ON EFFECTIVE DIMENSIONAL REDUCTION IN HYPERBOLIC SPACES

E.V. GORBAR

Bogolyubov Institute for Theoretical Physics
(14-b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: gorbar@bitp.kiev.ua)

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It is shown that the classical motion of massive particles in hyperbolic spaces H^D has a bounded character in $D-1$ coordinates. Studying the Dirac equation, it is found that a bounded character of the classical motion corresponds to the effective dimensional reduction $D+1 \to 1+1$ for fermions in the infrared region in the quantum problem. This effective dimensional reduction leads to the zero critical value of coupling constant for the dynamical symmetry breaking in hyperbolic spaces.

1. Introduction

It is well known that the dynamical symmetry breaking (DSB) and the mass (gap) generation for fermions usually require the presence of a strong attractive interaction [1] in order to break a symmetry that makes the quantative study of DSB a difficult problem. Therefore, it is very interesting to consider the cases where DSB takes place in the regime of weak coupling. Three such examples are known.

The first is breaking the $U(1)$ gauge symmetry in the presence of a Fermi surface when some fermion states are filled. According to the Bardeen–Cooper–Schrieffer theory of superconductivity [2], the Fermi surface is crucial for the formation of a bound state and a symmetry breaking condensate in the case of arbitrary small attraction between fermions. Indeed, according to [3, 4], the renormalization group scaling in this case is connected only with the direction perpendicular to the Fermi surface. Therefore, the effective dimension of the spacetime is $1+1$ from the viewpoint of renormalization group scaling. Since a bound state can be generated for any attraction in the $1+1$ dimension, this implies that the critical coupling constant is zero in this case.

The other example of DSB in the regime of weak coupling is DSB in a constant magnetic field [5–9], where the chiral symmetry is also dynamically broken for an arbitrary weak attraction. The physical reason for this is the effective dimensional reduction of the spacetime for fermions in the infrared region by 2 units in a constant magnetic field (DSB in a magnetic field in spacetimes of dimension higher than four was considered in [10]). Indeed, electrons being charged particles cannot propagate in directions perpendicular to the magnetic field when their energy is much less than the Landau gap $\sqrt{|eB|}$. This leads to the effective dimensional reduction in the infrared region by 2 units.

The dynamics of fermions in hyperbolic spaces H^D gives the third known example of DSB with zero critical coupling constant [11–15]. Analyzing the heat kernel in spacetimes $R \times H^D$, it was shown in [16] that the zero value of critical coupling constant for DSB in this case is also related to the effective dimensional reduction $D+1 \to 1+1$ for fermions in the infrared region. In a recent paper [17], the chiral and diquark condensates were studied in the extended NJL model in the hyperbolic space H^3, and it was shown that a negative curvature increases the values of both condensates.

The dynamics of quantum fields in hyperbolic spaces has received a great deal of attention in the literature (see, e.g., [18, 19] and references therein). Callan and Wilczek advocated hyperbolic spaces as a geometric means to regularize the infrared divergences of quantum field theories [20]. Further, the spatial sections of open Friedmann–Robertson–Walker models are hyperbolic spaces. The near-horizon optical metric of non-extreme black holes is asymptotically hyperbolic as well [21–23].

Although the analysis of a heat kernel in [16] clearly showed the presence of the effective dimensional reduction in spacetimes $R \times H^D$, physical reasons for such a reduction remained unknown, because the heat kernel is only an integral characteristic of the dynamics of a system. Therefore, it would be certainly desirable to clarify physical reasons for such a reduction. Fortunately, there exists a physically transparent way to demonstrate the occurrence of the effective dimensional reduction in H^D. According to [24], the effective dimensional reduction in the infrared region takes place in a quantum problem only when the classical motion in the corresponding problem has a bounded character with respect to the coordinates over which the dimensional reduction occurs. Let us consider, for example, the effective dimensional reduction in a constant magnetic field. In this case, a charged classical particle moves on circular orbits in the plane perpendicular to the constant magnetic field. Since the orbit radius is proportional to energy, the particle can go to infinity only if it has infinite energy. Therefore, a charged particle of finite energy always moves only in a finite region of the plane perpendicular to the constant magnetic field. This bounded character of motion means that the system is effectively of a finite size and leads to the effective dimensional reduction by 2 units in the infrared region in the quantum problem [5]. Therefore, it is interesting to consider the classical motion in hyperbolic spaces and see whether this motion has indeed a bounded character with respect to the coordinates over which the effective dimensional reduction takes place. Briefly, this motion and the Dirac equation in hyperbolic spaces were considered in a recent paper [25]. In the present paper, we consider these problems in more details.

2. Classical Motion in Hyperbolic Spaces

In this section, we study the classical motion in hyperbolic spaces. Let us first recall what hyperbolic spaces are and introduce the necessary notation. Hyperbolic spaces H^D are symmetric Riemannian spaces of constant negative curvature, whose interval in the Poincaré coordinates is given by

$$dl^2 = \frac{a^2}{x_1^2}(dx_1^2 + dx_2^2 + ... + dx_D^2), \tag{1}$$

where $x_1 > 0$, and a is the curvature radius. Further, the spacetime that we consider is $R \times H^D$, and the corresponding spacetime interval is

$$ds^2 = c^2 dt^2 - dl^2. \tag{2}$$

Relativistic motion of free particles proceeds along geodesics (see, e.g., [26]) which are extrema of the action

$$S = -mc\int ds = -mc^2 \int \sqrt{1 - \sum_{k=1}^{D} \frac{a^2 \dot{x}_k^2}{c^2 x_1^2}}\; dt. \tag{3}$$

Since the Lagrangian in (3) does not depend explicitly on t, the energy

$$E = \sum_k p_k \dot{x}_k - L = \frac{mc^2}{\sqrt{1 - \sum_k \frac{a^2 \dot{x}_k^2}{c^2 x_1^2}}} \tag{4}$$

is an integral of motion. Using this fact, it is not difficult to find the equations of motion. They are

$$\frac{d^2 x_1}{dt^2} = \frac{\dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2 - ... - \dot{x}_D^2}{x_1}\,, \tag{5}$$

$$\frac{d}{dt}\left(\frac{\dot{x}_2}{x_1^2}\right) = ... = \frac{d}{dt}\left(\frac{\dot{x}_D}{x_1^2}\right) = 0\,. \tag{6}$$

Equations (6) are easily integrated, by taking the form

$$\dot{x}_2 = C_2 x_1^2\,,\ \ \ldots\ ,\ \ \dot{x}_D = C_D x_1^2, \tag{7}$$

where $C_2\,, \ldots, C_D$ are integration constants. Substituting (7) in (5), the equation for x_1 takes the form

$$\frac{d^2 x_1}{dt^2} - \frac{\dot{x}_1^2}{x_1} = -C^2 x_1^3, \tag{8}$$

where $C^2 = C_2^2 + ... + C_D^2$. Changing the variable $z = \ln x_1$ (note that this change is unambiguous because $x_1 > 0$), we find the equation for z

$$\frac{d^2 z}{dt^2} = -C^2 e^{2z}. \tag{9}$$

Since this equation does not depend explicitly on t, we introduce $y(z) = \dot{z}$ and obtain the equation

$$\frac{d\,y^2}{dz} = -2C^2 e^{2z} \tag{10}$$

which is easily integrated, so that we have

$$y^2 = -C^2 e^{2z} + A, \tag{11}$$

where A is an integration constant. Since y^2 is positive, the right-hand side of Eq. (11) should be also positive.

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Therefore, $A > 0$. Since $y = \dot{z}$, it follows from Eq. (11) for $A = u^2$ that

$$\dot{z} = \pm\sqrt{u^2 - C^2 e^{2z}}. \tag{12}$$

Without loss of generality, we assume the sign minus in Eq. (12), because, in view of the change $t \to -t$, both signs are equivalent. Then, integrating Eq. (12), we obtain the sought solution

$$z(t) = \ln \frac{u}{C\cosh(ut+b)}, \tag{13}$$

where b is an integration constant. Using (13), it is not difficult to integrate Eqs. (7) and get the law of motion in other $D-1$ space coordinates $x_2, ..., x_D$. Thus, we find the following classical trajectories of motion in H^D (geodesics):

$$z(t) = \ln(x_1(t)) = \ln \frac{u}{C\cosh(ut+b)},$$

$$x_2(t) = \frac{uC_2}{C^2}\tanh(ut+b) + \tilde{C}_2,$$

$$x_D(t) = \frac{uC_D}{C^2}\tanh(ut+b) + \tilde{C}_D. \tag{14}$$

Here, $C = \sqrt{C_2^2 + C_3^2 + ... + C_D^2}$ and $u, b, C_2, \tilde{C}_2, ..., C_D, \tilde{C}_D$ are arbitrary constants (there are 2D of them and, for a particular trajectory, they are fixed by initial conditions). Finally, it is not difficult to check that energy (4) that corresponds to this motion equals $E = mc^2/(1 - v^2/c^2)^{1/2}$, where $v = au$. This means that u cannot exceed c/a.

At this point, we would like to comment why we prefer to work with the $z = \ln x_1$ coordinate rather than with x_1. In a certain sense, the z coordinate is more natural from the viewpoint of metric (1) because then the corresponding contribution to the interval has the flat space form dz^2, and z takes values on the whole real axis unlike the x_1 coordinate which takes values only on the positive semiaxis. It is easy to see from Eqs. (14) that the motion along the coordinate z has the same character as the usual flat space motion except a time interval of order $1/u$. Indeed, according to Eqs. (14), the classical particle moves like $z(t) = ut + z_1$ for $t \ll -\frac{1}{u}$. For $|t| \leq \frac{1}{u}$, its motion differs from the familiar inertial motion in a flat space. For $t \gg \frac{1}{u}$, the particle moves like $z(t) = -ut + z_2$, i.e., it changes the direction of motion and goes back to $-\infty$, where it started its motion. Thus, the motion along the coordinate z has a familiar inertial character except a time interval of order $1/u$ with the resulting change of the direction of motion.

On the other hand, according to Eqs. (14), the motion in $x_2, ..., x_D$ coordinates has a completely different character. The particle is practically motionless for almost the whole period of time except the time interval of order $1/u$, when it passes some finite distance. Therefore, the classical motion in these coordinates has a bounded character. On the other hand, coordinates can be arbitrarily chosen in the curved spacetime. Therefore, we should search for an invariant way to demonstrate a bounded character of the classical motion in hyperbolic spaces. To this end, we consider Killing vector fields connected with translations in the x_2, x_3, ..., x_D coordinates.

The hyperbolic space has $D-1$ "translational" Killing vector fields ξ_i ($i = \overline{2, D}$) which are given in the coordinates employed in this paper by

$$\xi_i = \frac{\partial}{\partial x_i}, \quad i = 2, \dots, D. \tag{15}$$

Note that these Killing vector fields realize translations in the corresponding coordinates and form a commuting subalgebra of Killing vector fields in the hyperbolic space. Using the normalized vector fields

$$n_i = \frac{\xi_i}{(-\xi_i \cdot \xi_i)^{1/2}}, \tag{16}$$

where the central dot denotes the scalar product in metric (1), we calculate the following reparametrization-invariant integral along a geodesic:

$$l_i^{(\text{geo})} = \int |(u_t \cdot n_i)| dt = a\int \sqrt{\frac{\dot{x}_i^{\,2}}{x_1^2}} dt = \frac{\pi a |C_i|}{C}, \tag{17}$$

where u_t is the tangent vector to the geodesic, and the integration over t goes from $-\infty$ to $+\infty$. Clearly, Eq. (17) describes the geodesic motion in x_i ($i = \overline{2, D}$) coordinates and, what is important, it does not depend on the choice of coordinates, because n_i are expressed through the Killing vector fields. Since $|C_i| < C$, the quantities $l_i^{(\text{geo})}$ are always less than $L = \pi a$. This is an important result. First of all, note that if we fix x_1 by hand and integrate Eq. (17) over dx_i (of course, the corresponding curve is not a geodesic), then the integral in (17) will be infinite. Therefore, one may expect *a priori* that the space $l_2^{(\text{geo})} \times ... \times l_D^{(\text{geo})}$ is a subspace of the $(D-1)$-dimensional Euclidean space R^{D-1} unbounded in all $D-1$ directions. However, according to Eq. (17), this is not correct, and the space $l_2^{(\text{geo})} \times ... \times l_D^{(\text{geo})}$

connected with geodesic motions is a $(D-1)$-dimensional cube L^{D-1}. Since $l_i^{(\mathrm{geo})}$ are invariant with respect to a change of the coordinates, we conclude that the geodesic motion in hyperbolic spaces has a bounded character in $D-1$ directions. Finally, let us emphasize that this result does not mean that the geodesic motion takes place in a certain finite region in the $x_2, ..., x_D$ coordinates. Equations (14) imply, for example, that one can connect any two points with different x_i by a geodesic, because $|C_i|/C^2$ can be arbitrary large unlike $|C_i|/C$ in Eq. (17) which is always bounded by unity.

At this point, it is instructive to calculate the spatial interval of a geodesic. Obviously, it is invariant with respect to the coordinate transformations that do not involve the time coordinate. Using Eqs. (1) and (14), we find

$$l^{(\mathrm{geo})} = \int_{t_a}^{t_b} \sqrt{\sum_{k=1}^{D} \frac{a^2 \dot{x_k}^2}{x_1^2}}\, dt = ua \int_{t_a}^{t_b} dt. \tag{18}$$

Obviously, it diverges as t_a or t_b tends to infinity. Finally, note that we can consider $l_1^{(\mathrm{geo})} = a \int \sqrt{\frac{\dot{x_1}^2}{x_1^2}} dt$ which is an analog of (17) for the motion along the x_1 coordinate and which, like $l^{(\mathrm{geo})}$, diverges. Using $\zeta_1 = \partial_{x_1}$, this integral can be presented also as

$$l_1^{(\mathrm{geo})} = \int |\,(u_t \cdot n_1)\,| dt, \tag{19}$$

where $n_1 = \zeta_1/(-\zeta_1 \cdot \zeta_1)^{1/2}$ is the normalized vector field which is orthogonal to the vector fields n_i $(i = \overline{2, D})$. Although ζ_1 is similar to ξ_i given by Eq. (15), it is not a Killing vector field in hyperbolic spaces. Nevertheless, being orthogonal to all ξ_i vector fields, it is, in fact, uniquely defined. Therefore, $l_1^{(\mathrm{geo})}$ has a clear geometric meaning. Note that the vector fields ζ_1 and ξ_i form a commuting algebra of vector fields equivalent to the algebra of "translational" Killing vector fields of the D-dimensional Euclidean space. Therefore, perhaps, the most clear signature of a bounded character of motion in hyperbolic spaces is provided by the ranges of values of $l_1^{(\mathrm{geo})}$ and $l_i^{(\mathrm{geo})}$.

3. Solutions of the Dirac Equation and the Effective Dimensional Reduction

In the previous section, we showed that the classical motion in hyperbolic spaces has a bounded character in $D-1$ coordinates. According to [24], this should lead to the effective dimensional reduction $D+1 \to 1+1$ in the quantum problem. In order to show this, we consider, in this section, solutions of the Dirac equation in hyperbolic spaces (see also [18,22], where different approaches to the derivation of solutions were used).

The Dirac equation in hyperbolic spaces H^D has the form

$$\Big(i\gamma^0 \partial_t + \frac{i x_1}{a}\gamma^1 \partial_1 + ... + \frac{i x_1}{a}\gamma^D \partial_D -$$

$$- \frac{i(D-1)}{2a}\gamma^1 - m\Big)\psi = 0. \tag{20}$$

In order to solve this equation, we multiply it by $i\hat{D}+m$ and obtain the second-order differential equation

$$\Big(-\partial_t^2 + \frac{(D-1)^2}{4a^2} - \frac{D-2}{a^2} x_1 \partial_1 +$$

$$+ \frac{x_1^2}{a^2}(\partial_1^2 + ... + \partial_D^2) - \frac{x_1}{a^2}\gamma^1\gamma^2\partial_2 -$$

$$- \frac{x_1}{a^2}\gamma^1\gamma^3\partial_3 - ... - \frac{x_1}{a^2}\gamma^1\gamma^D\partial_D - m^2\Big)\psi = 0. \tag{21}$$

Obviously, we can seek a solution in the form $\psi = e^{-iEt + ip_2x_2 + ... + ip_Dx_D} f(x_1)$. Then we have

$$\Big(E^2 + \frac{(D-1)^2}{4a^2} - \frac{D-2}{a^2} x_1 \partial_1 + \frac{x_1^2 \partial_1^2}{a^2} -$$

$$- \frac{x_1^2}{a^2}(p_2^2 + ... + p_D^2) - i\frac{x_1}{a^2}\gamma^1\gamma^2 p_2 -$$

$$- i\frac{x_1}{a^2}\gamma^1\gamma^3 p_3 - ... - i\frac{x_1}{a^2}\gamma^1\gamma^D p_D - m^2\Big) f(x) = 0. \tag{22}$$

Further, the Dirac γ matrices are present in this equation only in the operator

$$A = -i\frac{x_1}{a^2}\gamma^1\gamma^2 p_2 - i\frac{x_1}{a^2}\gamma^1\gamma^3 p_3 - ... - i\frac{x_1}{a^2}\gamma^1\gamma^D p_D.$$

Since this is a Hermitian operator and its square

$$A^2 = \frac{x_1^2}{a^4}(p_2^2 + ... + p_D^2)$$

is a unit matrix, it can be diagonalized, and its eigenvalues are obviously equal to

$$\sigma \frac{x_1}{a^2}\sqrt{p_2^2 + ... + p_D^2}, \tag{23}$$

where $\sigma = \pm$. Changing the variable $x = x_1\sqrt{p_2^2+...+p_D^2}$, we obtain

$$\left(E^2+\frac{x^2}{a^2}(-1+\partial_x^2)+\frac{(D-1)^2}{4a^2}\right.$$

$$\left.-\frac{(D-2)x}{a^2}\partial_x+\frac{\sigma x}{a^2}-m^2\right)f(x)=0. \quad (24)$$

The absence of any dependence on $p_2,...,p_D$ in this equation is remarkable, because it means that the energy does not depend on them, i.e., the energy is the same for any $p_2,...,p_D$. Equation (24) has form of the equation of the $(1+1)$-dimensional problem, and it is not difficult to find its spectrum, $E=\pm\sqrt{\nu^2/a^2+m^2}$, where ν takes values in $(0,+\infty)$. The effective $(1+1)$-dimensional form (24) of the Dirac equation means the effective dimensional reduction of the dynamics of fermions in hyperbolic spaces. From the mathematical viewpoint, this reduction is related to the spherical and scale symmetries of the H^D metric written in the Poincaré coordinates. Indeed, the spherical symmetry of the $x_2,...,x_D$ part of metric (1) reduces the dependence of the energy on $p_2,...,p_D$ to the dependence on only one invariant $p^2=p_2^2+...+p_D^2$. Then the symmetry of metric (1) with respect to the scale transformations $x_k\to\lambda x_k\,(k=\overline{1,D})$ (recall the change of the variable $x=x_1\sqrt{p_2^2+...+p_D^2}$ that we performed after Eq. (23)) eliminates any dependence on $p_2,...,p_D$ in Eq. (24) for eigenfunctions.

4. Conclusion

Studying the classical motion of massive particles and solving the Dirac equation in hyperbolic spaces H^D, we have clarified the physical reasons for the effective dimensional reduction $D+1\to1+1$ for fermions in the infrared region in hyperbolic spaces.

We have showed that the classical motion has a bounded character in $D-1$ coordinates and, according to Eq. (17), the physical system is effectively of a finite size with respect to these coordinates. On the other hand, the classical motion along the $z=\ln x_1$ coordinate has a familiar inertial character except a finite interval of time, when the particle changes its direction of motion and goes back to $-\infty$, where it started its motion. Therefore, for any classical trajectory with nonzero velocity, the motion along the x_1 coordinate leads to infinite $l_1^{(\mathrm{geo})}$ unlike finite $l_i^{(\mathrm{geo})}$ connected with a geodesic motion in other $D-1$ coordinates.

Solving the Dirac equation in spacetimes $R\times H^D$, we have showed that, due to the spherical and scale symmetries, the initial Dirac problem is reduced to an effective $(1+1)$-dimensional problem. This result agrees with the conclusions of [24] that the effective dimensional reduction in a quantum problem is related to the bounded character of the classical motion with respect to the coordinates over which the reduction takes place.

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ЕФЕКТИВНА РЕДУКЦІЯ РОЗМІРНОСТІ В ГІПЕРБОЛІЧНИХ ПРОСТОРАХ

Е.В. Горбар

Р е з ю м е

Показано, що класичний рух масивних частинок у гіперболічних просторах H^D має обмежений характер відносно $D-1$ координат. Аналізуючи рівняння Дірака знайдено, що обмежений характер класичного руху відповідає наявності ефективної редукції розмірності простору-часу $D+1 \to 1+1$ в інфрачервоній області для ферміонів у квантовій задачі. Ефективна редукція розмірності приводить до нульового значення константи зв'язку для динамічного порушення симетрії в гіперболічних просторах.