NONLINEAR DYNAMICS OF THE DIPOLE MOMENTUM OF A TWO-LEVEL ATOM IN THE SEMICLASSICAL JAYNES–CUMMINGS MODEL

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The nonlinear dynamics of expectation values for observables in the integrable Jaynes–Cummings model has been studied. The model describes the interaction between a two-level atom and a single-mode classical electromagnetic field. Explicit formulas have been obtained for the evolution of transverse components of the atomic dipole momentum and the inverse population of atomic levels. A comparison between the solutions obtained in the quantum-mechanical and semiclassical versions of the Jaynes–Cummings model has been made.

1. Introduction

The Jaynes–Cummings model [1] is one of the simplest ones in quantum optics. It describes the interaction between an idealized two-level atom and a single-mode electromagnetic field (a standing electromagnetic wave in a resonator). As a semiclassical version of the model, we adopt the system of equations that describes the dynamics of the expectation values of the atomic dipole moment and the probability for an atom to stay in the ground or excited state (the inverse population), when the atom interacts with a classical electromagnetic field.

In the quantum-mechanical model, the electromagnetic field is a time-dependent operator. Such a model can be considered either in the Heisenberg or the Schrödinger representation. In the former case, the system of Heisenberg operator equations can be linearized, so that the solutions can be expressed in terms of elementary functions [2]. Linearization is possible, because there emerge the unphysical identities between dipole moment operators in the two-level approximation. A similar picture is obviously observed in the Schrödinger representation as well.

The semiclassical version of the model is inherently nonlinear, and, therefore, the solutions expressed in terms of elementary functions can be obtained only in a few degenerate cases. Though the full integrability of the classical model, as well as its relation to the classical Gaudin magnet [3], was established rather long ago, we have found no explicit solutions in the scientific literature that would satisfy arbitrary, physically admissible initial conditions. At the same time, despite the fact that the nonlinear oscillations of energy level populations were described as early as in the first relevant publication [1], the description of the dynamics of average values for the atomic dipole moment remains incomplete, in our opinion. In this work, we present explicit formulas for all dynamical variables of the Jaynes–Cummings model.

Let us briefly discuss the physical meaning of consequences given by the Jaynes–Cummings quantum-mechanical model. Besides the well established and experimentally testified description of oscillations of the energy level populations, their collapse, and revival, some authors insist that this model also describes phenomena of spontaneous radiation emission. In so doing, the latter is assumed to be stimulated by "zero" oscillations of the electromagnetic field. Authoritative physicists [5] repeatedly emphasize that such a statement is wrong. Quantum electrodynamics asserts that spontaneous radiation emission arises because atomic energy states are quasistationary. The natural width of an energy level (and, therefore, its finite lifetime) is governed by quantum-mechanical fluctuations of a Coulomb field in the atom. These fluctuations are associated with polarization effects in vacuum. Polarization corrections to stationary energy values are complex values. Their real parts describe the Lamb shift of the levels, and the imaginary ones do their natural width [6] and, hence, the effect of spontaneous radiation emission. In the Jaynes–Cummings model, the energies of a two-level atom are strictly fixed; therefore, the origin of spontaneous radiation emission remains obscured there. Quantum electrodynamics eliminates zero oscillations of the transverse electromagnetic field in the case of unlimited space by varying the energy reference mark. But if they are left in the Hamiltonian of the electromagnetic field, the energy should obviously be reckoned from $\frac{1}{2}\hbar\omega$, where ω is the electromagnetic radiation frequency. Then, the energy calculated per one photon with "zero" oscillations amounts to $\frac{3}{2}\hbar\omega$. When the atom absorbs a photon and gets excited, an energy of $\frac{1}{2}\hbar\omega$ remains in the system, and it is this energy that stimulates radiation emission.

In the framework of a semiclassical model, one can imitate spontaneous radiation emission by introducing the frequency detuning

$$\Delta = \Omega - \omega,$$

where Ω is the frequency of the radiation emission by an atom. This quantity, being multiplied by Planck's constant, effectively plays the role of excited level width and makes the process nonlinear.

As was marked before, the main purpose of our research was to explicitly describe the dynamics of expectation values of the atomic dipole moment. The corresponding formulas are given in Section 4. In that Section, we pay attention to an interesting phenomenon of the dipole moment phase incursion, when an atom returns back to the ground state from an excited one. The formulas obtained for the dipole moment dynamics play an important role, e.g., when the pressure of light on an atom is calculated [10]. This aspect of the problem dealing with the light–atom interaction will be considered in the next work.

2. Model of Interaction between a Two-level Atom and a Single-mode Electromagnetic Field

If an electromagnetic field is close to a monochromatic one with frequency ω, it interacts only with those atomic states, the energies of which resonate with the field. This means that the atomic transition frequency is close to the field frequency:

$$\Omega = \frac{E_2 - E_1}{\hbar} \simeq \omega. \quad (1)$$

Bearing this circumstance in mind, one can approximately consider a multilevel atom as a two-level system.

Provided that the electromagnetic field amplitudes are small and the electromagnetic field wavelengths are longer than atomic dimensions, the radiation absorption and radiation emission processes can be described in the dipole approximation. Supposing the atom being immovable (such a situation can be realized if the wave in the resonator is standing), the interaction Hamiltonian looks like

$$H_{\text{int}} = -e\hat{\mathbf{r}}(t)\hat{\mathbf{E}}(t), \quad (2)$$

where $\hat{\mathbf{r}}(t)$ is the operator of electron coordinate in the atom, and $\hat{\mathbf{E}}(t)$ is the quantized electromagnetic field.

The electromagnetic field is considered in the Hamiltonian gauge, i.e.

$$\mathbf{E}(\mathbf{x}, t) = -\frac{\partial \mathbf{A}(\mathbf{x}, t)}{\partial t},$$

where $\mathbf{A}(\mathbf{x}, t)$ is the vector potential (div $\mathbf{A} = 0$). Let a single-mode quantized field be described at a fixed point $\mathbf{x}_0$ by the Heisenberg operator

$$\hat{\mathbf{E}}(\mathbf{x}_0, t) = \imath\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\mathbf{u}(\mathbf{x}_0)[\hat{a}(t) - \hat{a}^+(t)], \quad (3)$$

where V is the resonator volume, $\mathbf{u}$ is the polarization vector proportional to the field amplitude, and $\hat{a}$ and $\hat{a}^+$ are, respectively, the annihilation and creation operators of quantum field states, for which the following relation is true:

$$[\hat{a}, \hat{a}^+] = 1.$$

Let us denote the atomic states with the energies E_1 and E_2 ($E_2 > E_1$) as $|1\rangle$ and $|2\rangle$, respectively. The atomic Hamiltonian calculated for these states is evidently looks like a diagonal matrix,

$$\hat{H}_A = \begin{pmatrix} E_2 & 0 \\ 0 & E_1 \end{pmatrix} = (E_2 - E_1)\hat{S}_3 + \frac{1}{2}(E_1 + E_2)\hat{\mathbf{1}}_2, \quad (4)$$

where $\hat{S}_3 = \frac{1}{2}\hat{\sigma}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In its turn, the dipole moment operator looks like an antidiagonal matrix,

$$\hat{\mathbf{p}} = \begin{pmatrix} 0 & \mathbf{p}_{21} \\ \mathbf{p}_{12} & 0 \end{pmatrix} = \mathbf{p}_{21}\hat{S}_+ + \mathbf{p}_{12}\hat{S_-},$$

where $\mathbf{p}_{21} = e\langle 2 \mid \hat{\mathbf{r}} \mid 1\rangle = \mathbf{p}_{12}^*$, and $\hat{S}_+ = \hat{S}_-^{\mathrm{T}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then the operator of dipole interaction reads

$$\hat{H}_{\rm int} = -\imath\frac{\hbar\omega}{2\varepsilon_0 V}[(\mathbf{u},\mathbf{p}_{21})\hat{S}_+ + (\mathbf{u},\mathbf{p}_{12})\hat{S}_-](\hat{a}-\hat{a}^+). \quad (5)$$

Since the electromagnetic field is transverse, only the transverse (with respect to the quantization axis Oz) components of the dipole moment take part in the interaction. Assuming that

$$(\mathbf{u},\mathbf{p}_{21}) = \imath|\mathbf{u}||\mathbf{p}_{21}|,$$

we obtain the following expression for the interaction Hamiltonian:

$$\hat{H}_{\rm int} = \hbar g(\hat{S}^+ - \hat{S}^-)(\hat{a}-\hat{a}^+),$$

where the notation $g = \sqrt{\frac{\omega}{2\hbar\varepsilon_0 V}}|\mathbf{u}||\mathbf{p}_{21}|$ is used. The full "atom–field" Hamiltonian looks like

$$\hat{H} = \hbar\Omega\hat{S}_3 + \hbar\omega\hat{a}^+\hat{a} + \hbar g(\hat{S}^+ - \hat{S}^-)(\hat{a}-\hat{a}^+) \quad (6)$$

to within insignificant constant terms [2].

Rotation wave approximation (RWA)

The interaction Hamiltonian (6) contains the terms $(\hat{S}^+\hat{a} + \hat{S}^-\hat{a}^+)$ and $-(\hat{S}^+\hat{a}^+ + \hat{S}^-\hat{a})$ characterized by different oscillation frequencies. If the interaction is absent, $\hat{S}_\pm(t) = \hat{S}_\pm(0)e^{\pm i\Omega t}$ and $\hat{a}(t) = \hat{a}(0)e^{i\omega t}$. Therefore, the first term oscillates with the frequency $\Delta = \Omega - \omega$, and the second one with the frequency $\Omega + \omega$. Averaging the interaction Hamiltonian over fast oscillations [7], we obtain – in addition to the term $\hbar g(\hat{S}^+\hat{a}+\hat{S}^-\hat{a}^+)$ – the averaged term $\frac{\hbar g}{\Omega+\omega}(\hat{S}^+\hat{a}+\hat{S}^-\hat{a}^+)$, which we consider as a small correction due to a large frequency $\Omega+\omega$ in its denominator and neglect in what follows. Hence, we adopt the Hamiltonian in the form

$$\hat{H} = \hbar\Omega\hat{S}_3 + \hbar\omega\hat{a}^+\hat{a} + \hbar g(\hat{S}^+\hat{a} + \hat{S}^-\hat{a}^+) \quad (7)$$

which corresponds to the Jaynes–Cummings model in the "rotation wave approximation" [1,2]. The Heisenberg equations associated with Hamiltonian (7) look like

$$\imath\frac{d\hat{S}_+}{dt} = -\Omega\hat{S}_+ + 2g\hat{a}_+\hat{S}_3,$$

$$\imath\frac{d\hat{S}_-}{dt} = \Omega\hat{S}_- - 2g\hat{a}\hat{S}_3,$$

$$\imath\frac{d\hat{S}_3}{dt} = g(\hat{S}_+\hat{a} - \hat{S}_-\hat{a}^+),$$

$$\imath\frac{d\hat{a}}{dt} = \omega\hat{a} + g\hat{S}_-,$$

$$\imath\frac{d\hat{a}^+}{dt} = -\omega\hat{a}^+ - g\hat{S}_+. \quad (8)$$

The quantum-mechanical Hamiltonian (7) together with all its dynamical variables is defined in a Hilbert space. The latter is a tensor product of the two-dimensional space of atomic states and the infinite-dimensional space of electromagnetic field states. As basic field states, we choose photonic states $|n\rangle$, for which

$$\hat{a}^+\hat{a}|n\rangle = n|n\rangle, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad (9)$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (10)$$

An arbitrary normalized quantum state of the "atom–field" system can be written down in the form

$$|\Psi\rangle = \sum_n \Big(c_1(n)|1,n\rangle + c_2(n)|2,n\rangle\Big), \quad (11)$$

where

$$\sum_n \Big(|c_1(n)|^2 + |c_2(n)|^2\Big) = 1.$$

The states are evidently orthogonal to one another and satisfy the completeness criterion. Let us average the right- and left-hand sides of Eqs. (8) over states (11). The "semiclassical" character of such an averaging consists in that we neglect the quantum-mechanical correlations between the field variables and the atomic operators and put

$$\langle\Psi|\hat{a}^+\hat{S}_3|\Psi\rangle = \langle\Psi|\hat{a}^+|\Psi\rangle\langle\Psi|\hat{S}_3|\Psi\rangle.$$

Then, for the averaged $S_\pm = \langle\Psi|\hat{S}_\pm|\Psi\rangle$, $a^+ = \langle\Psi|\hat{a}^+|\Psi\rangle$, and $a^- = \langle\Psi|\hat{a}|\Psi\rangle$, we obtain a system of equations, the form of which is identical to that of system (8).

Now, we intend to pass from the complex-valued functions $S_\pm$ and $a^\pm$ to the real-valued quantities

$$S_1 = \frac{1}{2}(S_+ + S_-), \qquad S_2 = \frac{1}{2\imath}(S_+ - S_-),$$

$$a_1 = \frac{1}{2}(a + a^+), \qquad a_2 = \frac{1}{2\imath}(a - a^+).$$

We fix the classical Hamiltonian

$$\mathcal{H} = \Omega S_3 + \omega(a_1^2 + a_2^2) + g(a_1 S_1 + a_2 S_2).$$

Then, in the five-dimensional space of variables S_1, S_2, S_3, a_1, and a_2, we obtain the system of Hamiltonian equations

$$\frac{dS_i}{dt} = \{S_i, H\}, \qquad \frac{da_\nu}{dt} = \{a_\nu, H\}. \tag{12}$$

The Poisson bracket for this system is non-canonical (degenerate), and it has the form

$$\{f_1, f_2\} = \sum_{ij} W_{ij} \frac{\partial f_1}{\partial S_i}\frac{\partial f_2}{\partial S_j} + \frac{1}{2}\Big[\frac{\partial f_1}{\partial a_1}\frac{\partial f_2}{\partial a_2} - \frac{\partial f_2}{\partial a_1}\frac{\partial f_1}{\partial a_2}\Big],$$

where $W_{ij} = \varepsilon_{ijk} S_k$ (det $[W_{ij}] = 0$). System (12) is the main object of our investigations.

3. Integration of Hamiltonian Equations for the Jaynes–Cummings Semiclassical Model

The system of Hamiltonian equations (12), owing to the presence of two commuting integrals of motion

$$h_0 = S_3 + a^+ a^-, \tag{13a}$$

$$h_{-1} = \lambda_0 S_3 + S_+ a^+ + S_+ a^-, \qquad g\lambda_0 = \Delta, \tag{13b}$$

$$\mathcal{H} = \omega h_0 + g h_{-1}, \qquad \omega + g\lambda_0 = \Omega \tag{13c}$$

and the constraint

$$S_1^2 + S_2^2 + S_3^2 = S_+ S_- + S_3^2 = \frac{1}{4} \tag{14}$$

is integrable in the sense of the Liouville theorem [7]. The phase space of this system is a four-dimensional manifold, being a direct product of a sphere and a two-dimensional plane space $\mathbb{R}^2$:

$$\mathcal{M} \simeq S^2 \times \mathbb{R}^2.$$

The variables on the sphere are the projections of the dipole moment, S_1 and S_2, and the inverse level population S_3 which are coupled by relation (14). In accordance with the Liouville theorem, a joint level surface for the integrals of motion $h_0 = \text{const}$ and $h_{-1} = \text{const}$ is diffeomorphic to a two-dimensional torus T^2 (we refer to this object as *Liouville torus*). The system of equations (12) can be presented in the Lax form:

$$\imath \frac{d\hat{L}}{dt} = [\nabla h, \hat{L}], \tag{15}$$

where the Lax operator $\hat{L}$ looks like

$$\hat{L} = \begin{pmatrix} \frac{\lambda}{2} + \frac{1}{\lambda-\lambda_0}\widehat{S}_3 & \frac{1}{\lambda-\lambda_0}\widehat{S}_+ + a^+ \\ \frac{1}{\lambda-\lambda_0}\widehat{S}_- + a^- & -(\frac{\lambda}{2} + \frac{1}{\lambda-\lambda_0}\widehat{S}_3), \end{pmatrix},$$

and ∇h is the matrix gradient of the function

$$h = g(S_3^2 + S_+ S_-) + \omega(a^+ S_- + a S^+ + a_3 S_3).$$

This gradient is calculated by the formula

$$\nabla h = (\lambda - \lambda_0)^{-1}\big(\frac{\partial h}{\partial S_+}\hat{S}_- \frac{\partial h}{\partial S_-}\hat{S}_+\big) +$$

$$+ (\lambda - \lambda_0)^{-2}\big(\frac{\partial h}{\partial a_+}\hat{S}_- + \frac{\partial h}{\partial a}\hat{S}_+ + \frac{\partial h}{\partial a_3}\hat{S}_3\big), \ a_3 \to 0.$$

The validity of the Lax representation allows one, while studying system (12), to apply the methods of the theory of finite-zone integration of nonlinear equations of the soliton type [8]. On the basis of representation (15), we can make a few general statements about system (12). In particular, we can assert that it is algebraically integrable. This means that the phase space of the system can be extended on the complex dynamical variables and the complex time $t + i\tau$. In this case, the Liouville torus T^2 extends to a complex Abel torus $T^2_{\mathbb{C}}$. The latter, according to the Abel map and the Jacobi reciprocity formula, is identified with a direct symmetrized product of two copies of a Riemann surface:

$$T^2_{\mathbb{C}} \rightleftarrows (\mathfrak{R} \times \mathfrak{R})_{\text{sym}}.$$

The surface $\mathfrak{R}$ is defined as an algebraic curve by the algebraic equation

$$\mathfrak{R}: \quad \det[\hat{L}(\lambda) - \mu] = 0$$

or

$$w^2 = \lambda^4 - 2\lambda^3\lambda_0 + (4h_0 + \lambda_0^2)\lambda^2 +$$

$$+ 4(h_{-1} - 2\lambda_0 h_0)\lambda + 4(h_{-2} + \lambda_0^2 h_0 - \lambda h_{-1}), \tag{16}$$

where $w = 2\mu(\lambda - \lambda_0)$. The genus of the surface $\mathfrak{R}$ is equal to unity. The torus $T^2_{\mathbb{C}}$ is a generalized Jacobian with respect to the surface $\mathfrak{R}$; therefore, the Abel map is given by integrals of the first and second kinds.

We intend to use these facts in our next work for the construction of canonical variables “action–angle” and for the development of perturbation theory; this will allow us to go beyond the limits of RWA. But now, let us integrate system (12) in a simpler way. We separate two variables, and each of them will obey a separate closed equation. Such variables are $S_3(t)$ and a new variable $\xi(t) = -S_-(t)/a^-(t)$.

Closed equation for the variable S_3 and its solutions

For the variable $S_3(t)$, the differential equation is

$$\frac{d^2S_3}{d(gt)^2} = 6S_3^2 - (4h_0 + \lambda_0^2)S_3 - \left(\frac{1}{2} - \lambda_0 h_{-1}\right). \qquad (17)$$

It can be interpreted as a Newton equation for a particle with the mass $m = 1$ in a potential field

$$U(S_3) = -2S_3^3 + \frac{4h_0 + \lambda_0^2}{2}S_3^2 + \left(\frac{1}{2} - \lambda_0 h_{-1}\right)S_3.$$

For such a system, the conservation law of energy is fulfilled:

$$\left[\frac{dS_3}{d(gt)}\right]^2 = 2[E - U(S_3)]. \qquad (18)$$

On the other hand, Eqs. (12) give rise to

$$\left[\frac{dS_3}{d(gt)}\right]^2 = -[(S_+a^-)^2 + (S_-a^+)^2] + 2a^+a^-S_+S_- =$$

$$= -(h_{-1}^2 + \lambda_0^2 S_3^2 - 2\lambda_0 h_{-1}S_3) + 4(h_0 - S_3)(\frac{1}{4} - S_3^2). \qquad (19)$$

By comparing Eqs. (18) and (19), we find that

$$E = \frac{1}{2}(h_0 - h_{-1}^2).$$

Let e_1, e_2, and e_3 $(e_1 \geq e_2 \geq e_3)$ be the zeros of the polynomial on the right-hand side of Eq. (18). Then, for the function $S(gt)$, we have a solution

$$S(gt) = \wp(gt + \imath u_2) + \frac{1}{3}(e_1 + e_2 + e_3),$$

where $\wp(u)$ is the Weierstrass elliptic function determined by the parameters

$$E_i = e_i - \frac{1}{3}(e_1 + e_2 + e_3), \qquad E_1 + E_2 + E_3 = 0.$$

Let us pass over from the Weierstrass function to the Jacobi elliptic function taking advantage of the formula

$$\wp(u) = E_3 + \frac{E_1 - E_3}{\mathrm{sn}^2(\sqrt{E_1 - E_3}u; k)},$$

where $k = \sqrt{\frac{E_2 - E_3}{E_1 - E_3}} = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}$ and $u = u_1 + iu_2$. We select the imaginary part of the phase of the function $\mathrm{sn}(z; k)$ making use of the condition

$$\sqrt{e_1 - e_3}\ u_2 = K',$$

where K' is a quarter of the imaginary period of the elliptic function. Taking advantage of the formula

$$\mathrm{sn}(u + \imath K') = \frac{1}{k\ \mathrm{sn}(u)},$$

we ultimately obtain the expression

$$S_3(gt) = e_3 + (e_2 - e_3)\ \mathrm{sn}^2(\sqrt{e_1 - e_3}gt; k) \qquad (20)$$

which coincides, at $\lambda_0 = 0$, with the corresponding formula from work [1] to within the choice of the parameters e_1, e_2, e_3 and the phase.

Formula (20) can be used to describe three characteristic modes in the behavior of a two-level atom: the excitation of the atom by a weak electromagnetic field ("single-photon" excitation), spontaneous radiation emission, and Rabi oscillations of the inverse population of atomic energy levels. The zeros e_1, e_2, and e_3 are connected with the integrals h_0 and h_{-1} and the parameter λ_0 by the obvious relations

$$e_1 + e_2 + e_3 = h_0 + \frac{\lambda_0^2}{4},$$

$$e_1e_2 + e_2e_3 + e_1e_3 = \frac{\lambda_0}{2}h_{-1} - \frac{1}{4},$$

$$e_1e_2e_3 = \frac{1}{4}(h_{-1}^2 - h_0). \qquad (21)$$

3.1. Excitation of an atom by a weak electromagnetic field

Let $S_3(t = 0) = -\frac{1}{2}$ at the initial time moment, and $S_3(t) \to \frac{1}{2}$ as $t \to \infty$. Then, we have to put

$$e_3 = -\frac{1}{2}, \qquad e_2 = \frac{1}{2}, \qquad e_1 \geq e_2.$$

It follows from Eq. (21) that

$$e_1 = h_0 + \frac{\lambda_0^2}{4}, \qquad h_0 \geq \frac{1}{2}, \qquad h_0 = -\frac{1}{2} + n,$$

where $n = a^+a^-$ is the square of the electromagnetic field amplitude (or the "number of photons", in quantum-mechanical theory). Let $\lambda_0 = 0$, $n = 1$, and $h_0 = \frac{1}{2}$. In this case, $k = 1$, $e_1 - e_3 = 1$, and

$$S_3(gt) = -\frac{1}{2} + \tanh^2(gt). \qquad (22)$$

The behavior of the function $S_3(gt)$ is illustrated in Fig. 1,*a*. If the two-level atom is considered strictly resonant (with the zero width of the excited level), then, according to the uncertainty relation, an infinitely long time is required for the transition to take place.

3.2. Nonlinear Rabi oscillations

If the excited state $|2\rangle$ is characterized by a finite (irrespective how narrow) width ΔE, the transition time is finite. As was indicated in Introduction, the parameter $\hbar\lambda_0 g$ plays effectively the role of level width. Therefore, at $\lambda_0 \neq 0$ and $n = 1$, we obtain nonlinear oscillations of the inverse population (see Fig. 1,*b*)

$$S_3(gt) = -\frac{1}{2} + \mathrm{sn}^2\left(\sqrt{1+\frac{\lambda_0^2}{4}}gt; k\right) \tag{23}$$

with the period $\tau = 2K/\left(g\sqrt{1+\lambda_0^2/4}\right)$, where $4K$ is the real period of the elliptic function.

3.3. Spontaneous radiation emission

Provided that $S_3(0) = \frac{1}{2}$ and $n = 0$ at the initial time moment, we have $h_0 = S_3(0)$. Since h_0 is the integral of motion, the atom will stay in the state $S_3 = \frac{1}{2}$ for the infinitely long time. But if $\lambda_0 \neq 0$, formula (23) also describes the radiation emission processes; for this purpose, the initial moment of time should be shifted by half a period, i.e. by the value of $t_1 = 2K/\left(g\sqrt{1+\lambda_0^2/4}\right)$. It is equivalent to a motion with the initial condition

$$S_3(0) = \frac{1}{2} - \frac{\lambda_0}{2}.$$

This condition corresponds to a situation where the atom is already excited and there is a field, the square of the amplitude of which is $\langle a^+ a\rangle = \frac{\lambda_0}{2}$ (Fig. 1,*c*).

4. Dipole Moment Dynamics for a Two-level Atom

The operators $\hat{S}_+$ and $\hat{S}_-$ are directly coupled with the transverse components of the atomic dipole moment. To find the explicit formulas that describe the relevant averages $S_+(t)$ and $S_-(t)$, let us introduce a new separation variable

$$\xi(t) = -\frac{S_-(t)}{a^-(t)}. \tag{24}$$

Its direct differentiation, with Eqs. (12) being taken into account, gives rise to

$$\imath\frac{d\xi}{d(gt)} = \xi^2 + \lambda_0\xi + 2S_3, \tag{25}$$

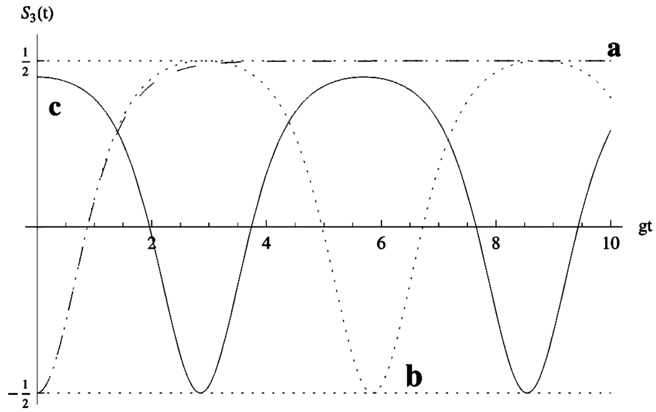

Fig. 1. $S_3(gt)$-dynamics under various initial conditions

where $\lambda_0 = \frac{\Omega-\omega}{g}$. Squaring both sides of Eq. (25) and applying the integrals of motion (13) and constraint (14), we obtain

$$\left[\imath\frac{d\xi}{d(gt)}\right]^2 = \xi^4 + 2\lambda_0\xi^3 + (4h_0 + \lambda_0^2)\xi^2 + 4h_{-1}\xi + 1. \tag{26}$$

It is easy to see that the substitution $\xi = \lambda - \lambda_0$ reduces the polynomial on the right-hand side of Eq. (26) to polynomial (16) which defines the curve $\mathfrak{R}$. Hence, the dynamical variable $\xi(t)$ evolves over the Riemann surface $\mathfrak{R}$.

Equation (26) can be rewritten in the form

$$\imath\frac{d\xi(t)}{d(gt)} = \pm\sqrt{(\xi-\varepsilon_1)(\xi-\varepsilon_2)(\xi-\varepsilon_1^*)(\xi-\varepsilon_2^*)},$$

where $\varepsilon_1 = R_1 e^{\imath\varphi_1}$, $\varepsilon_2 = R_1 e^{\imath\varphi_2}$, and ε_1^* and ε_2^* are the roots of the polynomial on the right-hand side of Eq. (26). In this case, the equalities

$$R_1\cos\varphi_1 + R_2\cos\varphi_2 = -\lambda_0,$$

$$R_1^2 + R_2^2 + 4R_1R_2\cos\varphi_1\cos\varphi_2 = 4h_0 + \lambda_0^2,$$

$$R_1R_2(R_1\cos\varphi_2 + R_2\cos\varphi_1) = -2h_{-1},$$

$$R_1^2R_2^2 = 1 + \lambda_0(h_0 - h_{-1}^2) \tag{27}$$

are fulfilled. Four, in conjugate pairs, roots are located along a circle with the center point C and radius R:

$$C = \frac{R_1^2 - R_2^2}{2(R_1\cos\varphi_1 - R_2\cos\varphi_2)},$$

$$R^2 = R_1^2 + C^2 - 2CR_1\cos\varphi_1 = R_2^2 + C^2 - 2CR_2\cos\varphi_2.$$

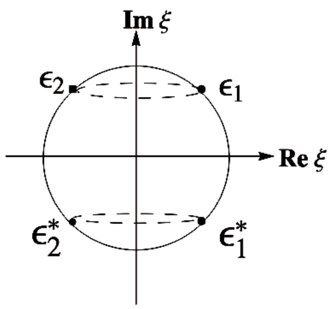

Fig. 2. Riemann sphere for the variable ξ

Using the homographic transformation

$$\xi = \frac{\alpha z + \gamma}{z + \delta},$$

let us map this circle onto the real axis. In so doing, we require that the root transformations $\varepsilon_1 \to -\frac{1}{k}$, $\varepsilon_2 \to -1$, $\varepsilon_2^* \to 1$, and $\varepsilon_1^* \to \frac{1}{k}$ occur. Then

$$\alpha = C + R, \delta = \imath \frac{(C+R) - R_2 \cos\varphi_2}{R - C + R_2 \cos\varphi_2}, \gamma = (C - R)\delta,$$

$$k^2 = \frac{(C + R - R_1 \cos\varphi_1)(C - R - R_2 \cos\varphi_2)}{(C - R - R_1 \cos\varphi_1)(C + R - R_2 \cos\varphi_2)}.$$

Now, we make cuts on the complex plane (the Riemann sphere) of the complex variable z, as is demonstrated in Fig. 2. Sticking two copies of the Riemann sphere together along these cuts, we obtain a Riemann surface of the first kind. It is equivalent to the algebraic curve

$$w^2 = (1 - z^2)(1 - k^2 z^2)$$

which is uniformized by the Jacobi elliptic function and its derivative:

$$z = \mathrm{sn}(u; k), \qquad w = \frac{\mathrm{d}}{\mathrm{du}} \mathrm{sn}(u; k).$$

Then, the solution of Eq. (26) is expressed by the formula

$$\xi(gt) = \frac{(C+R)\mathrm{sn}(u; k) + (C - R)\delta}{\mathrm{sn}(u; k) + \delta}, \tag{28}$$

where $u(gt) = agt + iu_2$,

$$a = \sqrt{(R + C - R_2 \cos\varphi_2)(R - C + R_1 \cos\varphi_1)},$$

and the choice of the parameter u_2 is connected with the choice of initial conditions.

To find the explicit forms of the functions $S_\pm(t)$, let us take advantage of Eqs. (12) directly. We seek the solutions in the form

$$S_\pm = |S_\pm| e^{\pm \imath \varphi_S}.$$

Then

$$\imath \frac{d}{dt} ln S_- = \Omega + 2g\xi^{-1} S_3,$$

so that the equations for the phase and the absolute value of the dipole moment are

$$\frac{d\varphi_S}{dt} = -\Omega + 2S_3 \mathrm{Re}\xi^{-1}, \qquad \frac{d}{dt} ln|S_\pm| = 2gS_3 \mathrm{Im}\xi^{-1}.$$

On the other hand, it follows from definition (24) of the variable ξ that

$$\mathrm{Re}\xi = -\frac{1}{2} \frac{h_{-1}}{h_0 - S_3}, \tag{29}$$

$$\mathrm{Im}\xi = \frac{1}{2g} \frac{1}{h_0 - S_3} \frac{dS_3}{dt}. \tag{30}$$

The absolute values of the functions $S_\pm(t)$ can be easily calculated from the algebraic relation (14)

$$|S_-| = \sqrt{\frac{1}{4} - S_3^2}.$$

Moreover, it can be shown that the phases of the dipole moment and the electromagnetic field are not independent, because they are interconnected by means of one of integrals of motion (13); namely,

$$h_{-1} = 2|a^\pm||S_\pm| \cos(\phi_S - \phi_a). \tag{31}$$

The electromagnetic field phase is expressed in terms of the integral of the real part of ξ (29):

$$\phi_a = \omega t - g \int \mathrm{Re}\ \xi dt, \tag{32}$$

while the corresponding absolute value can be found in terms of h_0 (13):

$$|a^\pm| = \sqrt{h_0 - S_3}.$$

In order to integrate expression (32), let us pass to the Weierstrass function

$$\phi_a = \omega t - \frac{h_{-1}}{2} \int \frac{[\alpha \wp(u) + \beta] du}{\gamma \wp(u) + \delta},$$

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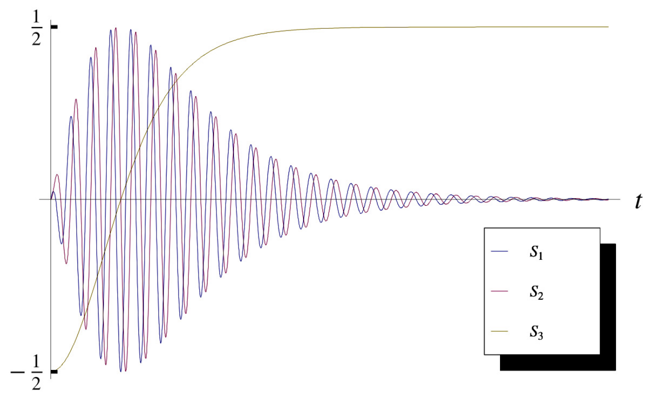

Fig. 3. Dynamics of dipole moment components in the degenerate case ($\lambda_0 = 0$ and $h_1 = 0$)

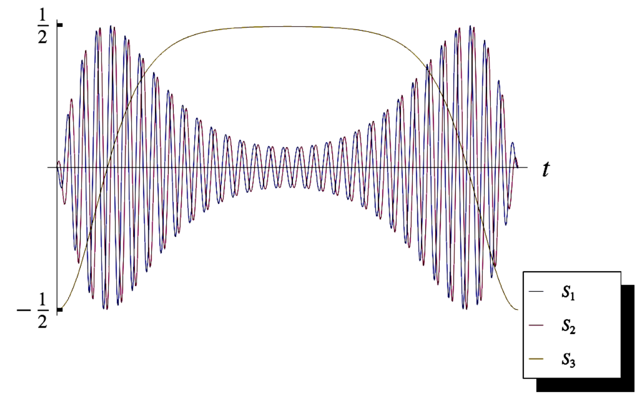

Fig. 4. Dynamics of dipole moment components at $h_1 = 0$ and $\lambda_0 \neq 0$

where $\alpha = 1$, $\beta = -E_3$, $\gamma = e_1$, $\delta = e_3(E_1 - E_3) - e_1 E_3$, and $u = gt$. Ultimately, the expression for the electromagnetic field phase looks like

$$\phi_a = \omega t + h_{-1}\Big(C_1 gt + C_2 \ln \frac{\sigma(gt+v)}{\sigma(gt-v)}\Big), \qquad (33)$$

where $C_1 = \frac{1}{2e_1} - \zeta(v)C$, $C_2 = \frac{C}{2}$, $C = \frac{(E_1-E_3)e_3}{\delta}$, and the parameter v is determined from the condition $\wp'(v) = -\frac{\delta}{\gamma}$. According to expression (31), the dipole moment phase is expressed in terms of the electromagnetic field phase as follows:

$$\phi_S = \arccos \frac{h_{-1}}{2|a^{\pm}||S_{\pm}|} + \phi_a. \qquad (34)$$

Degenerate case. Let the roots be located on the imaginary axis: $\varepsilon_1 = \varepsilon_2 = i$. Then,

$$\xi(gt) = \imath \mathrm{th}(gt),$$

and the inverse population $S_3(gt)$ is described by formula (22). This solution corresponds to the process which was considered before, namely, the atom is excited by a weak electromagnetic field (provided that $\lambda_0 = 0$). In this case, the dynamics of dipole moment components is described by the formulas

$$S_+ = \frac{\mathrm{sh}(gt)}{\mathrm{ch}^2(gt)} e^{\imath\Omega t}, \qquad S_- = \frac{\mathrm{sh}(gt)}{\mathrm{ch}^2(gt))} e^{-\imath\Omega t}.$$

For the components of the electromagnetic field, we have

$$a^+ = -\imath \frac{1}{\mathrm{ch}(gt)} e^{\imath\omega t}, \qquad a^- = \imath \frac{1}{\mathrm{ch}(gt)} e^{-\imath\omega t}.$$

Symmetrically arranged roots. This situation corresponds to the initial condition, when the atom is in the ground state. Let $h_1 = 0$. Then, the following restrictions are imposed on the roots: $R_1 = R_2 = 1$ and $\cos\varphi_1 = -\cos\varphi_2$. Taking into account that the shift of the imaginary part of the phase is $u_2 = -\frac{K'}{2}$, we obtain

$$\xi = \imath \frac{2(1+k)[1+k\ \mathrm{sn}^2(u;k)]\mathrm{sn}(u;k)}{(1+k)^2\ \mathrm{sn}^2(u;k) + [1+k\ \mathrm{sn}^2(u;k) + |\alpha|]^2},$$

where $\alpha = \mathrm{cn}(u;k)\mathrm{dn}(u;k)$, and the inverse population $S_3(gt)$ is described by formula (23). Such an arrangement of the roots corresponds to the process of light absorption by an atom, provided that the frequency detuning differs from zero ($\lambda_0 \neq 0$).

Dipole Moment Phase Incursion. In the system under consideration, there are two characteristic frequencies: the frequency of atomic transitions between the ground and excited states, $\Omega_{\mathrm{Atom}} \sim g$, and the frequency of interaction processes between the atom and the field (i.e. the frequencies of the dipole moment and the electromagnetic field), $\omega_{\mathrm{int}} \approx \omega$. It is easy to get convinced that the processes of level population variation are much slower than the processes of energy exchange between the atomic dipole moment and the electromagnetic field (Figs. 3 and 4):

$$\Omega_{\mathrm{Atom}} \ll \omega_{\mathrm{int}}.$$

We know the time interval needed for an atom to transit from state $|1\rangle$ to state $|2\rangle$ is

$$T_{\mathrm{Atom}} = \frac{2K}{g\sqrt{e_1 - e_3}}.$$

Within this period, the dipole moment executes a lot of cycles, and a certain finite phase incursion emerges,

$$\varphi_S(T_{\mathrm{Atom}}) = \omega T_{\mathrm{Atom}} + h_{-1}(gC_1 T_{\mathrm{Atom}} + 4C_2 v\eta_1),$$

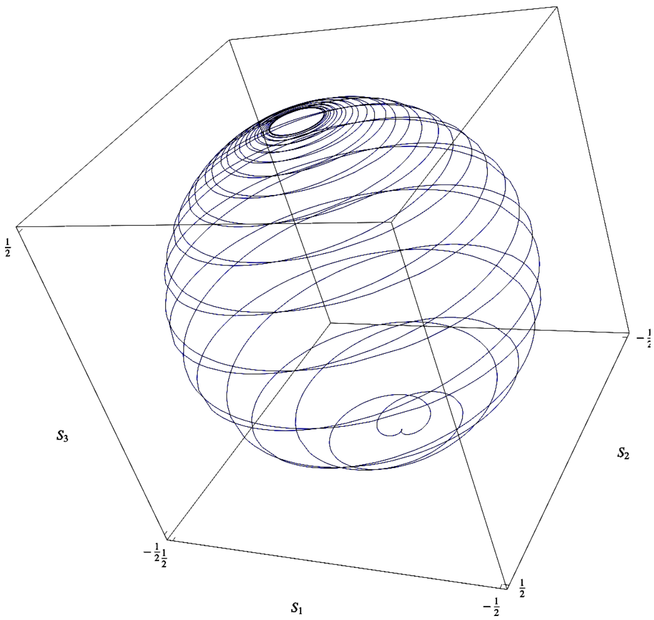

Fig. 5. Dynamics of the dipole moment vector at $h_1 = 0$ and $\lambda_0 \neq 0$

where $\eta_1 = \zeta(T_{\text{Atom}})$. As a result, the atom returns back to the initial state for the time T_{Atom}, whereas its dipole moment does not (Fig. 5). Hence, the interaction between the atom and the field changes the atom itself (in fact, the dipole moment of the atom characterizes—to a certain extent—its spatial distribution) in such a way that a deformation remains even after the interaction has terminated.

It is also important to emphasize that, even in the absence of a direct interaction, i.e. if $h_{-1} = 0$, the finite incursion of the dipole moment phase

$$\varphi_S(T_{\text{Atom}}) = \omega T_{\text{Atom}}$$

will be observed in any case.

5. Dynamics of the Jaynes–Cummings Model in the Quantum-mechanical and Semiclassical Versions. A Comparative Analysis

An important result of the quantum-mechanical Jaynes–Cummings model is the phenomenon of collapse and the revival of atomic inversion [4]. It is associated with a discreteness of photonic excitations and considered as a direct confirmation of the quantum-mechanical nature of radiation emission.

The solutions for the semiclassical version of the Jaynes–Cummings model, which were obtained above, are periodic functions, and they do not include such phenomena. On the other hand, the formulation of the inverse problem is pertinent: Does the inverse level population behave periodically in the quantum-mechanical case? The latter problem invokes the issues on the electromagnetic field states that can provide such a behavior and, more generally, on the correspondence between the quantum-mechanical and semiclassical theories of radiation emission.

The basic formulas of the quantum-mechanical Jaynes–Cummings theory are as follows [2]. Let the vector of the system state have form (11), where we mean that $|\alpha, n\rangle \equiv |\alpha\rangle \otimes |n\rangle$, $|\alpha\rangle$ is the atomic state ($\alpha = 1, 2$), and $|n\rangle$ is the n-photon state of the electromagnetic field (see Eqs. (9) and (10)). The solution of the Schrödinger equation with Hamiltonian (7) looks like

$$c_{1,\ n+1}(t) = c_{1,\ n+1}(0)[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{\imath\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)] -$$

$$-\frac{2\imath g\sqrt{n+1}}{\Omega_n} c_{2,\ n}(0)\sin\left(\frac{\Omega_n t}{2}\right) e^{\imath\frac{-\Delta t}{2}},$$

$$c_{2,\ n}(t) = c_{2,\ n}(0)[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{\imath\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)] -$$

$$-\frac{2\imath g\sqrt{n+1}}{\Omega_n} c_{1,\ n+1}(0)\sin\left(\frac{\Omega_n t}{2}\right) e^{\imath\frac{\Delta t}{2}},$$

where $\Omega_n^2 = \Delta^2 + 4g^2(n+1)$ and $n = 0, 1, 2, \ldots$. Then, for the average value of the operator $\hat{S}_3$, we have

$$\langle\Psi(t)|\hat{S}_3|\Psi(t)\rangle =$$

$$= \sum_n |c_{2,\ n}(0)|^2\Big[\cos^2\left(\frac{\Omega_n t}{2}\right) + \frac{\Delta^2}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right)\Big] +$$

$$+|c_{2,\ n-1}(0)|^2\Big[\frac{4g^2 n}{\Omega_{n-1}^2}\sin^2\left(\frac{\Omega_{n-1} t}{2}\right)\Big]. \quad (35)$$

Let the atom be in the ground state $|1, n\rangle$ at the initial time moment. Then, $c_{2,\ n}(0) = 0$, and, according to formula (35),

$$\tilde{S}_3(t) = -\frac{1}{2}\sum_{n=0}^{\infty} |c_{1,\ n+1}(0)|^2\Big[\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2}\cos(\Omega_n t)\Big]. \quad (36)$$

In the general case, function (36) is quasiperiodic, but we can find such field states that provide its periodicity. One of them can be constructed on the basis of the coherent state

$$|\alpha\rangle = \sum_n c_n |n\rangle,$$

by zeroing those coefficients c_n which do not satisfy the periodicity condition for expression (36)

$$\sqrt{n+1} = l, \quad l \in \mathbb{Z}.$$

Then, the coefficients in expression (36) are described by the distribution function

$$|c_{1,\ n+1}(0)|^2 = \frac{e^{-\langle n\rangle}\langle n\rangle^{n+1}}{(n+1)!}, \quad n = 0,\ 3,\ 8,\ 15,\ \ldots$$

Putting $\Delta = 0$, we obtain

$$\tilde{S}_3(t) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{e^{-\langle n\rangle}\langle n\rangle^{n+1}}{(n+1)!}\cos(2lgt). \tag{37}$$

For the sake of comparison, let us express the semiclassical result for the same initial condition in terms of the Weierstrass function and expand it in a series [9]

$$S_3(t) = -\frac{\eta_1}{\omega_1} + \frac{1}{3}h_0 - \frac{2\pi^2}{\omega_1^2}\sum_{m=1}^{\infty}\frac{mq^m}{1-q^{2m}}\cos\left[\frac{\pi m}{\omega_1}gt\right], \tag{38}$$

where $q = \exp\left(-\imath\pi\frac{\omega_2}{\omega_1}\right)$, ω_1 and ω_2 are the real and imaginary periods of the Weierstrass function, and $\eta_1 = \zeta(\omega_1)$. The equality between the periods of functions (37) and (38) dictates the following restriction on the eligible ω_1-values:

$$\omega_1 = \frac{\pi}{2\alpha},$$

where $\alpha = \frac{l}{m}$ and $\alpha \in \mathbb{Z}$, and, consequently, restrictions on h_0 and $\langle n\rangle$, the average number of photons in the field. In particular, the results practically coincide at $\langle n\rangle = 1$. In Fig. 6, the dashed curve illustrates the semiclassical result $S_3(t)$, and the solid curve characterizes the average of the quantum-mechanical one $\tilde{S}_3(t)$.

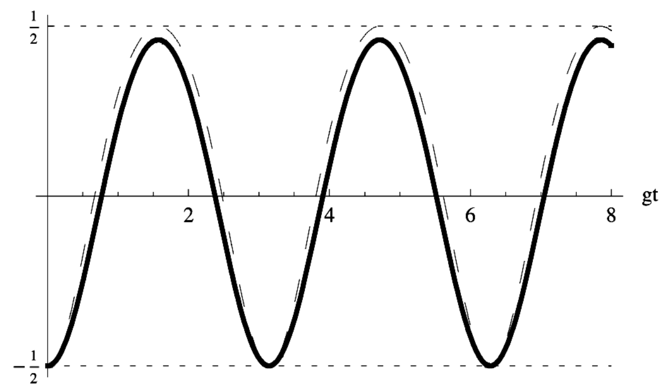

Fig. 6. Quantum-mechanical (solid curve) and semiclassical (dashed curve) dynamics of atomic inversion

6. Conclusions

The semiclassical Jaynes–Cummings model, where the frequency detuning was taken into account, was shown to adequately describe the dynamics of the atomic dipole moment. This vector, moving over the Bloch sphere ($S_1^2 + S_2^2 + S_3^2 = \frac{1}{4}$), precesses about the wave vector **k** with a frequency close to that of the field and nutates with period $T_{\rm Atom}$. The latter is governed by the initial condition and the electromagnetic field energy. Since the system is characterized by two characteristic periods, the transition of an atom from one state into another one and backwards is accompanied by the incursion of the dipole moment phase.

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НЕЛІНІЙНА ДИНАМІКА ДИПОЛЬНОГО МОМЕНТУ ДВОРІВНЕВОГО АТОМА У НАПІВКЛАСИЧНІЙ МОДЕЛІ ДЖЕЙНСА–КАММІНГСА

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Р е з ю м е

Досліджено нелінійну динаміку середніх значень спостережуваних величин в інтегровній моделі Джейнса–Камінгса, яка описує взаємодію дворівневого атома з одномодовим класичним електромагнітним полем. Отримано явні формули еволюції поперечних компонент дипольного моменту та інверсної заселеності рівнів атома. Проведено порівняння між розв'язками напівкласичної та квантової версій моделі Джейнса–Каммінгса.