BOLTZMANN–PICARD–VOLTERRA HEREDITARY DEFINING RELATIONS IN THE ELECTRODYNAMICS OF PHYSICAL SYSTEMS WITH MEMORY

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UDC 539.2

By the example of ferromagnetics with their clearly pronounced nonlinearity and hysteresis, we show the fundamental role of material relations supplementing the Maxwell's equations in the electrodynamics of continua and bodies with memory in the form given by Boltzmann–Picard–Volterra. The analytic formula for S-like loops of hysteresis is constructed within a developed phenomenological model. Analogous models can be developed also for the other physical systems, whose character is similar to that of ferromagnetics; for example, for ferroelectrics. We present a number of motivated arguments concerning the transformation of trial models to "working ones" and discuss the complete system of original evolutionary relations of the electrodynamics of physical systems with memory.

1. On the First Principles of the Electrodynamics of Continua and Bodies

It is known that the fundamental system of Maxwell's equations in the electrodynamics of continua and bodies is incomplete and requires the supplement by material relations which are different for different physical systems [1–4]. As distinct from the exactly established postulates such as Maxwell's equations, the material relations have usually a model character in view of that of the very conception of continuity in physics. Therefore, the searches for adequate material relations in the electrodynamics of continua cannot be removed from the agenda due to a significant growing variety of actual physical systems and processes.

Unlike the Nature's laws, a model can hardly describe all the nuances of phenomena with the proper correspondence with experiment. A successful model must correctly represent the defining factors of processes and their peculiarities [5, 6].

In the general form, the material relations of the electrodynamics of continua can be written as follows [7, Chap. 29, Sect. 3]:

$$\vec{B}(t) = \mu_0 \left[\vec{H}(t) + \vec{J}(t)\right], \tag{1a}$$

$$\vec{D}(t) = \varepsilon_0 \left[\vec{E}(t) + \vec{P}(t)\right]. \tag{1b}$$

Here, $\vec{B}(t), \vec{D}(t)$ – vectors of magnetic and electric induction; $\vec{H}(t), \vec{E}(t)$ – vectors of strength of magnetic and electric fields; $\vec{J}(t), \vec{P}(t)$ – vectors of magnetization and polarization of the medium, respectively; μ_0, ε_0 – magnetic and electric constants (we use the international system of units).

There exist the other material relations, whose discussion is omitted here.

For magnetic materials and materials polarized by an electric field, the essential role is played, respectively, by the vector $\vec{J}(t)$ (1a) and vector $\vec{P}(t)$ (1b). These vectors are objects for the adequate mathematical modeling. It is known from experiments that $\vec{J}(t)$ and $\vec{P}(t)$ are functionally connected with the fields $\vec{H}(t)$ and $\vec{E}(t)$, respectively. In order to study such connections, the materials are positioned in external fields. Therefore, we will consider, in what follows, $\vec{H}(t)$ and $\vec{E}(t)$ in (1a) and (1b) to be given external fields.

In this case, the functional dependence of $\vec{J}(t)$ on $\vec{H}(t)$ in ferromagnetics is of great interest in view of its nonlinearity and hysteresis which will be discussed. In authors' opinion, just the similar problems should be based on the Boltzmann–Picard–Volterra hereditary defining relations (BPV HDRs) [6], the interest in which begins to be gradually restored [8, 9]. They convincingly clarify the defining macroscopic factors of phenomena.

2. On the Boltzmann–Picard–Volterra[1] hereditary defining relations

Introduced in science as early as in 1875 by L. Boltzmann (and independently by J. Maxwell in the other form), the hereditary defining relations were

[1] They are called as the relations with memory, aftereffect, and delay. All denominations reflect the essence of BPV HDRs at different angles of vision.

a novel form of cause-effect connections in physics which described the development of processes in the course of time and were distinct from the form of Cauchy differential evolutionary boundary-value problems (CEBPs) dominating at that time and now in theoretical and mathematical physics (TMP).

But it was historically built up so that HDRs were firmly fixed only in the Boltzmann's creep (viscoelasticity) theory, rather than in the other fields of the theory of continua, though such leading figures of science as E. Picard and V. Volterra, as well as the others (see [6, 10–13]), were engaged with their active development and popularization.

It is sufficient to describe the essence and the form of HDRs by an example of subsystems with a single lumped parameter of state $y(t)$, because their generalization on multiparametric systems (including those with distributed parameters) gives no basic difficulties [6].

In the general form, the BPV HDR for systems with a single lumped parameter of state can be written as [12]

$$y(t) = f_1(x(t)) + \int_0^t K(x|t,\tau) f_2(x(\tau)) d\tau. \qquad (2)$$

Here, f_1 $(x(\mathrm{t}))$ – the component of a reaction of the system to the action of an external excitation of the process $x(t)$, both being synchronous; the integral component in (2) reflects the retarded (hereditary) reaction of the system to the external action; the response kernel $K(x|t,\tau)$ describes phenomenologically the internal inertial properties of the system, by integrally defining, together with $f_2(x(\tau))$, its evolution for the whole history of the process $(0 \le \tau \le t)$. The experience of the dealing with HDR shows that, as usual, $f_1(x(t)) = f_2(x(t)) \equiv f(x(t))$ [6]. Just this variant is realized in the below-developed phenomenological model of ferromagnetic hysteresis.

In contrast to HDRs of type (1), the solutions of CEBPs do not contain any components of a reaction of the system which would be synchronous with the external action, by reflecting always the influence of the internal and external factors on the evolution of the system for the whole history of processes[2]. Hence, one cannot omit HDRs or their equivalents in those problems of the theory of continua and bodies, where reactions of systems which are synchronous with the external action are of great significance. That is, it is necessary to introduce HDRs into tools of TMP. The retarded (integral) component in (2) responsible, in particular, for the hysteresis plays no less important role. We note that a similar component is also present in solutions of CEBPs [6]. The difficulty to work with HDRs consists only in that all functions on the right-hand side of (2) (f_1, f_2, K) should be modeled beforehand on the basis of some physical and formal consideration. At first glance, this makes HDRs to be insufficiently constructive. But this is a too superficial look. In the vast majority of real physico-technical problems solved with the help of CEBPs, it is necessary firstly to perform a simplifying (as usual, it is made intuitively and radically) modeling of the basic initial equations. Otherwise, the problem could be solved neither qualitatively not quantitatively. Hence, HDRs contain really not more difficulties than CEBPs. In both cases, the professional experience, trained intuition, and selection and analysis of version are of primary importance.

In the opinion of Picard, Volterra, and their followers, the initial first principles of the theory of continua and bodies must combine CEBP and HDR, by leading eventually to integro-differential boundary-value problems [6, 10–13]. We not only share this viewpoint but have supplemented HDR by the consideration of nonlocal effects [6], by constructing the nonlocally hereditary defining relations (NHDRs). In this case, the problems of modeling become more complicated. But we have shown [6] that the difficulties can be overcome and have demonstrated that NHDRs correspond to macroscopically spatially inhomogeneous material objects.

3. A Proposed Model of BPV HDRs for Ferromagnetics

First, we consider the simplest, but realistic version: a macroscopically homogeneous and isotropic ferromagnetic. Then the vector relation (1a) can be replaced by a scalar one

$$B(t) = \mu_0[H(t) + J(t)], \qquad (3)$$

which will be studied below.

Due to the clearly manifested heredity in ferromagnetics, we may set, in the frame of the BPV approach:

$$J(t) = \hat{V} J_0(t) \equiv J_0(t) + \int_0^t K(H(t)|t,\tau) J_0(\tau) d\tau, \qquad (3')$$

[2]See the literature on mathematical physics (e.g., [7] and also [6]).

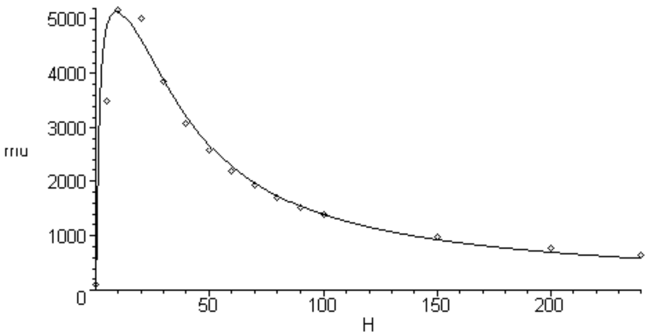

Fig. 1. Experimental points and curve (8) approximating the experimental dependence of the magnetic permeability $\mu(H)$ ($\mu_{\max}$= 5170, H [A/m])

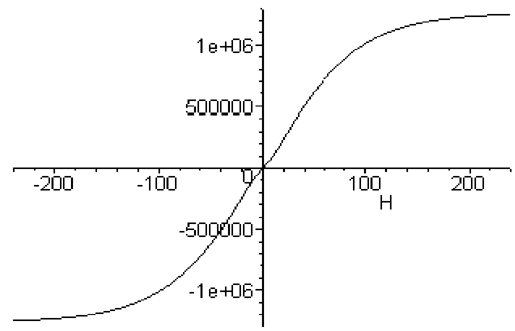

Fig. 2. Basic curve of magnetization $J_0(H)$ corresponding to the magnetic permeability shown in Fig. 1 (J, H [A/m])

where $\hat{V}$ – Volterra operator, $J_0(t) = J_0(H(t))$ – basic curve of magnetization; and $K(H(t)|t,\tau)$ – Volterra operator kernel.

We are faced with the task to adequately model both the basic curve of magnetization and the Volterra operator kernel ($3'$).

At least for a number of typical ferromagnetics, including iron (> 95% Fe [14]), the function $J_0(H)$ can be sufficiently well modeled by a superposition of hyperbolic tangents,

$$J_0(H) = A\,\mathrm{th}(aH) - B\,\mathrm{th}(bH), \tag{4}$$

where A, B, a, and b – fixed parameters of the approximation which require no physical interpretation, because the basic curve of magnetization $J_0(H)$ is approximated in phenomenology by experimental data.

As will be shown, the kernel $K(H(t)|t,\tau)$ of Volterra operator ($3'$) yields also to a quite certain modeling. But we consider first model (4).

We will determine the parameters of model (4) with regard for both the above-mentioned typical ferromagnetic [14] and the relation traditionally used in the theory of magnetism

$$J_0 = (\mu - 1)H, \tag{5}$$

where $\mu(H)$ – magnetic permeability approximated in model (4) and (5) by the relation

$$\mu(H) = 1 + A\frac{\mathrm{th}(aH)}{H} - B\frac{\mathrm{th}(bH)}{H}. \tag{6}$$

In Figs. 1 and 2, we present the experimental points, curve (6) approximating the experimental behavior of the magnetic permeability $\mu(H)$, and the corresponding basic curve of magnetization $J_0(H)$.

From four parameters of approximation (4), only three ones are independent, because the curve in Fig. 1 has the ascending and descending branches and the maximum point:

$$\frac{d\mu}{dH}\begin{cases} > 0 \text{ at } H < H_{\max}; & \text{(4a)} \\ = 0 \text{ at } H = H_{\max}; & \text{(4b)} \\ < 0 \text{ at } H > H_{\max}. & \text{(4c)} \end{cases}$$

Condition (4b) sets a functional connection between the parameters, whereas conditions (4a) and (4c) give a correlative connection between them.

Hereditary properties of the medium are determined by the kernel $K(H(t)|t,\tau)$ of the integral Volterra operator ($3'$). These properties must obviously reflect

a) the dependence of the kernel (in the general case) on an external magnetic field $H(t)$, since the latter magnetizes a substance and changes its properties, the kernel being the internal determinative of the process [4];

b) the kernel must be a monotonically nullifying function of $H(t)$,

$$\lim_{H\to\infty} K(H(t)|t,\tau) = 0, \tag{7}$$

because the nonmonotone behavior can be motivated by nothing, and the growth contradicts the saturation;

c) the kernel must be even and always bounded functional of the field $H(t)$ for its arbitrary dependence on the time.

Based on this reasoning and on the analysis of versions (see, for example, a version in [6]), we constructed a kernel of the following form:

$$K(H(t)|t,\tau) = \frac{\eta_0 \exp(-\alpha(t-\tau))}{1 + 2\beta\left|H(t)\frac{dH(t)}{dt}\right|}. \tag{8}$$

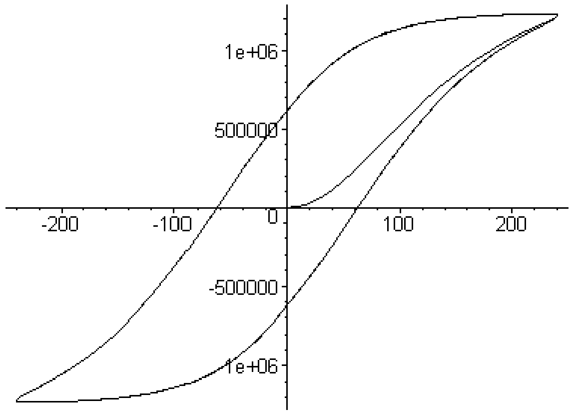

Fig. 3. Dynamical hysteresis loop in a typical ferromagnetic in the coordinates $J-H$ [A/m]

Kernel (8) is sufficiently simple, separable, and even in $H(t)$ and $\frac{dH}{dt}$ and satisfies the criterion of decaying memory (or the criterion of closed cycle [11, 12]);

d) the dependence of kernel (8) on the derivative $\frac{dH(t)}{dt}$ in the case of a periodic field,

$$H(t) = H_m \sin \omega t, \tag{9}$$

determines its explicit dependence on the frequency and the amplitude.

As a result of the substitution of (4) and (8) in (3′), we get the integral defining BPV relation for the magnetization which reproduces quite well a dynamical loop of the hysteresis in a typical ferromagnetic in the coordinates $J-H$[A/m] (Fig. 3).[3]

In this case, the parameters of model (3′), (6) determine the residual magnetization, the coercive force, and the loop area which is proportional, as known, to losses on the remagnetization, including the losses on Foucault currents and magnetic viscosity. Obviously, this point gives the possibility for the subsequent development and correction of the given approach.

It is worth noting that the separability of kernel (8),

$$K(...) = \varphi(H(t)|t)\psi(\tau), \tag{10}$$

where

$$\begin{cases} \varphi(H(t)|t) \equiv \dfrac{\exp(-\alpha t)}{1+2\beta\left|H(t)\frac{dH(t)}{dt}\right|}; \\ \psi(\tau) = \eta_0 \exp(\alpha\tau); \end{cases} \tag{11}$$

allows us to replace the BPV relation (2) by an equivalent evolutionary Cauchy boundary-value problem of the form [6]

$$\begin{cases} \left[\dfrac{d}{dt} + L(t)\right] z(t) = \eta(t)x(t); \\ z(0) = 0, \end{cases} \tag{12}$$

where

$$z(t) \equiv J(t) - J_0(t); \quad x(t) \equiv J_0(t);$$

$$L(t) \equiv -\frac{d}{dt}\left\{\ln \varphi(H|t)\right\}; \tag{12a}$$

$$\eta(t) \equiv K(H(t)|t,t) = \frac{\eta_0}{1+2\beta\left|H(t)\frac{dH(t)}{dt}\right|}. \tag{12b}$$

All this is verified by direct substitutions.

While analyzing problem (12), it becomes clear that to guess its existence is hardly possible without a preliminary construction of the very transparent phenomenological model (2), (4), (5), (8). This indicates once more the important role of hereditary defining BPV relations in theoretical and mathematical physics.

The above-considered problem shows clearly one more important peculiarity of BPV relations, namely, the possibility to describe, with their help, the essentially nonlinear problems on the basis of formally linear relations of type (2) or equivalent boundary-value problems of type (12) which are linear in form.

By referring the model developed in this section to iron (>95 % Fe [14]) for its computer-based approbation, we obtain a reassuring result (Fig. 3). But, in order to transform the trial model to a “working one,” we need an adequate physical construction of its parameters. This task requires the significant efforts hardly realizable by individuals. It is necessary to possess a significant personnel and technical facilities almost absent in higher schools at the present time, though skilled persons and the desire to work do not else disappear. Therefore, in what follows, we present the starting heuristic consideration of the subsequent ways to study the subject with the purpose to attract the attention of experts to it. We will give some draft of the program which should be corrected in the further studies. But even the trial phenomenological models properly reflecting the character of processes are of a certain interest, by revealing the macroscopic defining factors of physical phenomena.

[3]Formally, it corresponds to the elimination of the time from relations (3′) and (9). The coordinates (H, J) are an analog of the phase coordinates (x, p) in mechanics [15], whose role is limited. The basis of a comprehensive description of the physical systems is given by evolutionary (temporal) relations.

4. On the Determination of Parameters of the Developed Model

This problem can be divided into two stages: 1) the determination of parameters $A,\ a;\ B,\ b$ (4) of the basic curve of magnetization $J_0(H)$ and 2) the determination of parameters of the kernel $K(H|t,\tau)$: α, β, η_0 (10).

The realization of the model developed in Section 3 in the electrodynamics of continua and bodies with memory will give a practical possibility to represent all important physical quantities as functions of the time. In particular, this concerns both quasiphase coordinates (H,J). Our description is more general than that in terms of the quasiphase coordinates, because it widens, in particular, the possibilities for a comprehensive analysis of model (2), (3), (5), (8). The model can be tested in various modes of variation of an external magnetic field with the purpose to compare the calculated results with experimental data.

We propose three following modes as basic ones:
1) Sinusoidal mode (9),

$$H(t) = H_m \sin\omega t,$$

used in the previous section;

$$2)\quad H(t) = H_0 = \text{const}; \tag{13}$$

$$3)\quad \begin{cases} \lim\limits_{t\to\infty} H(t) = \infty; \\ \lim\limits_{t\to\infty} \frac{dH}{dt} = 0; \end{cases} \tag{14}$$

and relations (14) hold at a sufficiently small positive derivative during the whole process of magnetization. Case 2 corresponds to the sharp inclusion of a constant external field. On the contrary, case 3 corresponds to a sufficiently smoothly unboundedly increasing external magnetic field. The consistency of conditions (14) is verified by the following academic, but a quite rightful example:

$$H(t) = H_0 \ln(1+\gamma t), \tag{15}$$

where γ – sufficiently small value. According to (15), we have

$$\frac{dH}{dt} = \frac{H_0\gamma}{1+\gamma t} \underset{t\to\infty}{\to} 0. \tag{16}$$

By selecting γ, we can make derivative (16) to be a small value at any t. Therefore, conditions (14) can be realized in practice for various versions.

It is easy to verify that version (13) admits a simple analytic calculation in model (8). Indeed, according to (2) and (8) with regard for the condition $J_0(H_0) = \text{const}$, we get

$$J(t) = J_0(H_0)\left\{1 + \frac{\eta_0}{\alpha}\left[1 - e^{-\alpha t}\right]\right\}. \tag{17}$$

That is, $J(t)$ approaches asymptotically (as $t\to\infty$) a constant

$$J_\infty = \lim_{t\to\infty} J(t) = J_0(H_0)\left\{1 + \frac{\eta_0}{\alpha}\right\}. \tag{18}$$

An asymptotics of type (18) can be judged not only on the basis of the condition $J_0(H_0) = \text{const}$, but also by virtue of the following general upper bound for model (3′), (4), (8):

$$|J(t)| \le |J_0(H(t))| + e^{-\alpha t}\frac{\eta_0}{\alpha}\int\limits_0^{\alpha t} e^{\theta}\left|J_0\left(H\left(\frac{\theta}{\alpha}\right)\right)\right| d\theta, \tag{19}$$

because $1/(1+2\beta\,|...|) \le 1$.

Then, by taking the inequality $|\mathrm{th}\, x| \le 1$ and formula (4) into account, we get

$$|J_0(H(t))| \le |A| + |B|\,. \tag{20}$$

By virtue of (3′), (4), (8), the last inequality yields

$$|J(t)| \le \{|A| + |B|\} \times$$

$$\times\left\{1 + \frac{\eta_0}{\alpha}(1 - e^{-\alpha t})\right\} \underset{t\to\infty}{\overset{acc}{\to}} \{|A|+|B|\}\left\{1 + \frac{\eta_0}{\alpha}\right\}. \tag{21}$$

Hence, the less the ratio η_0/α, the less significant is the role of the retarded component in (3′) in the model under consideration. Moreover, the parameter α plays the role of the inverse time of relaxation ($\alpha = 1/\tau_0$). The parameter η_0 can be called the coefficient of delay.

Thus, the physically important parameters of the model can be interpreted in the following way: α – inverse time of relaxation, η_0 – coefficient of delay, β – coefficient of internal nonlinearity, because the kernel $K\,(H(t)|t,\tau)$ – internal determinative of the process.

In order to improve the model of the magnetization, it is of interest to perform the qualitative analysis and to compare the theoretical relation $J(t)$ in (3) with experiment for the three modes (9), (13), and (14).

For a sinusoidal external field, the dependence $J(t)$ corresponding to a hysteresis loop is presented in Fig. 4. The corresponding results for the second and third cases, (13) and (14), are shown, respectively, in Figs. 5 and 6.

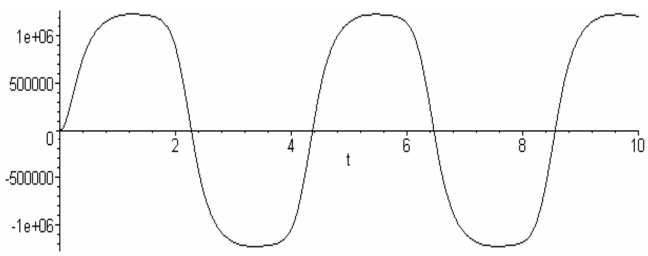

Fig. 4. Dependence $J(t)$ for a sinusoidal external field corresponding to the hysteresis loop shown in Fig. 3

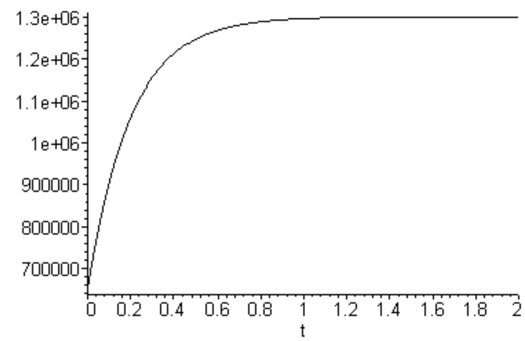

Fig. 5. Dependence $J(t)$ for an external field set by (13)

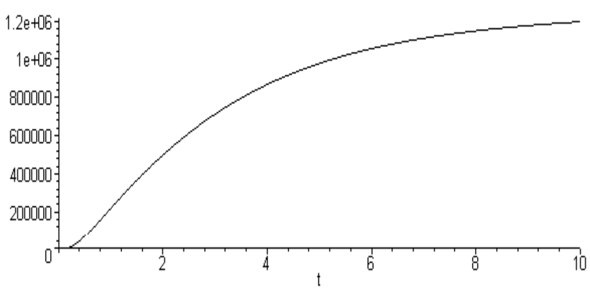

Fig. 6. Dependence $J(t)$ for an external field set by (14)

As seen, even the problem of the practical determination of independent parameters of the model is not simple and requires to carry out special experimental studies. We should like to attract the attention of experts to this point.

For the complete solution of the problem, we may propose two alternative ways to continue the study.

A) In the ideal case, the independent parameters of the model should be constructed with the help of the laws of thermodynamics, statistical physics, and, possibly, other known physical regularities. But the history of science teaches that it is not, first, a simple task. Second, its solution is related to an additional and usually illegibly controlled modeling, which is able to significantly hide the possible disagreements of theory and experiment. That is, it will remain unclear what should be corrected: the initial BPV model or the additional modeling of its parameters; or, possibly, both. We believe that the situation is very ambiguous.

B) The second way was successfully tested long ago in science and engineering practice. This is the way of a tabulation (discretization) of characteristic and independent physical quantities such, for example, as the specific heat capacity and the latent heat of phase transformations of various materials (in various temperature intervals); Young moduli and other characteristic parameters of deformable solids; specific capacities and inductances of power lines, and a lot of other physical quantities.

It is a very labor-consuming way. But it is more reliable, because it is not related to the risky additional modeling. Since the number of most usable materials and the scope of conditions of their operation are limited, such an approach seems quite realistic. After the determination of independent parameters within the BPV model developed in Section 3, it will be necessary to know the characteristic, subjected to the tabulation, physical quantities.

In conclusion, the authors thank Prof. S.M. Ryabchenko and Prof. E.D. Belokolos for the benevolent and constructive discussion of the present work, which led to a change of its form and structure. But we themselves are completely responsible for the possibly risky reasoning.

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Received 14.01.09.
Translated from Russian by V.V. Kukhtin

СПАДКОВІ ВИЗНАЧАЛЬНІ СПІВВІДНОШЕННЯ БОЛЬЦМАНА–ПІКАРА–ВОЛЬТЕРРИ В ЕЛЕКТРОДИНАМІЦІ ФІЗИЧНИХ СИСТЕМ З ПАМ'ЯТТЮ

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Р е з ю м е

На прикладі феромагнетиків з їх яскраво вираженими нелінійністю та гістерезисом показана першочергова роль матеріальних співвідношень, які доповнюють рівняння Максвелла в електродинаміці суцільних середовищ та тіл з пам'яттю у формі Больцмана–Пікара–Вольтерри, що аналітично визначають S-подібну петлю гістерезису. Із цією метою розроблено та апробовано відповідну фономенологічну модель. Аналогічні моделі можуть бути побудовані і для інших фізичних систем, формально подібного з феромагнетиками характеру, наприклад, для сегнетоелектриків. Висунуто низку мотивованих міркувань щодо перетворення пробних моделей в "робочі", а також відносно повної системи вихідних еволюційних співвідношень електродинаміки фізичних систем з пам'яттю.