ON COORDINATE AND MOMENTUM UNCERTAINTIES OF THE CENTER OF MASSES OF A COHERENT ELECTRON PAIR

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Manifestations of the uncertainties that are inherent to the coordinates and the momentum of the center of masses of a coherent electron pair have been studied. It has been demonstrated that there exists a nonzero probability to register a pair of electrons on the same side of their center of masses. The dependence of this quantity on the state parameters has been calculated, and the conditions, under which it equals 1/2, have been determined.

1. Introduction

The center of masses is one of the most important physical characteristics. If the concept of the center of masses is applied to quantum systems, it is necessary to bear in mind that every continuous observable in such systems is characterized by an uncertainty. The uncertainties of observable quantities, in their turn, give rise to an uncertainty of the corresponding center of masses. In this work, the uncertainty of the center of masses of a pair of coherent electrons is considered.

The properties and characteristics of quantum-mechanical systems composed of a small number of particles have been of a great scientific interest recently, first of all, from the viewpoint of the quantum-mechanical information theory, because the components of such systems are the carriers of information in quantum channels. This work is devoted to studying the properties of the momentum and the coordinates of the center of masses of a pair of coherent electrons.

A pair of coherent electrons has been regarded as a candidate for the role of a data carrier since the times of the Einstein–Podolsky–Rosen discussion [1], and the issue on quantum-mechanical entanglement remains of a huge interest since then. Information estimations of the quantum entanglement – such as the Shannon and von Neumann entropies – are calculated in the center-of-mass reference frame, especially in the relativistic case [2].

Gaussian packets and coherent states are useful for our problem, because they make it possible to fulfill all necessary calculations. Gaussian wave packets are also applied in a lot of other problems (see, e.g., work [3]). In particular, the position of a Gaussian wave packet at the center of masses is considered. The center of masses of a system of particles is one of the main characteristics of the Gaussian wave packet state.

The center-of-mass reference frame plays its role in problems dealing with the optical pumping which is a powerful tool for the creation of specific electron states in atoms. In particular, the motion of the reference frame of the atomic center of mass in the course of pumping is analyzed [4]. Strictly speaking, the noncoherent nature of spontaneous radiation emission makes it expedient to take the decoherence into account, together with the quantum state of the center of masses. Such systems find application in the mechanics of quantum states [5,6] and the problems of quantum logic and quantum calculations [7,8].

The conventional interpretation of the Stern–Gerlach experiment testifies that the atomic center-of-mass reference frame plays the role of a "quantum detector" for the atomic spin. The relevant researches are devoted to studying whether the creation of such a detector is possible [9].

2. Wave Function of a Separate Electron

Considering the state of a separate coherent electron, we see that a plane wave does not satisfy us as a tool for its description. The same is true for functions localized in space, like the Dirac delta function $\delta(x-x_0(t))$. Instead, let us consider a superposition of plane waves

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \Psi(p) \exp\left(i\frac{px}{\hbar}\right) dp. \tag{1}$$

The probability amplitudes $\Psi(x)$ and $\Psi(p)$ correspond to the Gauss distribution of probabilities in the coordinate and momentum spaces, respectively,

$$\rho(x) = e_0 \left|\Psi(x)\right|^2, \tag{2}$$

where e_0 is the electron charge. Such probability distributions are characterized by the coordinate, σ_q, and momentum, σ_p, uncertainties. Making use of the Heisenberg uncertainty relation, which is minimal for coherent states at a definite time moment, we can write

$$\sigma_q \sigma_p = \frac{\hbar}{2}. \tag{3}$$

For a free particle, the momentum uncertainty is constant, $\sigma_p = \text{const}$, whereas the coordinate one depends on the time. Equality (3) is not more true, i.e. the wave packet gets "smeared" as the time increases. The coordinate and momentum uncertainties are isotropic quantities.

Hence, taking all those properties of the wave function parameters into account, we can write down the three-dimensional wave function of a coherent electron in the form

$$\Psi(\mathbf{r}, \mathbf{r}_0, \mathbf{p}_0) = \frac{1}{\left(\sqrt{\sigma\sqrt{2\pi(1+\omega^2(t)^2)}}\right)^3} \times$$

$$\times \exp\left(-\frac{\left(\mathbf{r} - \mathbf{r}_0 - \frac{\mathbf{p}_0}{m}(t)\right)^2}{4\sigma^2(1+\omega^2(t)^2)} + i\frac{\mathbf{p}_0\mathbf{r}}{\hbar}\right), \tag{4}$$

where σ is the coordinate uncertainty at the initial time moment, r_0 the average coordinate of the electron at the initial time moment, and p_0 the average electron momentum. We can transfer the coordinate origin into a point that is described by the radius-vector $\mathbf{r}_0$ and rewrite the wave function of a coherent electron in the form

$$\Psi(\mathbf{r}, \mathbf{p}_0) = \frac{1}{\left(\sqrt{\sigma\sqrt{2\pi(1+\omega^2 t^2)}}\right)^3} \times$$

$$\times \exp\left(-\frac{\left(\mathbf{r} - \frac{\mathbf{p}_0}{m}t\right)^2}{4\sigma^2(1+\omega^2(t)^2)} + i\frac{\mathbf{p}_0\mathbf{r}}{\hbar}\right). \tag{5}$$

At the initial time moment, when the correlation between the coordinate and the momentum is absent (this moment is referred to as the culmination moment [10]), the wave function of a coherent electron looks like

$$\Psi(\mathbf{r}, \mathbf{p}_0) = \frac{1}{\left(\sqrt{\sigma\sqrt{2\pi}}\right)^3} \exp\left(-\frac{\mathbf{r}^2}{4\sigma^2} + i\frac{\mathbf{p}_0\mathbf{r}}{\hbar}\right). \tag{6}$$

The average momentum $\mathbf{p}_0$ of a free particle characterizes its velocity $\mathbf{V} = \frac{\mathbf{p}_0}{m}$. Hereafter, we use the following notations for wave functions:

$$\Psi(p, \alpha) = \langle p \mid \alpha \rangle, \tag{7}$$

$$\Psi(r, \alpha) = \langle r \mid \alpha \rangle, \tag{8}$$

$$|\alpha\rangle = \int \langle p \mid \alpha \rangle |p\rangle. \tag{9}$$

3. Wave Function of a Coherent Electron Pair

The wave function of a pair of coherent electrons is a combination of products of the wave functions of each electron. In the case where the electrons have antiparallel or parallel spins, the wave function has to be a symmetric or antisymmetric (according to the Pauli exclusion principle) combination, respectively:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{\Psi_1(\mathbf{r}_2)\Psi_2(\mathbf{r}_1) \pm \Psi_1(\mathbf{r}_1)\Psi_2(\mathbf{r}_2)}{\sqrt{1 \pm N^2}}. \tag{10}$$

Here, N is the overlap integral which is a scalar product of the wave functions and is defined as

$$N = \int \Psi_1^*(r)\Psi_2(r) d^3r = \langle \Psi_1(r) \mid \Psi_2(r) \rangle. \tag{11}$$

Substituting relation (3) into formula (10), we obtain an explicit expression for the wave function of a coherent electron pair. The average coordinates are

$$\langle \mathbf{r}_1 \rangle + \langle \mathbf{r}_2 \rangle = \langle \mathbf{r}_1 + \mathbf{r}_2 \rangle.$$

Hence, we can define a parameter which is the average coordinate of the center of masses in classical mechanics.

Let us choose a reference frame which coincides with the reference frame of the center of masses. Hence, $\mathbf{r}_{01} = \mathbf{r}_0$, $\mathbf{r}_{02} = -\mathbf{r}_0$, $\mathbf{p}_{01} = -\mathbf{p}_0$, and $\mathbf{p}_{02} = \mathbf{p}_0$. Then the wave function is

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{1 \pm N^2}} \frac{1}{\left(\sigma\sqrt{2\pi(1+\omega^2 t^2)}\right)^3} \times$$

$$\times \left[\exp\left(-\frac{\left(\mathbf{r}_1 - \mathbf{r}_0 - \frac{\mathbf{p}_0}{m}t\right)^2}{4\sigma^2(1+\omega^2 t^2)} - \frac{\left(\mathbf{r}_2 + \mathbf{r}_0 - \frac{\mathbf{p}_0}{m}t\right)^2}{4\sigma^2(1+\omega^2(t-t_0)^2)} + \right.\right.$$

$$\left. + i\frac{\mathbf{p}_0(\mathbf{r}_1 - \mathbf{r}_2)}{\hbar} \right) \pm \exp\left(-\frac{\left(\mathbf{r}_2 - \mathbf{r}_0 - \frac{\mathbf{p}_0}{m}t\right)^2}{4\sigma^2(1+\omega^2 t^2)} - \right.$$

$$\left.\left. - \frac{(\mathbf{r}_1 + \mathbf{r}_0 - \frac{\mathbf{p}_0}{m}t)^2}{4\sigma^2(1+\omega^2 t^2)} - i\frac{\mathbf{p}_0(\mathbf{r}_1 - \mathbf{r}_2)}{\hbar} \right) \right], \quad (12)$$

where $N = \exp\left(-\frac{2p_0^2\sigma^2}{\hbar^2} - \frac{r_0^2}{2\sigma^2}\right)$. The parameters $\mathbf{r}_0$ and $\mathbf{p}_0$ acquire the meaning of the relative coordinate and the relative momentum of the pair of coherent electrons, respectively [11].

Let us consider a coherent electron pair, as well as a separate coherent electron, at the culmination moment. The two-particle wave function reads

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{1 \pm N^2}} \frac{1}{\left(\sigma\sqrt{2\pi}\right)^3} \times$$

$$\times \left[\exp\left(-\frac{(\mathbf{r}_1 - \mathbf{r}_0)^2}{4\sigma^2} - \frac{(\mathbf{r}_2 + \mathbf{r}_0)^2}{4\sigma^2} + i\frac{\mathbf{p}_0(\mathbf{r}_1 - \mathbf{r}_2)}{\hbar} \right) \pm \right.$$

$$\left. \pm \exp\left(-\frac{(\mathbf{r}_2 - \mathbf{r}_0)^2}{4\sigma^2} - \frac{(\mathbf{r}_1 + \mathbf{r}_0)^2}{4\sigma^2} + i\frac{\mathbf{p}_0(\mathbf{r}_1 - \mathbf{r}_2)}{\hbar} \right) \right]. \quad (13)$$

The charge density of the coherent electron pair looks like

$$\rho(\mathbf{r}) = \frac{e_0}{1+N^2} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{2\sigma^2} \right) + \exp\left(-\frac{(\mathbf{r} + \mathbf{r}_0)^2}{2\sigma^2} \right) +$$

$$+ \exp\left(-\frac{r^2 + 2r_0^2}{2\sigma^2} \right) \exp\left(-\frac{2p_0^2\sigma^2}{\hbar^2} \right) \times$$

$$\times \left[\exp\left(\frac{2i\mathbf{p}_0\mathbf{r}}{\hbar} \right) + \exp\left(-\frac{2i\mathbf{p}_0\mathbf{r}}{\hbar} \right) \right]. \quad (14)$$

4. Center of Masses of a Coherent Electron Pair

In contrast to the center of masses of a pair of classical material points, the momentum and the coordinates of the center of masses of a system of quantum particles, as well as every other observable, are random quantities. Consequently, they are characterized not only by their average values, but also by all other properties of random variables, in particular – in our case – the uncertainty.

The reference frame, the origin of which is fixed at the center of masses, can be defined only by using the average values for the momenta and coordinates of particles in the system. In a specific case of an electron pair, the origin coordinates of the reference frame fixed at the center of masses are defined by the expressions

$$\mathbf{R}_c(t) = \frac{1}{2}\langle \mathbf{r}_1 \rangle + \frac{1}{2}\langle \mathbf{r}_2 \rangle + \frac{t}{m}\langle \mathbf{p}_1 \rangle + \frac{t}{m}\langle \mathbf{p}_2 \rangle.$$

Since the electrons are identical, the average coordinate values for each of them are identical too. As a result, we have

$$\mathbf{R}_c(t) = \langle \mathbf{r}_{1/2} \rangle + 2\frac{t}{m}\langle \mathbf{p}_{1/2} \rangle. \quad (15)$$

Here, $\langle \mathbf{r}_{1/2} \rangle$ and $\langle \mathbf{p}_{1/2} \rangle$ are the average coordinate and momentum, respectively, of the first or the second electron.

In the center-of-mass reference frame, the average values for the momentum and coordinate operators of each electron in the pair are equal to zero. This fact prohibits one from defining the operator of either the relative coordinate or the relative momentum. The usual replacement of the sum of coordinate operators by the center-of-mass coordinate operator,

$$\hat{R}_c = \frac{1}{2}\hat{r}_1 + \frac{1}{2}\hat{r}_2,$$

the sum of momentum operators by the operator of system momentum,

$$\hat{P}_s = \hat{p}_1 + \hat{p}_2,$$

the difference between the coordinate operators by the relative coordinate operator,

$$\hat{r} = \hat{r}_1 - \hat{r}_2,$$

and the half-difference between momentum operators by the relative momentum operator,

$$\hat{p} = \frac{1}{2}\hat{p}_1 - \frac{1}{2}\hat{p}_2,$$

does not enable the observables to be related to the corresponding classical averages, because the operators obtained in such a way are characterized by zero average values.

In the phase space, the coherent state of an electron pair in the center-of-mass reference frame is described by two opposite points:

$$\alpha_1|_{\text{c.m.}} = \alpha; \;\; \alpha_2|_{\text{c.m.}} = -\alpha.$$

Instead of two points in the phase space, the state is characterized in the coordinate space by a single point only.

5. Density Matrices in the Center-of-mass Reference Frame

If we use the reference frame fixed at the center of masses of the electron pair, the general expressions for density matrices become a little bit simpler. In terms of notations used in expressions (7) and (8), the two-particle density matrix looks like

$$\hat{\rho}_{\text{pair}} = \frac{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| \otimes |-\boldsymbol{\alpha})(-\boldsymbol{\alpha}|}{2\left(1 \mp |N|^2\right)} + \frac{|-\boldsymbol{\alpha}\rangle\langle-\boldsymbol{\alpha}| \otimes |\boldsymbol{\alpha})(\boldsymbol{\alpha}|}{2\left(1 \mp |N|^2\right)} \mp$$

$$\mp \frac{|\boldsymbol{\alpha}\rangle\langle-\boldsymbol{\alpha}| \otimes |-\boldsymbol{\alpha})(\boldsymbol{\alpha}|}{2\left(1 \mp |N|^2\right)} \mp \frac{|-\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| \otimes |\boldsymbol{\alpha})(-\boldsymbol{\alpha}|}{2\left(1 \mp |N|^2\right)}.$$

The one-particle density matrix also becomes simpler:

$$\hat{\rho}_{\text{pair}} = \frac{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| + |-\boldsymbol{\alpha}\rangle\langle-\boldsymbol{\alpha}| \mp \overline{N}|\boldsymbol{\alpha}\rangle\langle-\boldsymbol{\alpha}| \mp N|-\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|}{2\left(1 \mp |N|^2\right)}. \tag{16}$$

6. Momentum Distributions

In the momentum representation, the expressions for probability distributions become much simpler. The two-electron distribution is obtained by averaging the projection operator

$$\hat{f}\left(\mathbf{p}_1, \mathbf{p}_2\right) = |\mathbf{p}_1\rangle\langle\mathbf{p}_1| \otimes |\mathbf{p}_2)(\mathbf{p}_2|.$$

The result of averaging is the squared modulus of the wave function of a coherent electron pair in the momentum representation. The two-electron wave function in this representation looks like

$$\psi\left(\mathbf{p}_1, \mathbf{p}_2, t; \alpha\right) = \frac{\exp\left(-i\frac{\mathbf{p}_1^2}{2m\hbar}t - i\frac{\mathbf{p}_2^2}{2m\hbar}t\right)}{\left(\sigma\sqrt{2\pi}\right)^3} \times$$

$$\times\left(\frac{\exp\left(-\frac{(\mathbf{p}_1-\mathbf{p}_\alpha)^2}{4\sigma^2} - \frac{(\mathbf{p}_2+\mathbf{p}_\alpha)^2}{4\sigma^2} - i\frac{\mathbf{r}_\alpha(\mathbf{p}_1-\mathbf{p}_2)}{\hbar}\right)}{\sqrt{1 \pm N^2}} \pm\right.$$

$$\left. \pm \frac{\exp\left(-\frac{(\mathbf{p}_1+\mathbf{p}_\alpha)^2}{4\sigma^2} - \frac{(\mathbf{p}_2-\mathbf{p}_\alpha)^2}{4\sigma^2} + i\frac{\mathbf{r}_\alpha(\mathbf{p}_1-\mathbf{p}_2)}{\hbar}\right)}{\sqrt{1 \pm N^2}}\right). \tag{17}$$

In the free electron approximation, the square of the absolute value of the two-electron wave function does not depend on the time – as a manifestation of the momentum conservation law – and reads

$$f\left(\mathbf{p}_1, \mathbf{p}_2; \boldsymbol{\alpha}\right) = \frac{1}{2\left(1 \mp N^2\right)} \times$$

$$\times[g\left(\mathbf{p}_1 - \mathbf{p}_\alpha\right) g\left(\mathbf{p}_2 + \mathbf{p}_\alpha\right) + g\left(\mathbf{p}_1 + \mathbf{p}_\alpha\right) g\left(\mathbf{p}_2 - \mathbf{p}_\alpha\right) \pm$$

$$\pm \phi\left(\mathbf{p}_1; \boldsymbol{\alpha}\right) \overline{\phi}\left(\mathbf{p}_2; \boldsymbol{\alpha}\right) \pm \overline{\phi}\left(\mathbf{p}_1; \boldsymbol{\alpha}\right) \phi\left(\mathbf{p}_2; \boldsymbol{\alpha}\right)]. \tag{18}$$

Here, $g\left(\mathbf{p}\right)$ is used to denote an ordinary normal distribution in the momentum space:

$$g\left(\mathbf{p}\right) = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^3} e^{\left(-\frac{\mathbf{p}^2}{2\sigma^2}\right)},$$

and $\phi\left(\mathbf{p}; \boldsymbol{\alpha}\right)$ to denote a contribution to the interference term:

$$\phi\left(\mathbf{p}; \boldsymbol{\alpha}\right) = \frac{N \exp\left(-\frac{\left(\mathbf{p} + i\mathbf{r}_\alpha \frac{2\sigma^2}{\hbar}\right)^2}{2\sigma^2}\right)}{\left(\sigma\sqrt{2\pi}\right)^3}. \tag{19}$$

Two first terms in expression (18) correspond to the normal distribution of the registration of one electron in the vicinity of momentum $\mathbf{p}_\alpha$ and the other in the vicinity of momentum $-\mathbf{p}_\alpha$, by taking the permutation symmetry of particles into account. The third and fourth terms are responsible for the interference between states. These terms give rise to a number of specific properties of the electron pair, and some of them will be analyzed below.

The one-electron distribution of probabilities in the momentum representation can be obtained by integrating the two-electron distribution over the momentum of either electron. The integration of the normal distribution over the whole momentum space produces unity. Therefore, the first two terms in expression (18) bring about the sum of distributions around the point $\mathbf{p}_\alpha$, and the last two around the point $-\mathbf{p}_\alpha$. As a result of the integration of expression (18) over all momenta, we obtain the overlap integral. Therefore, the result looks like

$$f(\mathbf{p};\boldsymbol{\alpha}) = \frac{g(\mathbf{p}-\mathbf{p}_\alpha)+g(\mathbf{p}+\mathbf{p}_\alpha)}{2(1\mp N^2)}\mp$$

$$\mp N\frac{\phi(\mathbf{p};\boldsymbol{\alpha})+\overline{\phi}(\mathbf{p};\boldsymbol{\alpha})}{2(1\mp N^2)}. \qquad (20)$$

The majority of interference phenomena that are characteristic of a pair of coherent electrons are described by the second fraction in this expression.

7. Coordinate Distributions

The two-electron distribution of probabilities in the coordinate representation is obtained by averaging the projection operator

$$\hat{f}(\mathbf{p}_1,\mathbf{r}_2) = |\mathbf{r}_1\rangle\langle\mathbf{r}_1| \otimes |\mathbf{r}_2)(\mathbf{r}_2|$$

over the coherent state. The result of averaging is the squared modulus of the wave function of a coherent electron pair in the coordinate representation. As formula (13) demonstrates, the two-electron wave function in this representation has a more complicated dependence on the time than it was in the momentum representation.

In contrast to the momentum distribution, the coordinate one depends on the time, though their forms are similar:

$$f(\mathbf{r}_1,\mathbf{r}_2;\boldsymbol{\alpha}) = \frac{1}{2(1\mp N^2)}\times$$

$$\times[h(\mathbf{r}_1;\boldsymbol{\alpha},t)\,h(\mathbf{r}_2;-\boldsymbol{\alpha},t)+h(\mathbf{r}_1;-\boldsymbol{\alpha},t)\,h(\mathbf{r}_2;\boldsymbol{\alpha},t)\pm$$

$$\pm\psi(\mathbf{r}_1;\boldsymbol{\alpha},t)\,\overline{\psi}(\mathbf{r}_2;\boldsymbol{\alpha},t)\pm\overline{\psi}(\mathbf{r}_1;\boldsymbol{\alpha},t)\,\psi(\mathbf{r}_2;\boldsymbol{\alpha},t)]. \qquad (21)$$

Here, the notation $h(\mathbf{r};\boldsymbol{\alpha},t)$ is used for the normal distribution in the coordinate space with the wave-packet smearing:

$$h(\mathbf{r};\boldsymbol{\alpha},t) = \frac{\exp\left(-\frac{\left(\mathbf{r}-\mathbf{r}_\alpha-\frac{\mathbf{p}_\alpha}{m}t\right)^2}{2\sigma_q^2(1+\omega^2t^2)}\right)}{\left(\sigma_q\sqrt{2\pi}\sqrt{1+\omega^2t^2}\right)^3},$$

and $\psi(\mathbf{r};\boldsymbol{\alpha},t)$ is used to denote the contribution to the interference term:

$$\psi(\mathbf{r};\boldsymbol{\alpha},t) = \frac{N\exp\left(-\frac{\left(\mathbf{r}-i\mathbf{r}_\alpha\omega t+i\frac{\mathbf{p}_\alpha\hbar}{2\sigma^2}\right)^2}{2\sigma_q^2(1+\omega^2t^2)}\right)}{\left(\sigma_q\sqrt{2\pi}\sqrt{1+\omega^2t^2}\right)^3}. \qquad (22)$$

Two first terms in expression (21) correspond to the normal distribution of the registration of one electron in the vicinity of the point $\mathbf{r}_\alpha+\frac{\mathbf{p}_\alpha}{m}t$ and another electron in the vicinity of the point $-\mathbf{r}_\alpha-\frac{\mathbf{p}_\alpha}{m}t$, taking the permutation symmetry of particles into account. The third and fourth terms are responsible for the interference of states. The contribution of the interference terms has the overlap integral as a weight; the one-particle distributions are similar to the normal one, but around the points with imaginary coordinates $i\mathbf{r}_\alpha\omega t - i\frac{\mathbf{p}_\alpha\hbar}{2\sigma^2}$. The time dependence of the interference term gives rise to a smearing of the interference pattern in the course of a detection procedure with the detector operation time close to ω^{-1}.

Depending on the duration of the one-electron distribution over the coordinate, two effects take place. The first effect consists in the translation of the distribution maximum point according to the classical motion law

$$\mathbf{r}_{\text{eff}}(t) = \mathbf{r}_\alpha + \frac{\mathbf{p}_\alpha}{m}t.$$

If the coordinate uncertainty can be neglected in the problem, the motion of electrons does not differ from the classical one. As the time increases, the initial coordinates can be neglected, and the law of distribution maximum motion gets simpler:

$$\mathbf{r}_{\text{eff}}(t) = \frac{\mathbf{p}_\alpha}{m}t.$$

The second effect is the smearing of the wave function. Its essence is an increase of the coordinate uncertainty with increase in the time according to the law

$$\sigma_q(t) = \sigma_{q0}\sqrt{1+\omega^2t^2}.$$

As the time increases, the initial uncertainty of the coordinate becomes insignificant, and the law of smearing becomes simpler:

$$\sigma_q(t) = \frac{\sigma_p}{m}t.$$

Hence, as the time increases, the relation between the momentum uncertainty and the momentum itself transforms into the relation between the position uncertainty and the coordinate.

The one-electron distribution of probabilities is obtained by the integration over the coordinates of another electron. The integration of the normal distribution over the whole momentum space gives unity. Therefore, the first two terms in expression (21) give the sum of distributions around the points $\mathbf{r}_\alpha + \frac{\mathbf{p}_\alpha}{m}t$ and $-\mathbf{r}_\alpha - \frac{\mathbf{p}_\alpha}{m}t$. The integration of expression (21) over all coordinates gives the overlap integral. Therefore, the resulting distribution looks like

$$f(\mathbf{r},t;\boldsymbol{\alpha}) = \frac{h(\mathbf{r};\boldsymbol{\alpha},t) + h(\mathbf{r};-\boldsymbol{\alpha},t)}{2(1\mp N^2)}\mp$$

$$\mp N\frac{\psi(\mathbf{r};\boldsymbol{\alpha},t) + \overline{\psi}(\mathbf{r};\boldsymbol{\alpha},t)}{2(1\mp N^2)}. \quad (23)$$

The spatial interference in the pair of coherent electrons is associated with the second fraction in this expression.

8. Uncertainty of the Center of Masses

In the classical concept of the center of masses for an electron pair, electrons are located on its opposite sides. In the own reference frame, this property is preserved in time, and the momenta of electrons have opposite directions. Due to the overlapping of electron wave functions in the coordinate representation, there appears a probability to observe electrons on the same side of the center of masses. Analogously, the overlapping of wave functions in the momentum representation makes it possible to register the momenta with identical signs for both electrons.

9. Uncertainty of the Total Momentum

Let us calculate the probability distribution for the momentum p of either electron, provided that the momentum of the other electron is negative. Should the sum of momenta be precisely equal to zero, this distribution would be zero at every negative value.

This distribution is determined as the ratio between the probability that the momentum of the first electron has the value p and the momentum of the second electron is negative, on the one hand, and the probability that the momentum of the first electron is arbitrary and the momentum of the second electron is negative, on the other hand:

$$P(p_1|p_2<0) = \frac{P(p_1,p_2<0)}{P(p_2<0)} = 2\int\limits_{-\infty}^{0}\rho(p_1,p_2)\,dp_2. \quad (24)$$

Confining the consideration to the one-dimensional case and taking the explicit expression for the overlap integral,

$$N = \exp\left(-2|\alpha|^2\right) = \exp\left(-\frac{p_\alpha^2}{2\sigma^2} - 2\frac{x_\alpha^2\sigma^2}{\hbar^2}\right),$$

into account, we obtain the following formula for the probability distribution of the second electron momentum, provided that the momentum of the first electron is negative:

$$P(p|p_2<0) = \frac{1}{1\pm N^2}\left[\frac{1}{2}g(p-p_\alpha)\times\right.$$

$$\times\left(1+\mathrm{erf}\left(\frac{p_\alpha}{\sigma\sqrt{2}}\right)\right) + \frac{1}{2}g(p+p_\alpha)\times$$

$$\times\left(1-\mathrm{erf}\left(\frac{p_\alpha}{\sigma\sqrt{2}}\right)\right)g(p)\,N\exp\left(-2\frac{x_\alpha^2\sigma^2}{\hbar^2}\right)\times$$

$$\left.\times\left(-\cos\left(\frac{px_\alpha}{2\hbar}\right) + \sin\left(\frac{px_\alpha}{2\hbar}\right)\mathrm{erf}i\left(\frac{\sigma\sqrt{2}x_\alpha}{\hbar}\right)\right)\right].$$

If the momentum p_α is much larger than the momentum uncertainty σ_p, the conditional probability that both electrons have identical momenta tends to zero. With a reduction of p_α, there appears a conditional probability to register electrons with identical directions of their momenta.

Hence, the momentum and the coordinates of the center of masses of a coherent electron pair are equal to zero only on the average. Really, the difference between the electron momenta is associated with the operator $\Delta\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$ which has a zero average value in the center-of-mass reference frame, but is characterized by a nonzero dispersion

$$\sigma^2_{\Delta\mathbf{p}} = 2\sigma_p^2. \quad (25)$$

Therefore, the measurement results for the momenta of a electron pair coincide only on the average, and the statistical uncertainty of the momentum difference is a characteristic of a quantum state of the pair.

There exists a nonzero probability to register both electrons with identical momentum directions (the both

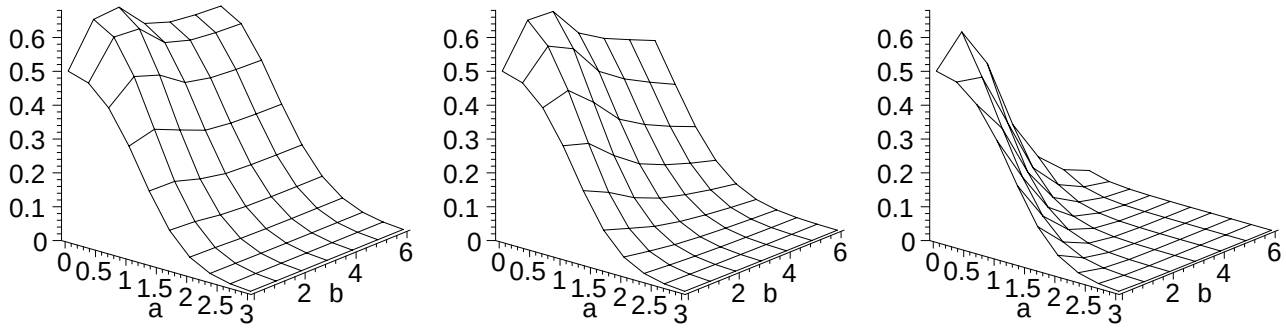

Fig. 1. Dependences of the probability for both electrons in a singlet pair to be registered on the same side of their center of masses on the state parameters. The momentum uncertainty of the particles $\sigma = 1$. The distributions are calculated for the time moments $\omega t = 0$, 0.1, and 0.4 (from left to right)

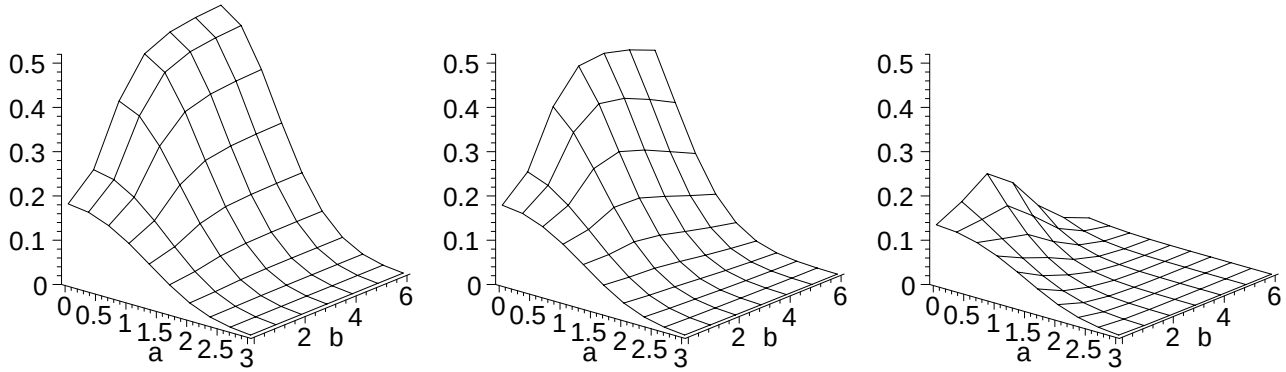

Fig. 2. The same as in Fig. 1, but for a triplet electron pair

are directed either to the left or to the right). This probability depends on the relationship between the state parameters and their uncertainties. In Figs. 1 and 2, the dependence of the probability for the electrons to have identical momentum directions on the state parameters is depicted for the singlet and triplet states, respectively. The growth of the difference between average momenta above the uncertainty value is accompanied by a fast reduction of the probability for the momenta to have identical directions. The spatial distance between states, on the contrary, reduces the probability only in the vicinity of the coordinate origin (triplet states) or even increases it a little (singlet states). The increase of the distance above the coordinate uncertainty practically does not affect the probability of identical momentum directions.

10. Probabilities of a Deviation from the Center of Masses

Among specific properties of an electron pair in the center-of-mass reference frame, there is a probability of the simultaneous registration of electrons either on the different sides or on the same side of the center of masses. Let us calculate the probabilities of such a registration.

The probability that an electron is to the left from the center of masses is calculated making use of the one-electron density matrix

$$P_l = \int\limits_{-\infty}^{0} \rho\left(x, t; x_\alpha, p_\alpha\right) dx.$$

Owing to its symmetry with respect to the center of masses, the result is $\frac{1}{2}$. The probability that both electrons are to the left from the center of masses is calculated making use of the two-electron wave function (21) in the coordinate representation by the formula

$$P_{ll} = \int\limits_{-\infty}^{0}\int\limits_{-\infty}^{0} \rho\left(x_1, x_2, t; x_\alpha, p_\alpha\right) dx_1 dx_2.$$

For the coherent state of an electron pair, the result is

$$P_{ll} = \frac{1 - \left(\mathrm{erf}\left(\frac{\mathbf{r}_\alpha + \frac{\mathbf{p}_\alpha}{m}t}{\sigma_q\sqrt{2\left(1+\omega^2t^2\right)}}\right)\right)^2}{2\left(1 \pm N^2\right)} \pm$$

$$\pm \frac{N^2 + N^2\left(\mathrm{erf}i\left(\frac{\mathbf{r}_\alpha\omega t - \frac{\mathbf{p}_\alpha\hbar}{2\sigma^2}}{\sigma_q\sqrt{2\left(1+\omega^2t^2\right)}}\right)\right)^2}{2\left(1 \pm N^2\right)}.$$

The conditional probability that the second electron is on the same side of the center of masses as the first one is the ratio between the probability to register both electrons on the left side to the probability to register at least one electron on the left side:

$$P_{l|l} = \frac{P_{ll}}{P_l}.$$

Therefore, we obtain the following expression for the probability to register both electrons on the same side of the center of masses:

$$P_{l|l} = \frac{1 - \left(\mathrm{erf}\left(\frac{\mathbf{r}_\alpha + \frac{\mathbf{p}_\alpha}{m}t}{\sigma_q\sqrt{2\left(1+\omega^2t^2\right)}}\right)\right)^2}{1 \pm N^2} \pm$$

$$\pm \frac{N^2 + N^2\left(\mathrm{erf}i\left(\frac{\mathbf{r}_\alpha\omega t - \frac{\mathbf{p}_\alpha\hbar}{2\sigma^2}}{\sigma_q\sqrt{2\left(1+\omega^2t^2\right)}}\right)\right)^2}{1 \pm N^2}. \qquad (26)$$

Unlike the probability that the momenta are directed identically, this probability depends on time, because the wave packets of free particles, first, smear and, second, move with the velocity $v_{cl} = \frac{p_\alpha}{m}$.

11. Conclusions

Besides the average value, the center of masses of a pair of coherent electrons is also characterized by the uncertainties of its position and velocity (the coordinates and the momentum). In the case where those uncertainties are of the order of the initial distance between electrons and the initial velocity of their relative motion, respectively, the behavior of the electron pair becomes nonclassical. The electrons can be registered with a nonzero probability on the same side of their center of masses. For an electron pair with small – in comparison with the relevant uncertainties – initial distance and relative velocity, the probability to register both electrons on the same side of the center of masses can grow up to 1/2. In the opposite case where the coordinates and momentum uncertainties are small, the relative motion in the electron pair almost does not differ from the classical one, and the probability distributions for the coordinates and the momentum are reduced to symmetrized combinations of the probability distributions for coordinates and momenta, respectively, of separate electrons.

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ПРО НЕВИЗНАЧЕНІСТЬ ІМПУЛЬСІВ І КООРДИНАТ ЦЕНТРА МАС КОГЕРЕНТНИХ ЕЛЕКТРОНІВ

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Р е з ю м е

Досліджено фізичні прояви невизначеності імпульсів і координат центра мас пари когерентних електронів. Показано, що існує ненульова ймовірність реєстрації пари електронів по один бік від центра мас. Розраховано залежність цієї ймовірності від параметрів стану і знайдено умови, за яких вона досягає 1/2.