ANALYSIS OF THE PROBE NULL STRING MOTION IN THE GRAVITATIONAL FIELD OF A CLOSED NULL STRING WITH CONSTANT RADIUS

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The exact solution of the equations describing the motion of a probe null string in the gravitational field of a closed null string with constant (in time) radius has been obtained and analyzed. The closed null string is assumed to move along the z-axis, being totally located in a plane orthogonal to this axis.

1. Introduction

An idea that the studies of multidimensional objects, the part of which are strings, can become a basis for our understanding of the nature has obtained rather a pronounced recognition in contemporary physics. Among the directions of the string theory development is the elucidation of the role that those objects play in cosmology [1, 2]. Gauge theories of grand unification, which are based on the idea of spontaneous symmetry violation, predict that one-dimensional topological defects, coined as cosmic strings, can be formed in the course of phase transitions in the early Universe [3–6]. Null strings are the zero tension limit in string theory [7]. Some possibilities to use null strings in cosmology are discussed in the scientific literature. For instance, in work [8], it was shown that, by considering the gas of null strings as a dominant gravitation source in D-dimensional Friedmann–Robertson–Walker spaces with $k = 0$, the mechanism of inflation characteristic of those spaces can be described. In work [1], the gas of relict null strings was considered among possible candidates to play the role of a carrier of the so-called "black" matter, the existence of which in the Universe can be adopted now as an established fact. Though the subject of inquiry in the given examples is a gas of null strings rather than a separate null string, the properties of this gas still remain unclear. In our opinion, the first step to the understanding of properties of the null string gas can be the solution of the problem concerning the gravitational fields that are generated by null strings moving along various paths, as well as finding the trajectories of probe null strings in the gravitational field of another null string.

In this work, we studied the probable dynamics of a probe null string in the gravitational field of a closed null string with constant (time-invariant) radius that moves along the z-axis and, at every moment, lies completely in a plane orthogonal to this axis. The analysis of the Einstein equations, which was started in work [9], testifies that the quadratic form, which describes the gravitational field generated by a closed null string, can be presented in this case as

$$dS^2 = e^{2\nu}\left(dt^2 - dz^2\right) - Ad\rho^2 - Bdz^2, \tag{1}$$

where the functions $\nu = \nu(q)$, $B = B(q)$, $A = L(\rho)F(q)$, and $q = t + z$ are the solutions of the equation

$$\frac{\partial}{\partial q}\left(\frac{A_{,q}}{A} + \frac{B_{,q}}{B}\right) - 2\nu_{,q}\left(\frac{A_{,q}}{A} + \frac{B_{,q}}{B}\right) +$$

$$+\frac{1}{2}\left(\left(\frac{A_{,q}}{A}\right)^2 + \left(\frac{B_{,q}}{B}\right)^2\right) = 0. \tag{2}$$

Here, the notation $A_{,q} = \partial A/\partial q$ is introduced. The metric functions $B(q)$ and $A(q,\rho)$ should also satisfy the following asymptotic equality (the boundary conditions):

$$A\left(q,\rho\right),\, B\left(q\right)\big|_{q=0,\ \rho=R}=0,\qquad \left.\frac{A\left(q,\rho\right)}{B\left(q\right)}\right|_{q=0,\ \rho=R}=0. \tag{3}$$

We also note that the closed null string with radius R, which generates a gravitational field in the cylindrical coordinate system ($x^0=t$, $x^1=\rho$, $x^2=\varphi$, $x^3=z$), moves along the path

$$t=\tau,\quad z=-\tau,\quad \varphi=\sigma,\quad \rho=R=\mathrm{const}, \tag{4}$$

where τ and σ are the parameters on the light surface of the null string, namely, σ is the space-like parameter that marks points along the string (for closed strings, it usually ranges from 0 to 2π), and τ is the time-like parameter which is the characteristic time for an observer located at the string point with the coordinate σ. For form (1), the equations of motion for the probe closed null string can be presented in the form

$$q_{,\tau\tau}+2\nu_{,q}(q_{,\tau})^2=0, \tag{5}$$

$$\eta_{,\tau\tau}+e^{-2\nu}\left(L(\rho)F_{,q}(\rho_{,\tau})^2+B_{,q}(\varphi_{,\tau})^2\right)=0, \tag{6}$$

$$\rho_{,\tau\tau}+\left(\frac{F_{,q}}{F}\right)q_{,\tau}\rho_{,\tau}+\left(\frac{L_{,q}}{2L}\right)(\rho_{,\tau})^2=0, \tag{7}$$

$$\varphi_{,\tau\tau}+\left(\frac{B_{,q}}{B}\right)q_{,\tau}\varphi_{,\tau}=0, \tag{8}$$

and the constraint equations in the form

$$e^{2\nu}\left((t_{,\tau})^2-(z_{,\tau})^2\right)-A(\rho_{,\tau})^2-B(\varphi_{,\tau})^2=0, \tag{9}$$

$$e^{2\nu}\left(t_{,\tau}t_{,\sigma}-z_{,\tau}z_{,\sigma}\right)-A\rho_{,\tau}\rho_{,\sigma}-B\varphi_{,\tau}\varphi_{,\sigma}=0, \tag{10}$$

where $\eta=t-z$.

When integrating Eq. (5), two cases have to be considered:

$$q_{,\tau}=0\rightarrow q=q(\sigma) \tag{11}$$

and

$$q_{,\tau}\neq 0. \tag{12}$$

Comparing Eqs. (11) and (4) with each other, one can notice that the former describes a case where the probe null string moves in the same direction along the z-axis as the null string that generates the gravitational field does. We also note that the special cases of Eq. (11) are the case

$$q=\alpha=\mathrm{const}\neq 0, \tag{13}$$

where the probe null string lies – at every time t – completely in a plane parallel to the plane of the null string generating the gravitational field and the case

$$q=0, \tag{14}$$

where the planes of the probe and field-generating null strings coincide.

2. Dynamics of Probe Null String for Case (11)

In case (11), Eqs. (7) and (8) look like

$$\varphi_{,\tau\tau}=0, \tag{15}$$

$$\rho_{,\tau\tau}+\left(\frac{L_{,q}}{2L}\right)(\rho_{,\tau})^2=0. \tag{16}$$

The first and the second integral for Eq. (15) are

$$\varphi_{,\tau}=P_\varphi(\sigma),\quad \varphi(\tau,\sigma)=\varphi_0(\sigma)+P_\varphi(\sigma)\tau, \tag{17}$$

where $\varphi_0(\sigma)$ and $P_\varphi(\sigma)$ are integration "constants".

Two cases are possible for Eq. (16):

$$\rho_{,\tau}=0,\rightarrow \rho=\rho_0(\sigma), \tag{18}$$

and

$$\rho_{,\tau}\neq 0. \tag{19}$$

In case (19), the first and second integrals for Eq. (16) are

$$\rho_{,\tau}=\frac{P_\rho(\sigma)}{\sqrt{L}},\quad \int\sqrt{L(\rho)}d\rho=\rho_0(\sigma)+P_\rho(\sigma)\tau, \tag{20}$$

where $\rho_0(\sigma)$ and $P_\rho(\sigma)$ are integration "constants".

In the case $P_\rho(\sigma)=0$, Eqs. (18) and (20) are identical. Therefore, in the following analysis of the equations of motion for case (12), cases (18) and (19) will be considered together.

In view of expressions (11), (17), and (20), Eq. (6) looks like

$$\eta_{,\tau\tau}=-e^{-2\nu}\left(F_{,q}P_\rho^2+B_{,q}P_\varphi^2\right). \tag{21}$$

Since, according to expressions (1) and (11), the right-hand side of the last equation is a function of the parameter σ only, the first and second integrals of Eq. (21) are

$$\eta_{,\tau} = P_\eta - e^{-2\nu}\left(F_{,q}P_\rho^2 + B_{,q}P_\varphi^2\right)\tau, \tag{22}$$

$$\eta = \eta_0 + P_\eta\tau - e^{-2\nu}\left(F_{,q}P_\rho^2 + B_{,q}P_\varphi^2\right)\tau^2, \tag{23}$$

where $\eta_0 = \eta_0(\sigma)$ and $P_\eta = P_\eta(\sigma)$ are integration "constants".

In view of expressions (11), (17), and (20), constraint (9) takes the form

$$F(q)P_\rho^2 + B(q)P_\varphi^2 = 0. \tag{24}$$

We note that, according to expression (1), the functions $F(q)$ and $B(q)$ are positive definite for all $q \neq 0$. Therefore, Eq. (24) implies that if the probe null string corresponding to case (11) does not lie in the same plane with the null string which generates the gravitational field, there is a unique possibility for condition (24) to be fulfilled, namely,

$$P_\rho(\sigma) = P_\varphi(\sigma) = 0. \tag{25}$$

On the other hand, if the probe null string lies in the same plane with the null string generating the gravitational field, then, according to Eq. (3), constraint (24) becomes an identity for arbitrary initial momenta $P_\rho(\sigma)$ and $P_\varphi(\sigma)$. Hence, if $q \neq 0$, the solution of the equations of motion for the probe null string under condition (3) looks like

$$\varphi = \varphi_0(\sigma), \quad \rho = \rho_0(\sigma), \quad q = t + z = q_0(\sigma),$$

$$\eta = t - z = \eta_0(\sigma) + P_\eta(\sigma). \tag{26}$$

From expressions (26), it follows that, in case (11), the motion of the probe null string does not depend on the metric functions, being completely determined by initial conditions. It can be demonstrated that, for solution (26), constraint (10) takes the form

$$P_\eta(\sigma)q_{0,\sigma} = 0. \tag{27}$$

Since the static solutions of the equations of motion for the null string are unphysical (i.e. $P_\eta(\sigma) \neq 0$), a unique continuation for Eq. (27) is

$$q_{0,\sigma} = 0, \rightarrow q_0 = \alpha = \text{const}, \tag{28}$$

where $\alpha \neq 0$, because solution (26) was written down for the case $q \neq 0$. An important result follows from expression (28); namely, if the probe null string moves in the same direction with the null string generating the gravitational field, a unique opportunity for the former string arrangement is that this string should lie completely in one of the planes parallel to the plane of the latter string at any time moment (something like a polarization effect). In this case, as follows from the first of equalities (26), the initial shape of the probe null string – generally speaking, it can be arbitrary – does not change in time.

If $q = 0$ (this is the case where the probe and field-generating null strings are in the same plane), both constraints (9) and (10) are satisfied identically, so that the solution of the equations of motion for the probe null string is as follows:

$$q = 0, \quad \eta = \eta_0 + P_\eta\tau - e^{-2\nu}\left(F_{,q}P_\rho^2 + B_{,q}P_\varphi^2\right)\tau^2/2, \tag{29}$$

$$\varphi = \varphi_0 + P_\varphi\tau, \quad \int \sqrt{L(\rho)}d\rho = \rho_0 + P_\rho. \tag{30}$$

Expressions (29) and (30) testify that, in the case where the initial momenta $P_\rho(\sigma)$ and $P_\varphi(\sigma)$ equal zero (this is the case of the ideal gas of identical null strings), the dynamics of the probe null string in the plane $q = 0$ does not differ from that in planes $q = \alpha = \text{const}$. At the same time, if $P_\rho(\sigma) \neq 0$ and $P_\varphi(\sigma) \neq 0$ in expressions (29) and (30), the trajectory of the probe null string varies in time.

For instance, it is easy to show [10] that the following metric functions satisfy the asymptotic equalities (3) and belong to the solutions of Eq. (2):

$$A(q,\rho) = (\rho - R)^2\text{sh}^4(\lambda q),$$

$$B(q) = sh^2(\lambda q)\text{ch}^2(\lambda q), \quad e^{2\nu} = \text{ch}^2(\lambda q), \tag{31}$$

where $\lambda = \text{const}$. From expressions (31), we obtain

$$L(\rho) = (\rho - R)^2, \quad F(q) = \text{sh}^4(\lambda q). \tag{32}$$

In case (31) and (32), the solution of (29) and (30) looks like

$$q = 0, \quad t = (\eta_0 + P_\eta\tau)/2, \quad \varphi = \varphi_0 + P_\varphi\tau, \tag{33}$$

$$\rho = R \pm \sqrt{2(\rho_0 + P_\rho\tau)}. \tag{34}$$

From equality (34), it follows that, if the probe null string is located beyond the region confined by the field-generating null string at the initial moment (i.e. $\rho < R$; this case corresponds to the sign "+" in expression (34)), then, depending on the expression for the initial momentum $P_\rho(\sigma)$, it can either approach the null string that generates the gravitational field (but cannot penetrate into the region confined by the latter) or move away from it. If the probe null string is located in the region confined by the field-generating null string at the initial moment (i.e. $0 < \rho < R$; this case corresponds to the sign "$-$" in expression (34)), then the probe null string will remain in the internal region confined by the latter, irrespective of the expression for the initial momentum $P_\rho(\sigma)$.

3. Dynamics of Probe Null String for Case (12)

In case (12), the first and second integrals of Eq. (5) are

$$q_{,\tau} = P_q e^{2\nu}, \quad \int e^{2\nu} dq = q_0 + P_q, \tag{35}$$

where $q_0 = q_0(\sigma)$ and $P_q = P_q(\sigma)$ are integration "constants".

Since we consider case (12),

$$P_q \neq 0 \tag{36}$$

in Eq. (35). If $\rho_{,\tau} \neq 0$ and $\varphi_{,\tau} \neq 0$, the first integral of Eqs. (7) and (8)) for (12) is

$$\rho_{,\tau} = P_\rho / F(q)\sqrt{L(\rho)}, \quad \varphi_{,\tau} = P_\varphi / B(q). \tag{37}$$

By integrating Eq. (37), we obtain

$$\int \sqrt{L(\rho)} d\rho = \rho_0 + P_\rho \int \frac{d\tau}{F(q)},$$

$$\varphi(\tau, \sigma) = \varphi_0 + P_\varphi \int \frac{d\tau}{B(q)}. \tag{38}$$

From Eq. (35), we can see that

$$d\tau = \left(e^{2\nu}/P_q\right) dq. \tag{39}$$

We use this relation to change the integration variable in Eqs. (38). As a result, we ultimately obtain

$$\int \sqrt{L(\rho)} d\rho = \rho_0 + \frac{P_\rho}{P_q} \int \frac{e^{2\nu}}{F(q)} dq, \tag{40}$$

$$\varphi(q, \sigma) = \varphi_0 + \frac{P_\varphi}{P_q} \int \frac{e^{2\nu}}{B(q)} dq. \tag{41}$$

In the case

$$\rho_{,\tau} = 0; \quad \varphi_{,\tau} = 0, \quad \rightarrow \quad \rho = \rho_0(\sigma); \quad \varphi = \varphi_0(\sigma), \tag{42}$$

Eqs. (7) and (8) are satisfied identically. Again, since (42) can be regarded as a special case of solutions (40) and (41) (if we put $P_\rho = P_\varphi = 0$ in those solutions, we will obtain Eq. (42)), only solution (40) and (41) will be studied further.

Using transformation (39), the first and second integrals of the equation

$$\eta_{,\tau\tau} = -e^{-2\nu}\left(P_\rho^2 \frac{F_{,q}}{F^2} + P_\varphi^2 \frac{B_{,q}}{B^2}\right) \tag{43}$$

can be expressed as follows:

$$\eta_{,\tau} = P_\eta + \frac{1}{P_q}\left(\frac{P_\rho^2}{F(q)} + \frac{P_\varphi^2}{B(q)}\right), \tag{44}$$

$$\eta(\tau, \sigma) = \eta_0 + P_\eta \tau +$$

$$+\frac{1}{P_q^2}\left(P_\rho^2 \int \frac{e^{2\nu}}{F(q)} dq + P_\varphi^2 \int \frac{e^{2\nu}}{B(q)} dq\right). \tag{45}$$

Substituting expressions (35), (37), and (44) into constraint (9), we obtain

$$P_\eta P_q = 0. \tag{46}$$

Since $P_q \neq 0$ in case (12), we obtain unambiguously from Eq. (46) that

$$P_\eta = 0. \tag{47}$$

In view of Eq. (47), constraint (10) for (35), (37), (40), (41), and (45) reads

$$P_q \eta_{0,\sigma} - 2P_\varphi \varphi_{0,\sigma} - 2P_\rho \rho_{0,\sigma} = 0. \tag{48}$$

Using Eq. (47), let us present expression (45) in the following form:

$$z = t - \eta_0 - \frac{1}{P_q^2}\left(\int \frac{P_\rho^2 e^{2\nu}}{F(q)} dq + \int \frac{P_\varphi^2 e^{2\nu}}{B(q)} dq\right). \tag{49}$$

Since Eq. (35) connects the sheet parameters τ and σ with q, relation (49) can be used to determine the dynamics of the probe null string along the z-axis at various fixed time moments t. For instance, for space (31), equality (49) looks like

$$z = t - \eta_0 + \frac{1}{\lambda}\coth(\lambda q)\left(\frac{P_\rho^2}{3P_q^2}\text{cth}^2(\lambda q) + \frac{P_\varphi^2}{P_q^2}\right). \qquad (50)$$

One can see that the coordinate $z \to -\infty$, if $t \to -\infty$. Hence, for space (31), the probe null string is located at $z = -\infty$ at $t = -\infty$, and it moves in the positive direction of the z-axis with time. According to Eq. (4), the null string that generates the gravitational field is at $z = +\infty$ at $t = -\infty$, and it moves in the negative direction of the z-axis. Therefore, expression (12) describes the case where the probe and field-generating null strings move toward each other. By considering Eq. (50) for a fixed time moment t, one can determine the z-distribution of the probe null string at this moment.

Solutions (40) and (41) for space (31) can be presented as

$$\varphi(q, \sigma) = \varphi_0 - \frac{P_\varphi}{\lambda P_q}\text{cth}(\lambda q), \qquad (51)$$

$$\rho(q, \sigma) = R \pm \sqrt{2\left(\rho_0 - \frac{P_\rho}{3\lambda P_q}\text{cth}^3(\lambda q)\right)}. \qquad (52)$$

Whence, it follows that, in the case where the momenta $P_\rho(\sigma)$ and $P_\varphi(\sigma)$ differ from zero at the initial time moment, the dynamics of the probe null string that moves to meet the field-generating null string can be rather complicated. For instance, from Eq. (51), it follows that, in the case $P_\varphi(\sigma) \neq 0$, the closer the probe null string approaches the plane $q = 0$ (it is the plane, where the null string generating the gravitational field is located), the more strongly it becomes twisted. From expression (52), it follows that, in the case $P_\rho(\sigma) \neq 0$, when the probe null string approaches the plane $q = 0$, either its infinite inflation, which accompanies by the shape change, or collapse is possible, depending on the functions $\rho_0(\sigma)$, $P_\rho(\sigma)$, and $P_q(\sigma)$, the choice of which restricts constraint (48). However, if

$$P_\rho(\sigma) = 0, \quad P_\varphi(\sigma) = 0 \qquad (53)$$

at the initial time moment, there is a unique opportunity to satisfy constraint (48) in case (12), namely,

$$\eta_{0,\sigma} = 0 \to \eta_0 = \text{const}. \qquad (54)$$

The solution of the equations of motion at (53) and (54) is

$$z = t - \eta, \quad \rho = \rho_0(\sigma), \quad \varphi = \varphi_0(\sigma). \qquad (55)$$

One can see that, in the case where the initial momenta of the probe null string equal zero, and this string moves toward the null string that generates the gravitational field, it is located completely in a plane parallel to the plane $q = 0$ at every time moment. Moreover, the initial shape of the probe null string is not changed in time. Thus, provided that conditions (53) are fulfilled, both zero strings, by moving toward each other, will meet in the plane $q = 0$. Then, they will fly away in the opposite directions, not affecting each other.

4. Conclusions

The analysis of the solution of the equations describing the motion of a probe null string in the gravitational field of a closed null string with constant radius gave rise to the following conclusions:

1. The probe null string can move in either direction with respect to the null string that generates the gravitational field.

2. If the probe null string moves in the same direction as the field-generating null string does, it is totally located in a plane parallel to the plane of the latter null string at any time moment, which can be interpreted as a polarization effect. In this case, if the probe null string does not lie in the plane $q = 0$, its initial shape – generally speaking, it can be arbitrary – is not changed in time. However, if the probe null string lies in the plane $q = 0$ and if the initial momenta $P_\rho(\sigma)$ and $P_\varphi(\sigma)$ equal zero (the case of the ideal gas of identical null strings), its dynamics does not differ from the dynamics of strings in planes $q = \alpha = \text{const}$. Otherwise, i.e. in the case $P_\rho(\sigma) \neq 0$ and $P_\varphi(\sigma) \neq 0$, the dynamics of the probe null string in the plane $q = 0$ can be rather complicated.

3. If the probe null string goes toward the null string that generates the gravitational field, then, in the case where the initial momenta $P_\rho(\sigma)$ and $P_\varphi(\sigma)$ differ from zero, its dynamics can be rather complicated. For instance, in the case $P_\varphi(\sigma) \neq 0$, the closer the probe null string approaches the plane $q = 0$ (it is the plane, where the null string that generates the gravitational field is located), the more strongly it becomes twisted. At the same time, if $P_\rho(\sigma) \neq 0$, then, depending on the form of the functions ρ_0, $P_\rho(\sigma)$, and $P_q(\sigma)$, the choice of which restricts the constraint equation, either an infinite inflation of the probe null string together with its shape change or its collapse is possible, when the probe null string approaches the plane $q = 0$ (in this case, the probe

null string can be not located in a plane parallel to the plane $q = 0$). If the initial momenta of the probe null string equal zero, the latter is completely located in a plane parallel to the plane $q = 0$ at any time moment, and the initial shape of the probe null string does not vary in time. This means that, in this case, both null strings, moving toward each other, meet in a plane and, afterwards, fly away in opposite directions, not affecting each other.

By analyzing the results of this work, we may suppose that, since different regions of a gas of null strings with constant radius are not coupled causally at the initial time moment, there can appear a domain structure of this gas. That is, we may suggest that there are a large number of isolated regions, where null strings are strictly polarized and move in the same direction. Within every domain, this direction of motion is random and does not correlate with the directions of motion in other domains. The conditions for such domains to emerge, as well as their subsequent existence and the physical processes in regions that separate domains from one another, can be a topic of following researches of the gas of null strings.

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АНАЛІЗ РУХУ ПРОБНОЇ НУЛЬ-СТРУНИ В ГРАВІТАЦІЙНОМУ ПОЛІ ЗАМКНЕНОЇ НУЛЬ-СТРУНИ ПОСТІЙНОГО РАДІУСА

О.П. Леляков

Р е з ю м е

Отримано і проаналізовано точний розв'язок рівнянь руху пробної нуль-струни в гравітаційному полі замкненої нуль-струни постійного (незмінного з часом) радіуса, що прямує уздовж осі z й у кожний момент часу цілком знаходиться у площині, що ортогональна до цієї осі.