2; x_1, \ldots, x_s),$$

with the initial data

$$G_s^{(1)}(t, \xi_1; x_1, \ldots, x_s) \mid_{t=0} = \sum_{i=1}^{s} \delta(\xi_1 - x_i)\, \delta_{s,1},$$

where $\delta_{s,1}$ is the Kronecker symbol, $s \geq 1$.

In a similar manner for the marginal microscopic phase densities of k-ary type $G^{(k)}(t) = (0, \ldots, 0, G_k^{(k)}(t), \ldots, G_s^{(k)}(t), \ldots)$, we derive

$$\frac{\partial}{\partial t} G_s^{(k)}(t) = \Big(-\sum_{i=1}^{k} \Big\langle v_i, \frac{\partial}{\partial r_i}\Big\rangle +$$

$$+ \sum_{i \neq j=1}^{k} \Big\langle \frac{\partial}{\partial r_i}\Phi(r_i - r_j), \frac{\partial}{\partial v_i}\Big\rangle \Big) G_s^{(k)}(t) +$$

$$+ \sum_{i=1}^{k} \int d\xi_{k+1} \Big\langle \frac{\partial}{\partial r_i}\Phi(r_i - r_{k+1}), \frac{\partial}{\partial v_i}\Big\rangle G_s^{(k+1)}(t) \qquad (17)$$

with the initial data

$$G_s^{(k)}(t) \mid_{t=0} = \sum_{i_1 \neq \ldots \neq i_k = 1}^{s} \prod_{l=1}^{k} \delta(\xi_l - x_{i_l}) \delta_{s,k}. \qquad (18)$$

Here, $1 \leq r < s$, and if $k = s$, the marginal microscopic phase density $G_s^{(s)}(t)$ is governed by the Liouville equation.

Thus, in terms of the variables $\xi_1, \ldots, \xi_k$, the dual BBGKY hierarchy (17) for marginal microscopic phase densities (15) is represented as the Bogolyubov system of equations with respect to the arity index $k \geq 1$, while evolution equations (17) have a structure of a sequence of equations with respect to the index of the number of particles $s \geq k$.

We note that, for every s, a solution of initial-value problem (17)-(18) of the system of equations (17) can be represented as the expansion

$$G_s^{(k)}(t, \xi_1, \ldots, \xi_k; x_1, \ldots, x_s) =$$

$$= \sum_{n=0}^{s-k} \frac{1}{n!} \int d\xi_{k+1} \ldots d\xi_{k+n}\, \mathfrak{A}_{1+n}(-t)\, G_s^{(k+n)}(0),$$

where $G_s^{(k+n)}(0)$ is initial data (18), and $\mathfrak{A}_{1+n}(-t) \equiv \mathfrak{A}_{1+n}(-t, Y_1, \xi_{k+1}, \ldots, \xi_{k+n})$ is the $(1+n)th$-order cumulant (34) of the groups of evolution operators (31) defined further.

2.4. Evolution of states of many-particle systems

We furnish further comments about the evolution of microscopic phase densities in the framework of the evolution of states of many-particle systems.

In this case, the microscopic phase densities are defined at the initial time moment $N(0) = (N^{(1)}(0), \ldots, N^{(k)}(0), \ldots)$, where $N^{(k)}(0) = (0, \ldots, 0, N_k^{(k)}(0), \ldots, N_n^{(k)}(0), \ldots)$ and

$$N_n^{(k)}(0) \equiv N_n^{(k)}(0, \xi_1, \ldots, \xi_k; x_1, \ldots, x_n) =$$

$$= \sum_{i_1 \neq \ldots \neq i_k = 1}^{n} \prod_{l=1}^{k} \delta(\xi_l - x_{i_l}), \quad n \geq k \geq 1.$$

The evolution of states is described by the Cauchy problem of a sequence of the Liouville equations for the sequence $D(t) = (I, D_1(t), \ldots, D_n(t), \ldots)$ of distribution functions $D_n(t) \equiv D_n(t, x_1, \ldots, x_n)$

$$\frac{\partial}{\partial t} D_n(t) = \Big(-\sum_{i=1}^{n} \Big\langle p_i, \frac{\partial}{\partial q_i} \Big\rangle +$$

$$+ \sum_{i \neq j = 1}^{n} \Big\langle \frac{\partial}{\partial q_i} \Phi(q_i - q_j), \frac{\partial}{\partial p_i} \Big\rangle \Big) D_n(t). \tag{19}$$

$$D_n(t)|_{t=0} = D_n(0), \quad n \geq 1. \tag{20}$$

A solution of the Cauchy problem (19)–(20) was constructed in [15]. Average values of the microscopic phase densities $N(0)$ are determined by the expressions

$$\langle N^{(k)} \rangle(t, \xi_1, \ldots, \xi_k) = (1, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \times$$

$$\times \int dx_{k+1} \ldots dx_{k+n} D_{k+n}(t, \xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_{k+n}), \tag{21}$$

where $D_k(t, \xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_{k+n})$ is the value of a solution of initial-value problem (19)-(20) at a point $\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_{k+n}$. Thus, the evolution of the functions $\langle N^{(k)} \rangle(t)$, $k \geq 1$, will be governed by the BBGKY hierarchy with respect to the arity index k (see the next section).

In the thermodynamic limit, the evolution of states is described in terms of the marginal distribution functions governed by the BBGKY hierarchy [14],[15]. From Liouville equations (19) according to functional (4) for the marginal distribution functions $F(t) = (I, F_1(t), \ldots, F_s(t), \ldots)$, we derive

$$\frac{\partial}{\partial t} F_s(t) = \Big(-\sum_{i=1}^{s} \Big\langle p_i, \frac{\partial}{\partial q_i} \Big\rangle +$$

$$+ \sum_{i \neq j = 1}^{s} \Big\langle \frac{\partial}{\partial q_i} \Phi(q_i - q_j), \frac{\partial}{\partial p_i} \Big\rangle \Big) F_s(t) +$$

$$+ \sum_{i=1}^{s} \int dx_{s+1} \Big\langle \frac{\partial}{\partial q_i} \Phi(q_i - q_{s+1}), \frac{\partial}{\partial p_i} \Big\rangle F_{s+1}(t), \tag{22}$$

$$F_s(t) \mid_{t=0} = F_s(0), \quad s \geq 1. \tag{23}$$

We remark that, for a system of N particles, the BBGKY hierarchy (22) is a system of equations, and the equation for $F_N(t)$ is the Liouville equation (19).

A solution of Cauchy problem (22)-(23) was constructed in [15],[17]. The microscopic phase densities in this case are given by marginal microscopic phase densities (11) and their average values are determined by the expressions

$$\langle G^{(s)} \rangle(t) = F_s(t, \xi_1, \ldots, \xi_s), \quad s \geq 1,$$

where $F_s(t, \xi_1, \ldots, \xi_s)$ is the value of a solution of initial-value problem (22)-(23) at a point $\xi_1, \ldots, \xi_s$. The evolution equations for the average values of microscopic phase densities will be considered in the next section.

3. Evolution of Average Values of Microscopic Phase Densities

We derive the evolution equations of average values of microscopic phase densities and construct a solution of the initial-value problem of the obtained hierarchy of equations.

3.1. BBGKY hierarchy for average values of marginal microscopic phase densities

We introduce the evolution equations for average values of marginal microscopic phase densities (15).

Due to (16), the evolution equation of average value (2) of the additive-type marginal microscopic phase density has the form

$$\frac{\partial}{\partial t} \langle G^{(1)} \rangle(t, \xi_1) = -\Big\langle v_1, \frac{\partial}{\partial r_1} \Big\rangle \langle G^{(1)} \rangle(t, \xi_1) +$$

$$+ \int d\xi_2 \Big\langle \frac{\partial}{\partial r_1} \Phi(r_1 - r_2), \frac{\partial}{\partial v_1} \Big\rangle \langle G^{(2)} \rangle(t, \xi_1, \xi_2). \tag{24}$$

Let us consider the general case, i.e. the k-ary type microscopic phase density. According to (2) and (15), the dual BBGKY hierarchy (17) yields

$$\frac{\partial}{\partial t} \langle G^{(k)} \rangle(t) = -\sum_{i=1}^{k} \Big\langle v_i, \frac{\partial}{\partial r_i} \Big\rangle \langle G^{(k)} \rangle(t) +$$

$$+\sum_{i\neq j=1}^{k}\Big\langle\frac{\partial}{\partial r_i}\Phi(r_i-r_j),\frac{\partial}{\partial v_i}\Big\rangle\langle G^{(k)}\rangle(t)+$$

$$+\sum_{i=1}^{k}\int d\xi_{k+1}\Big\langle\frac{\partial}{\partial r_i}\Phi(r_i-r_{k+1}),\frac{\partial}{\partial v_i}\Big\rangle\langle G^{(k+1)}\rangle(t), \quad (25)$$

with the initial data

$$\langle G^{(k)}\rangle(t,\xi_1,\dots,\xi_k)|_{t=0}=\langle G^{(k)}\rangle(0), \quad k\geq 1. \quad (26)$$

Due to functional (2), initial data (26) are given as the functions

$$\langle G^{(k)}\rangle(0,\xi_1,\dots,\xi_k)=F_k(0,\xi_1,\dots,\xi_k),$$

where $F_s(0,\xi_1,\dots,\xi_s)$ is the value of the initial marginal state at a point $\xi_1,\dots,\xi_s$.

We note that, according to the definition of functionals (1) and (2), the equality

$$\langle G^{(k)}\rangle(t)=\langle N^{(k)}\rangle(t)$$

holds in the case of finitely many particles. In the thermodynamic limit, the value $\langle N^{(k)}\rangle(t)$ tends to $\langle G^{(k)}\rangle(t)$, i.e. to the solution of Cauchy problem (25)-(26).

3.2. BBGKY hierarchy for average values of microscopic phase densities

For comparison, we derive hierarchy (25) from equations (10).

In the general case, i.e. for the k-ary type microscopic phase density (5), we have

$$\frac{\partial}{\partial t}\langle N^{(k)}\rangle(t)=\Big(-\sum_{i=1}^{k}\Big\langle v_i,\frac{\partial}{\partial r_i}\Big\rangle+$$

$$+\sum_{i\neq j=1}^{k}\Big\langle\frac{\partial}{\partial r_i}\Phi(r_i-r_j),\frac{\partial}{\partial v_i}\Big\rangle\Big)\langle N^{(k)}\rangle(t)+$$

$$+\sum_{i=1}^{k}\int d\xi_{k+1}\Big\langle\frac{\partial}{\partial r_i}\Phi(r_i-r_{k+1}),\frac{\partial}{\partial v_i}\Big\rangle\langle N^{(k+1)}\rangle(t), \quad (27)$$

with the initial data ($k\geq 1$)

$$\langle N^{(k)}\rangle(t,\xi_1,\dots,\xi_k)|_{t=0}= \quad (28)$$

$$=(1,D(0))^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_{k+1}\dots dx_{k+n}D_{k+n}(0),$$

where $D_{k+n}(0)\equiv D_{k+n}(0,\xi_1,\dots,\xi_k;x_{k+1},\dots,x_{k+n})$ is the value of the distribution function $D_{k+n}(0)$ of the initial state at a point $\xi_1,\dots,\xi_k;x_{k+1},\dots,x_{k+n}$.

As a result of the formal transition to the thermodynamic limit, Cauchy problem (27)–(28) gets form (25)-(26).

Let us transform Eqs. (27) to the form which is usually used in the plasma theory.

We find the covariation of the microscopic phase density $\langle\delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle$, where the observable $\delta N^{(1)}=N^{(1)}-\langle N^{(1)}\rangle$ is a fluctuation of the microscopic phase density.

Since $\langle\delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle=\langle N^{(1)}(\xi_1)N^{(1)}(\xi_2)\rangle-\langle N^{(1)}(\xi_1)\rangle\langle N^{(1)}(\xi_2)\rangle$, according to formula (1), we have

$$\langle N^{(1)}(\xi_1)N^{(1)}(\xi_2)\rangle(t)=$$

$$=(1,D(0))^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\dots dx_nD_n(0,x_1,\dots,x_n)\times$$

$$\times\sum_{i=1}^{n}\sum_{j=1}^{n}\delta(\xi_1-X_i(t))\delta(\xi_2-X_j(t)).$$

In accordance with the equality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\delta(\xi_1-x_i)\delta(\xi_2-x_j)=$$

$$=\sum_{i=1}^{n}\delta(\xi_1-x_i)\delta(\xi_2-x_i)+\sum_{i\neq j=1}^{n}\delta(\xi_1-x_i)\delta(\xi_2-x_j),$$

we find

$$\langle N^{(1)}(\xi_1)N^{(1)}(\xi_2)\rangle(t)=$$

$$=\delta(\xi_1-\xi_2)\langle N^{(1)}\rangle(t,\xi_1)+\langle N^{(2)}\rangle(t,\xi_1,\xi_2).$$

Thus, the covariation of the microscopic phase density $\langle\delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle$ is defined as follows:

$$\langle\delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle(t)=\delta(\xi_1-\xi_2)\langle N^{(1)}\rangle(t,\xi_1)+$$

$$+\langle N^{(2)}\rangle(t,\xi_1,\xi_2)-\langle N^{(1)}\rangle(t,\xi_1)\langle N^{(1)}\rangle(t,\xi_2).$$

For the regularized interaction potential, i.e. $\Phi^{'}(0)=0$, in terms of the covariation of the microscopic phase density $\langle\delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle$, the first equation from hierarchy (27) reduces to the Vlasov-type equation

$$\frac{\partial}{\partial t}\langle N^{(1)}\rangle(t,\xi_1)+\Big\langle v_1,\frac{\partial}{\partial r_1}\Big\rangle\langle N^{(1)}\rangle(t,\xi_1)-$$

$$\Big\langle\frac{\partial}{\partial r_1}\int d\xi_2\Phi(r_1-r_2)\langle N^{(1)}\rangle(t,\xi_2),\frac{\partial}{\partial v_1}\Big\rangle\langle N^{(1)}\rangle(t,\xi_1)=$$

$$=\int d\xi_2\Big\langle\frac{\partial}{\partial r_1}\Phi(r_1-r_2),\frac{\partial}{\partial v_1}\Big\rangle\langle\delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle(t). \tag{29}$$

This unclosed equation with respect to the average value of the additive-type microscopic phase density $\langle N^{(1)}\rangle(t)$ is the conventional equation for the description of a system of charged particles [7].

3.3. System of charged particles

If we consider a plasma, i.e. a system of charged particles, then $\Phi(r_1-r_2)=e^2\,|r_1-r_2|^{-1}$, and the macroscopic electric field $\langle E\rangle(t)$ is defined from the equation

$$\mathrm{div}\langle E\rangle(t,\xi)=e\,\langle N^{(1)}\rangle(t,\xi)=$$

$$=e\,(1,D(0))^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int dx_1\ldots dx_n N_n^{(1)}(t,\xi)D_n(0).$$

Here, $N_n^{(1)}(t,\xi)$ is the additive-type microscopic phase density (5).

Thus, in this case, Eq. (29) takes the form

$$\left(\frac{\partial}{\partial t}+v^\alpha\frac{\partial}{\partial r^\alpha}+e\,\langle E^\alpha\rangle(t,r)\frac{\partial}{\partial v^\alpha}\right)\langle N^{(1)}\rangle(t,\xi)=$$

$$=-e\,\frac{\partial}{\partial v^\alpha}\langle\delta E^\alpha(t,r)\delta N^{(1)}(t,\xi)\rangle, \tag{30}$$

where $\delta E=E-\langle E\rangle$ is a fluctuation of the electric field $E=(E_1,\ldots,E_n,\ldots)$, where

$$E_n\equiv E(t,r;x_1,\ldots,x_n)=$$

$$=-e\,\frac{\partial}{\partial r}\int d\xi'\,\frac{N_n^{(1)}(t,\xi';x_1,\ldots,x_n)}{|r-r'|},$$

and the macroscopic electric field $\langle E\rangle\equiv\langle E\rangle(t,r)$ is

$$\langle E\rangle(t,r)=-e\,\frac{\partial}{\partial r}\int d\xi'\,\frac{\langle N^{(1)}\rangle(t,\xi')}{|r-r'|}.$$

Equation (30) is an unclosed equation with respect to the average value of the microscopic phase density $\langle N^{(1)}\rangle(t)$.

3.4. On the hierarchy for average values of microscopic phase densities: evolution of states

According to (21) and Cauchy problem (19)-(20) of the Liouville equations, the evolution equations of the average values of k-ary type microscopic phase densities (8) get the form of hierarchy (27) with initial data (28).

Using the BBGKY hierarchy (22), due to formula (2) for marginal phase densities (11), we derive hierarchy (25) with initial data (26).

As was noted above, hierarchy (27) in the thermodynamic limit transforms to the hierarchy of equations (25).

3.5. On the solution of the initial-value problem

To determine a solution of hierarchy (25), we introduce some preliminaries.

On integrable functions, we define the group of operators

$$(S_k(-t)f_k)(\xi_1,\ldots,\xi_k):=$$

$$:=f_k(\Xi_1(-t,\xi_1,\ldots,\xi_k),\ldots,\Xi_k(-t,\xi_1,\ldots,\xi_k)), \tag{31}$$

where the functions $\Xi_i(t)\equiv(V_i(t),R_i(t))$ are the solution of the Cauchy problem of the Hamilton equations for "macroscopic variables" (dynamics of continuum)

$$\frac{d}{dt}R_i(t)=V_i(t), \tag{32}$$

$$\frac{d}{dt}V_i(t)=-\sum_{j\neq i,j=1}^{k}\frac{\partial}{\partial R_i(t)}\Phi(R_i(t)-R_j(t)),$$

with the initial data

$$R_i(0)=r_i,$$

$$V_i(0)=v_i,\qquad i=1,\ldots,k.$$

The generator of group (31) is defined by the Poisson bracket with respect to the variables $\xi_1,\ldots,\xi_k$ on the continuously differentiable functions $f_k\equiv f_k(\xi_1,\ldots,\xi_k)$

$$\lim_{t\to0}\frac{1}{t}(S_k(-t)-I)f_k=\Bigg(-\sum_{i=1}^{k}\Big\langle v_i,\frac{\partial}{\partial r_i}\Big\rangle+$$

$$+\sum_{i\neq j=1}^{k}\Big\langle\frac{\partial}{\partial r_i}\Phi(r_i-r_j),\frac{\partial}{\partial v_i}\Big\rangle\Bigg)f_k.$$

We also define the cumulants of groups of operators (31) like (14) on integrable functions. They have properties similar to those of (14).

A solution of Cauchy problem (25)–(26) is defined by the expansion over the arity index of the microscopic phase density, whose evolution is governed by the corresponding-order cumulant (semiinvariant) of the evolution operators (31), namely

$$\langle G^{(k)}\rangle(t,\xi_1,\ldots,\xi_k) = (U(t)\langle G\rangle(0))_k(\xi_1,\ldots,\xi_k) =$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int d\xi_{k+1}\ldots d\xi_{k+n}\mathfrak{A}_{1+n}(-t)\langle G^{(k+n)}\rangle(0), \qquad (33)$$

where $\langle G\rangle(0) = (0, \langle G^{(1)}\rangle(0), \ldots, \langle G^{(k)}\rangle(0), \ldots)$ is the sequence on integrable functions. If $n \geq 0$,

$$\mathfrak{A}_{1+n}(-t) \equiv \mathfrak{A}_{1+n}(-t, Y_1, \xi_{k+1}, \ldots, \xi_{k+n}) = \qquad (34)$$

$$= \sum_{\mathrm{P}:\{Y_1, X\backslash Y\}=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i\subset \mathrm{P}} S_{|X_i|}(-t, X_i)$$

is the $(1+n)th$-order cumulant of the groups of operators (31), $\sum_{\mathrm{P}}$ is the sum over all possible partitions P of the set $\{Y_1, \xi_{k+1}, \ldots, \xi_{k+n}\}$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset \{Y_1, X\backslash Y\} \equiv \{Y_1, \xi_{k+1}, \ldots, \xi_{k+n}\}$. The set Y_1 consists of one element of $Y \equiv (\xi_1, \ldots, \xi_k)$, i.e. the set $\xi_1, \ldots, \xi_k)$ is a connected subset of the partition P ($|\mathrm{P}| = 1$).

If $\langle G^{(k)}\rangle(0)$ are integrable functions [17], series (33) converges for small densities.

We remark that a solution of Cauchy problem (25)-(26) can be represented as the iteration series of the BBGKY hierarchy (25)

$$\langle G^{(k)}\rangle(t,\xi_1,\ldots,\xi_k) =$$

$$= \sum_{n=0}^{\infty}\int_0^t dt_1\ldots\int_0^{t_{n-1}} dt_n \int d\xi_{k+1}\ldots d\xi_{k+n} S_k(-t+t_1)\times$$

$$\times \sum_{i_1=1}^{k}(-\mathcal{L}_{\mathrm{int}}(i_1,k+1))S_{k+1}(-t_1+t_2)\ldots$$

$$\ldots S_{k+n-1}(-t_{n-1}+t_n)\sum_{i_n=1}^{k+n-1}(-\mathcal{L}_{\mathrm{int}}(i_n,k+n))\times$$

$$\times S_{k+n}(-t_n)\langle G^{(k+n)}\rangle(0), \qquad (35)$$

where

$$-\mathcal{L}_{\mathrm{int}}(i,j) \equiv \Big\langle \frac{\partial}{\partial r_i}\Phi(r_i-r_j), \frac{\partial}{\partial v_i}\Big\rangle.$$

If we apply the Duhamel formula to cumulants of groups of operators (31), solution expansion (33) reduces to iteration series (35) of the BBGKY hierarchy.

4. Evolution Equations of Non-Markovian Transport

Using the above-obtained results, we develop a new approach to the description of the evolution of the average value of microscopic phase density. Namely we formulate the non-Markovian evolution equation for the average value of microscopic phase density.

4.1. Generalized equation for the average value of microscopic phase density

Consider Cauchy problem (25), (26) with initial data which are completely defined by the average value of additive-type microscopic phase density $\langle G^{(1)}\rangle(0)$, for example,

$$\langle G^{(k)}\rangle(0) = \prod_{i=1}^{k}\langle G^{(1)}\rangle(0,\xi_i). \qquad (36)$$

We note that $\langle G^{(1)}\rangle(0,\xi_i) = F_1(0,\xi_i)$. Therefore, initial data (36) have the transparent sense; they satisfy the chaos condition.

In that case, the initial-value problem of the BBGKY hierarchy (25), (26) is not a completely well-defined Cauchy problem, because the generic initial data are not independent for every unknown functions $\langle G^{(k)}\rangle(t)$, $k \geq 1$, of the hierarchy of equations. Thus, it naturally arises the possibility of reformulating such initial-value problem as a new Cauchy problem for the independent unknown function, i.e. the average value of the additive-type microscopic phase density $\langle G^{(1)}\rangle(t)$, together with explicitly defined functionals $\langle G^{(k)}\rangle(t,\xi_1,\ldots,\xi_k \mid \langle G^{(1)}\rangle(t))$, $k \geq 2$, of the solution $\langle G^{(1)}\rangle(t)$ of this Cauchy problem instead other unknown average values of microscopic phase densities [15], [18].

The functionals $\langle G^{(k)}\rangle(t,\xi_1,\ldots,\xi_k \mid \langle G^{(1)}\rangle(t))$, $k \geq 2$, are represented by the expansions over the products with respect to the average value of the additive-type microscopic phase density $\langle G^{(1)}\rangle(t)$

$$\langle G^{(k)}\rangle(t,\xi_1,\ldots,\xi_k \mid \langle G^{(1)}\rangle(t)) := \qquad (37)$$

$$:= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\xi_{k+1} \dots d\xi_{k+n} \mathfrak{V}_{1+n}(t) \prod_{i=1}^{k+n} \langle G^{(1)} \rangle (t, \xi_i),$$

where the evolution operators $\mathfrak{V}_{1+n}(t) \equiv \mathfrak{V}_{1+n}(t, Y_1, \xi_{k+1}, \dots, \xi_{k+n})$ are defined from the condition that the functionals $\langle G^{(k)} \rangle (t)(t \mid \langle G^{(1)} \rangle (t))$ are congruent with solutions (33) of initial-value problem (25)–(26).

We give examples of first terms of series (37):

$$\mathfrak{V}_1(t, Y_1) = \widehat{\mathfrak{A}}_1(t, Y_1),$$

$$\mathfrak{V}_2(t, Y_1, \xi_{k+1}) =$$

$$= \widehat{\mathfrak{A}}_2(t, Y_1, \xi_{k+1}) - \widehat{\mathfrak{A}}_1(t, Y_1) \sum_{j=1}^{s} \widehat{\mathfrak{A}}_2(t, \xi_j, \xi_{k+1}),$$

where $Y_1 \equiv (\xi_1, \dots, \xi_k)$, and $\widehat{\mathfrak{A}}_k(t)$ is the kth-order cumulant (14) of the scattering operators

$$\widehat{S}_k(t) = S_k(-t, \xi_1, \dots, \xi_k) \prod_{i=1}^{k} S_1(t, \xi_i), \tag{38}$$

the operator $S_k(-t, \xi_1, \dots, \xi_k)$ is defined by formula (31), and $\widehat{S}_1(t) = I$ is the identity operator.

On integrable functions, the action of scattering operators (38) is defined by the formula

$$(\widehat{S}_k(t) f_k)(\xi_1, \dots, \xi_k) = f_k(\Xi_1(t, \Xi_1(-t, \xi_1, \dots, \xi_k)), \dots$$

$$\dots, \Xi_k(t, \Xi_k(-t, \xi_1, \dots, \xi_k))), \tag{39}$$

where the functions $\Xi_i(t) \equiv (V_i(t), R_i(t)), i = 1, \dots, n$ are solutions of the Cauchy problem of Hamilton equations (32) for "macroscopic variables" with corresponding initial data. The generator of group (38) of scattering operators is defined by the Poisson bracket with respect to the variables $\xi_1, \dots, \xi_k$ with an interaction potential Φ on the continuously differentiable functions $f_k \equiv f_k(\xi_1, \dots, \xi_k)$

$$\lim_{t \to 0} \frac{1}{t} (\widehat{S}_k(t) - I) f_k = \sum_{i \neq j=1}^{k} \Big\langle \frac{\partial}{\partial r_i} \Phi(r_i - r_j), \frac{\partial}{\partial v_i} \Big\rangle f_k.$$

In terms of scattering operators (38), the first terms of series (37) have the form

$$\mathfrak{V}_1(t, Y_1) = \widehat{S}_k(t, Y),$$

$$\mathfrak{V}_2(t, Y_1, \xi_{k+1}) = \widehat{S}_{k+1}(t, Y, \xi_{k+1}) -$$

$$- \widehat{S}_k(t, Y) \sum_{j=1}^{k} \widehat{S}_2(t, \xi_j, \xi_{k+1}) + (k-1) \widehat{S}_k(t, Y).$$

If $\langle G^{(1)} \rangle (t)$ is an integrable function, then, in case of low densities, expansion (37) is the converging series [18].

The average value of the additive-type microscopic phase density $\langle G^{(1)} \rangle (t)$ is governed by the following Cauchy problem (*the generalized evolution equation for the average value of microscopic phase density*):

$$\frac{\partial}{\partial t} \langle G^{(1)} \rangle (t, \xi_1) + \Big\langle v_1, \frac{\partial}{\partial r_1} \Big\rangle \langle G^{(1)} \rangle (t, \xi_1) = \tag{40}$$

$$= \int d\xi_2 \Big\langle \frac{\partial}{\partial r_1} \Phi(r_1 - r_2), \frac{\partial}{\partial v_1} \Big\rangle \langle G^{(2)} \rangle (t, \xi_1, \xi_2 \mid \langle G^{(1)} \rangle (t))$$

with the initial data

$$\langle G^{(1)} \rangle (t, \xi_1)|_{t=0} = \langle G^{(1)} \rangle (0, \xi_1). \tag{41}$$

The functional $\langle G^{(2)} \rangle (t, \xi_1, \xi_2 \mid \langle G^{(1)} \rangle (t))$ in the collision integral of Eq. (40) is defined by expansion (37)

$$\langle G^{(2)} \rangle (t)(t, \xi_1, \xi_2 \mid \langle G^{(1)} \rangle (t)) =$$

$$= \widehat{\mathfrak{A}}_1(t, Y_1) \prod_{i=1}^{2} \langle G^{(1)} \rangle (t, \xi_i) + \int d\xi_3 (\widehat{\mathfrak{A}}_2(t, Y_1, \xi_{k+1}) -$$

$$- \widehat{\mathfrak{A}}_1(t, Y_1) \sum_{j=1}^{s} \widehat{\mathfrak{A}}_2(t, \xi_j, \xi_{k+1})) \prod_{i=1}^{3} \langle G^{(1)} \rangle (t, \xi_i) + ..., \tag{42}$$

where $Y_1 \equiv (\xi_1, \xi_2)$.

We represent the first term of expansion (42) in an explicit form. According to the definition of two-particle scattering operator (39), we have

$$\widehat{\mathfrak{A}}_1(t, Y_1) \prod_{i=1}^{2} \langle G^{(1)} \rangle (t, \xi_i) =$$

$$= \langle G^{(1)} \rangle (t, \Xi_1(t, \Xi_1(-t, \xi_1, \xi_2))) \times$$

$$\times \langle G^{(1)} \rangle (t, \Xi_2(t, \Xi_2(-t, \xi_1, \xi_2))),$$

where the functions $(i = 1, 2)$

$$\Xi_i(t, \Xi_i(-t, \xi_1, \xi_2)) =$$

$$= (V_i(-t,\,\xi_1,\xi_2), R_i(-t,\,\xi_1,\xi_2) + t\,V_i(-t,\,\xi_1,\xi_2)) \qquad (43)$$

are defined as above by (32) in formula (39).

The following statement is true. If the initial data are completely defined by $\langle G^{(1)}\rangle(0)$, then Cauchy problem (25)–(26) is equivalent to initial-value problem (40)–(41) of the generalized evolution equation for the average value of the microscopic phase density and functionals $\langle G^{(k)}\rangle(t,\xi_1,\dots,\xi_k \mid \langle G^{(1)}\rangle(t))$, $k \geq 2$, defined by expansions (37) provided a low density of particles.

Thus, at an arbitrary time moment, the evolution of the average value of the microscopic phase density can be described by Eq. (40) without any approximations.

The solution of Cauchy problem (40)–(41) is defined by the expansion

$$\langle G^{(1)}\rangle(t,\xi_1) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\xi_2 \dots d\xi_{1+n}\, \mathfrak{A}_{1+n}(-t) \prod_{i=1}^{n+1} \langle G^{(1)}\rangle(0,\xi_i), \quad (44)$$

where $\mathfrak{A}_{1+n}(-t) \equiv \mathfrak{A}_{1+n}(-t,\xi_1,\dots,\xi_{n+1})$ is the $(1+n)th$-order cumulant (14) of the group of operators (31).

For the low densities, i.e. $\int |\langle G^{(1)}\rangle(0,\xi)| d\xi < e^{-1}$, series (44) converges. If $\langle G^{(1)}\rangle(0)$ is a continuously differentiable function with compact support, then, for $t \in \mathbb{R}$, it is a classical solution of generalized equation (40) for the average value of the microscopic phase density.

We observe the links of the introduced generalized equation for the average value of the microscopic phase density with Bogolyubov's method of the derivation of kinetic equations [14]. Functionals (37) are formally similar to the corresponding functionals of Bogolyubov's method [19] if they satisfy the principle of weakening of correlations [15]. The proof of this statement is completely similar to the proof of the equivalence of both representations of the BBGKY hierarchy solutions as iteration series (35) and functional series (33).

4.2. Example: regularized interaction potential

We separate the Vlasov term in Eq. (40) to represent it in the conventional form for the description of a system of charged particles.

We introduce the macroscopic characteristic of correlations in continuum

$$\langle \mathcal{G}\rangle(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t)) = \langle G^{(2)}\rangle(t)(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t)) -$$

$$-\langle G^{(1)}\rangle(t,\xi_1)\langle G^{(1)}\rangle(t,\xi_2), \qquad (45)$$

where the functional $\langle G^{(2)}\rangle(t)(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t))$ is defined by expansion (42). Then the generalized evolution equation (40) for the average value of microscopic phase density gets the form

$$\frac{\partial}{\partial t}\langle G^{(1)}\rangle(t,\xi_1) = -\Big\langle v_1, \frac{\partial}{\partial r_1}\Big\rangle + \qquad (46)$$

$$+\Big\langle \frac{\partial}{\partial r_1} \int d\xi_2 \Phi(r_1-r_2)\langle G^{(1)}\rangle(t,\xi_2), \frac{\partial}{\partial v_1}\Big\rangle \langle G^{(1)}\rangle(t,\xi_1) +$$

$$+\int d\xi_2 \Big\langle \frac{\partial}{\partial r_1}\Phi(r_1-r_2), \frac{\partial}{\partial v_1}\Big\rangle \langle \mathcal{G}\rangle(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t))$$

with initial data (41).

In the case of regularized interaction potential, i.e. $\Phi'(0) = 0$, the functional $\langle \mathcal{G}\rangle(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t))$ is the formal thermodynamic limit of the covariation of the microscopic phase density $\langle \delta N^{(1)}(\xi_1)\delta N^{(1)}(\xi_2)\rangle$ from the right-hand side of Eq. (29).

If we consider a plasma, i.e. a system of charged particles, the macroscopic electric field $\langle E\rangle(t)$ is determined from the equation

$$\mathrm{div}\langle E\rangle(t,\xi) = e\,\langle G^{(1)}\rangle(t,\xi).$$

In this case, Eq. (46) gets the form

$$\left(\frac{\partial}{\partial t} + v^\alpha \frac{\partial}{\partial r^\alpha} + e\,\langle E^\alpha\rangle(t,r)\frac{\partial}{\partial v^\alpha}\right)\langle G^{(1)}\rangle(t,\xi) =$$

$$= \int d\xi' \,\frac{\partial}{\partial r^\alpha}\,\frac{1}{|r-r'|}\,\frac{\partial}{\partial v^\alpha}\,\langle \mathcal{G}\rangle(t,\xi,\xi' \mid \langle G^{(1)}\rangle(t)),$$

where the functional $\langle \mathcal{G}\rangle(t \mid \langle G^{(1)}\rangle(t))$ in the collision integral is defined by expansions (45),(42).

4.3. On the Fokker–Planck representation of generalized collision integral

If we consider a plasma [14], then the small parameter is the plasma parameter associated with the weak interaction

$$\varepsilon = \beta e^2 / r_D,$$

where r_D is the Debye screening parameter, e is the electric charge, and β^{-1} is the value of the order of the average kinetic energy of electrons.

Let us consider the first term of the collision integral expansion of $\langle \mathcal{G}\rangle(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t))$ in Eq. (46), i.e. the first-order term with respect to the plasma parameter associated with the weak interaction,

$$\mathcal{I} \equiv \int d\xi_2 \Big\langle \frac{\partial}{\partial r_1}\Phi(r_1 - r_2), \frac{\partial}{\partial v_1}\Big\rangle (\widehat{S}_2(t,\xi_1,\xi_2) - I)\times$$

$$\times \prod_{i=1}^{2} \langle G^{(1)}\rangle(t,\xi_i). \quad (47)$$

We will construct a suitable approximation of the non-Markovian collision integral (47) for a plasma. Due to the fact that, for the scattering operator $\widehat{S}_2(t,1,2)$, the Duhamel formula is formally true (see also (43))

$$\widehat{S}_2(t,\xi_1,\xi_2) - I = \int_0^t d\tau S_2(-t+\tau,\xi_1,\xi_2)\times$$

$$\times\bigg(\Big\langle \frac{\partial}{\partial r_1}\Phi(r_1 - r_2), \frac{\partial}{\partial v_1}\Big\rangle + \Big\langle \frac{\partial}{\partial r_2}\Phi(r_1 - r_2), \frac{\partial}{\partial v_2}\Big\rangle\bigg)\times$$

$$\times S_1(-\tau+t,\xi_1)S_1(-\tau+t,\xi_2),$$

where $S_2(-t+\tau,\xi_1,\xi_2)$ is evolution operator (31), collision integral (47) can be given as follows:

$$\mathcal{I} = \int_0^t d\tau \int d\xi_2 \Big\langle \frac{\partial}{\partial r_1}\Phi(r_1 - r_2), \frac{\partial}{\partial v_1}\Big\rangle S_2(-t+\tau,\xi_1,\xi_2)\times$$

$$\times\bigg(\Big\langle \frac{\partial}{\partial r_1}\Phi(r_1 - r_2), \frac{\partial}{\partial v_1}\Big\rangle + \Big\langle \frac{\partial}{\partial r_2}\Phi(r_1 - r_2), \frac{\partial}{\partial v_2}\Big\rangle\bigg)\times$$

$$\times S_1(-\tau+t,\xi_1)S_1(-\tau+t,\xi_2)\langle G^{(1)}\rangle(t,\xi_1)\langle G^{(1)}\rangle(t,\xi_2).$$

Then, by expanding the obtained collision integral $\mathcal{I}$ in the Dyson—Phillips series in the plasma parameter and by restricting ourselves to the first term of the expansion (weak coupling limit), we find

$$\mathcal{I} = \int_0^t d\tau \int d\xi_2 \Big\langle \frac{\partial}{\partial r_1}\Phi(r_1 - r_2), \frac{\partial}{\partial v_1}\Big\rangle\times$$

$$\times\bigg\langle \frac{\partial}{\partial r_1}\Phi(r_1(-t+\tau) - r_2(-t+\tau)), \bigg(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}\bigg)\bigg\rangle\times$$

$$\times\langle G^{(1)}\rangle(t,\xi_1)\langle G^{(1)}\rangle(t,\xi_2),$$

where $r_i(-t+\tau) \equiv r_i(-t+\tau,\xi_1,\xi_2)$, $i = 1,2$, are the solutions of initial-value problem (32) for $k = 2$. We remark that this expression has similar structure to the generalized Landau collision integral in the kinetic theory [14].

Thus, we can finally rewrite Eq. (46) in the Fokker–Planck representation [12] as

$$\frac{\partial}{\partial t}\langle G^{(1)}\rangle(t,\xi_1) + \Big\langle v_1, \frac{\partial}{\partial r_1}\Big\rangle\langle G^{(1)}\rangle(t,\xi_1) - \quad (48)$$

$$-\Big\langle \frac{\partial}{\partial r_1}\int d\xi_2 \Phi(r_1 - r_2)\langle G^{(1)}\rangle(t,\xi_2), \frac{\partial}{\partial v_1}\Big\rangle\langle G^{(1)}\rangle(t,\xi_1) =$$

$$= \frac{\partial}{\partial v_1^\alpha \partial v_1^\beta}\mathcal{D}^{\alpha\beta}(t)\langle G^{(1)}\rangle(t,\xi_1) + \frac{\partial}{\partial v_1^\alpha}\mathcal{B}^\alpha(t)\langle G^{(1)}\rangle(t,\xi_1),$$

where the transport coefficients are defined by the expressions

$$\mathcal{D}^{\alpha\beta}(t) = \int_0^t d\tau \int d\xi_2 \frac{\partial}{\partial r_1^\alpha}\Phi(r_1 - r_2)\times$$

$$\times\frac{\partial}{\partial r_1^\beta}\Phi(r_1(-t+\tau) - r_2(-t+\tau))\langle G^{(1)}\rangle(t,\xi_2), \quad (49)$$

$$\mathcal{B}^\alpha(t) = \int_0^t d\tau \int d\xi_2 \frac{\partial}{\partial r_1^\alpha}\Phi(r_1 - r_2)\times$$

$$\times\bigg(\frac{\partial}{\partial r_2^\beta}\Phi(r_1(-t+\tau) - r_2(-t+\tau))\frac{\partial}{\partial v_2^\beta} -$$

$$-\frac{\partial}{\partial v_1^\beta}\frac{\partial}{\partial r_1^\beta}\Phi(r_1(-t+\tau) - r_2(-t+\tau))\bigg)\langle G^{(1)}\rangle(t,\xi_2). \quad (50)$$

Evolution equation (48) is non-Markovian for the average value of the microscopic phase density with the Fokker–Planck collision integral determining by nonlinear transport coefficients (49), (50). It can be reduced to the canonical Fokker–Planck equation as a result of further approximations [13].

We observe that the higher-order terms in (46) with respect to the plasma parameter from the functional $\langle \mathcal{G}\rangle(t,\xi_1,\xi_2 \mid \langle G^{(1)}\rangle(t))$ are equal to zero in the weak-coupling approximation.

5. Conclusion

The BBGKY hierarchy of equations (25) for the generalized microscopic phase densities is formulated, and the microscopic derivation of a non-Markovian collision term in the evolution equation for the average value of the additive-type microscopic phase density (40) is done.

The time-nonlocal generalization of the Fokker–Planck equation (48)–(50) on the basis of such collision term (47) is proposed. The evolution equation generated by such collision term can be applied for the further analysis of turbulent transport [7]. The memory effects can be important for the description of transport under saturated turbulence [8–10].

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ВІД ІЄРАРХІЇ ББГКІ ДО НЕМАРКІВСЬКИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ

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Р е з ю м е

Розглянуто проблему опису еволюції мікроскопічної фазової густини та її узагальнень. Для цього введено послідовність маргінальних мікроскопічних фазових густин та сформульовано відповідні ієрархії рівнянь ББГКІ для таких мікроскопічних розподілів та їх середніх значень. Дано мікроскопічний вивід узагальненого еволюційного рівняння для середнього значення мікроскопічної фазової густини та отримано немарківське узагальнення інтеграла зіткнень Фоккера–Планка.