NONRELATIVISTIC CASE OF GRAVITATIONAL LENSING BY SIMPLE COSMIC STRING LOOPS

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Cosmic strings are topologically stable, one-dimensional defects in vacuum which can appear during appropriate phase transitions in an adiabatically expanding early Universe which cools down from a very hot initial state. Their discovery would lead to advances in cosmology and fundamental physics. One of the most efficient ways to detect cosmic strings is related to their gravitational lensing signatures which appear to be different from those of standard lenses. We study a simple model of gravitational lensing by symmetric and asymmetric loops. An explicit form of the lens equation is obtained, and the relations for magnification are derived. We also discuss possible observational manifestations of cosmic strings within our model.

1. Introduction

The goal of the present work is to carry out a theoretical analysis of such a phenomenon as the topological defects – the cosmic strings. They are objects which could be formed as a result of the phase transitions of fields in the early Universe. The equations of the grand unification theory (GUT) with spontaneously broken symmetry have solutions in the form of topological defects, including strings [1]. The discovery and study of properties of these objects can answer such practically important questions as the origin of high-energy cosmic particles and the large-scale structure of the Universe [2–5] (there exist the theories, by which the cosmic strings in the early Universe formed great inhomogeneities of matter, by moving through it – galaxies and clusters of galaxies). Therefore, their search is an urgent problem requiring to be solved.

By the present day, several observations of objects which are possibly lensed or by a cosmic string or the loop of a string are known. But none of these facts was confirmed with absolute accuracy. One of the most known cases with rather strange behavior of a lens system is the Q0957+561 system in 1994–1996. For a quite long time period, the delay in time between fluctuations of two images was measured, and, at just the same period, the synchronous fluctuations which can be induced only by the passage of some object quite close to the observer were registered [6].

In this work, we analyze the properties of cosmic strings, evaluated their approximate number in the vicinity of our Galaxy, and determined analytically the parameters of the gravitational lens formed by a cosmic string.

2. Calculation of the Number of Topological Defects

Depending on the creation time, the cosmic strings are characterized by a dimensionless parameter $\alpha = \left(\frac{\Gamma G\mu}{c^2}\right)^n$ [7]. In this formula, $\Gamma = 50$ – dimensionless parameter which determines the ratio of the emitted (due to gravitational waves) energy to the mass (with some additional factors), G – gravitational constant, μ – linear density of a string, c – light velocity, n – dimensionless parameter which defines the spectrum of fluctuations of a string. In this case, n=3/2 (the spectrum is inversely proportional to the oscillation mode of a string) [8]. The length of a string is given by the relation

$$l = \alpha ct, \tag{1}$$

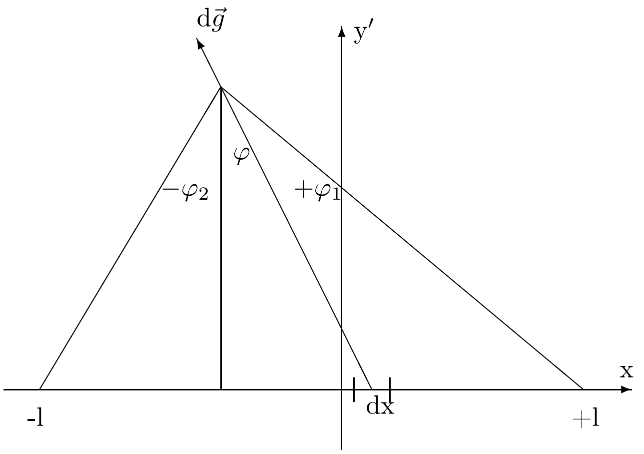

Fig. 1. Gravitational field of a string. All points lie in the plane of the figure

where t – lifetime of the Universe. Since we consider the near vicinity of our Galaxy, we take $t = 13.6 \times 10^9$ yr. The mass of a string

$$M = \mu l = \frac{\alpha^{2/3} c^2}{\Gamma G} \alpha ct = \frac{\alpha^{5/3} c^3 t}{\Gamma G}. \quad (2)$$

The distribution function of the concentration of strings over α has the form

$$n = \frac{p^{-1}}{\alpha (ct)^3}, \quad (3)$$

where p – probability of the separation of a loop on the self-intersection which does not depend on the parameters of a loop. We set $p = 10^{-3}$ [8, 9]. Consider 3 types of cosmic strings with $\alpha = \beta^{3/2}$ [7, 10]:

1. They were created in 10^{-35} s after the Big Bang on the decay of the unified interaction (by GUT), i.e. on the separation of the strong interaction, and are characterized by $\beta = 10^{-6}$.

2. They were created in 10^{-11} s on the decay of the electroweak interaction and are characterized by $\beta = 10^{-8}$.

3. They were created in 10^{-6} s on the creation of protons and neutrons from quarks and are characterized by $\beta = 10^{-11}$.

The quantity β depends on the density of a string by the relation $\beta = \frac{\Gamma G \mu}{c^2}$. The number of topological defects

$$N = V \frac{p^{-1}}{\alpha (ct)^3}. \quad (4)$$

We can determine the mean distance, at which such a string can be met:

$$d_S = n^{-1/3} = \left(\frac{p^{-1}}{\alpha (ct)^3} \right)^{-1/3} = \alpha^{1/3} p^{1/3} ct. \quad (5)$$

The mean angle, at which a terrestrial observer sees the string,

$$\theta = \frac{l}{d_s} = \frac{\alpha ct}{\alpha^{1/3} p^{1/3} ct} = \alpha^{2/3} p^{-1/3}. \quad (6)$$

The angle depends only on the type of a string (see Tables 1, 2).

As seen, only small topological defects of the third type must be met in the Galaxy, whereas the Andromeda Nebula can contain else several more massive defects of the second type. A part of these defects can be observed, because they are positioned against the background of stars or the Galaxy. The large strings which are positioned against the background of the Andromeda Nebula have angular sizes of the order of 0.02", whereas the Galaxy itself has a size of 3°. That is, their observation is possible, but it is rather difficult.

3. Lensing on a Loop of a Cosmic String

One of the most important methods of observation of cosmic strings is the search for the gravitational lensing on them. Consider two limiting versions of the form of a loop – symmetric and asymmetric.

The most asymmetric configuration of a string is a straight line. While approaching each other, such two strings can annihilate. However, we will use this configuration only as the limiting case.

All angles in Fig. 1 are measured to the right from the vertical. The y' axis is chosen in such a way that the point, in which the field is sought, be in the $x0y'$ plane. The strength vector is directed in the opposite side. The field components along the x and y' axes which are created by an element of the length dx are

$$dg = \frac{G \mu dx}{\left(\frac{y'}{\cos \varphi} \right)^2}, \quad (7)$$

T a b l e 1. Characteristics of various types of strings

Types of strings	Parameters of strings			
	Length, ps	Mass, in Sun's mass	Distance, ps	Visible angular size
1	4.3	1.75×10^6	4.29×10^5	2.1"
2	4.3×10^{-3}	17.5	4.29×10^4	0.02"
3	1.36×10^{-7}	5.52×10^{-7}	1.36×10^3	0.000021"

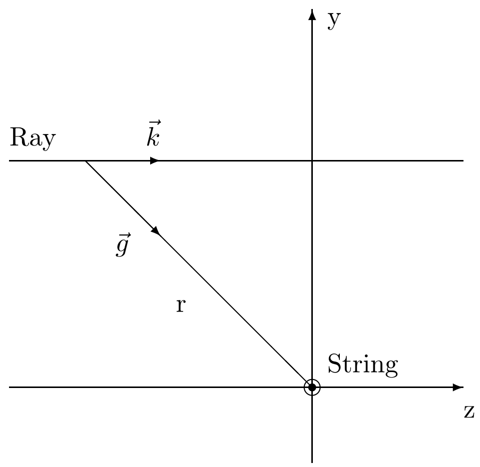

Fig. 2. Deviation of a ray by the field of a string. The projection on the $x0z$ plane; the string is directed along the x axis

where μ – linear density of the string, and $2l$ – length of the string. The increment of the coordinate as a function of the increment of the angle looks as

$$dx = \frac{y' d\varphi}{\cos^2 \varphi}. \tag{8}$$

This yields

$$dg_x = \frac{\mu G}{y'} \sin\varphi d\varphi, \tag{9}$$

$$dg_{y'} = -\frac{\mu G}{y'} \cos\varphi d\varphi. \tag{10}$$

By a simple integration, we get

$$g_x = \int\limits_{-\varphi_2}^{+\varphi_1} \frac{\mu G}{y'} \sin\varphi d\varphi = \frac{-\mu G}{y'}(\cos\varphi_2 - \cos\varphi_1), \tag{11}$$

$$g_{y'} = \int\limits_{-\varphi_2}^{+\varphi_1} -\frac{\mu G}{y'} \cos\varphi d\varphi = \frac{-\mu G}{y'}(\sin\varphi_1 + \sin\varphi_2). \tag{12}$$

T a b l e 2. Number of strings depending on their form

Types of strings	Numbers		
	Milky Way	M31 – Andromeda Nebula	Close vicinity of our Galaxy
1	0	0	53
2	0	4	53100
3	192	137000	1.68 bln

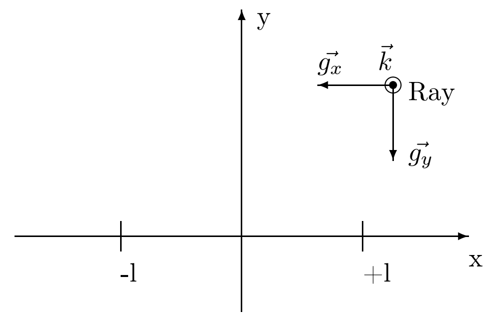

Fig. 3. Deviation of a ray by the field of a string. The projection on the $x0y$ plane, the string is directed along the y axis

By changing the variables, we obtain the formula for the gravitational field of a string:

$$g_x = \mu G \left(\frac{1}{\sqrt{(l+x)^2 + y'^2}} - \frac{1}{\sqrt{(l-x)^2 + y'^2}} \right), \tag{13}$$

$$g_{y'} = -\mu G \left(\frac{l+x}{y'\sqrt{(l+x)^2 + y'^2}} + \frac{l-x}{y'\sqrt{(l-x)^2 + y'^2}} \right). \tag{14}$$

Let us pass from the coordinates x, y' to x, y, z, and let r be the distance to the string. It is obvious that $r{=}y'$, $r = \sqrt{z^2 + y^2}$, $g_y = g_{y'} \frac{y}{r}$,

$$g_x = \mu G \left(\frac{1}{\sqrt{(l+x)^2 + r^2}} - \frac{1}{\sqrt{(l-x)^2 + r^2}} \right), \tag{15}$$

$$g_y = -\mu G \left(\frac{l+x}{r\sqrt{(l+x)^2 + r^2}} + \frac{l-x}{r\sqrt{(l-x)^2 + r^2}} \right) \frac{y}{r}. \tag{16}$$

It is convenient to rewrite the last formula as

$$g_y = -\mu G \left(\frac{y(l+x)}{r^2\sqrt{(l+x)^2 + r^2}} + \frac{y(l-x)}{r^2\sqrt{(l-x)^2 + r^2}} \right). \tag{17}$$

The deviation angle

$$\vec{\alpha} = \frac{2}{c^2} \int\limits_{-\infty}^{+\infty} \vec{g} dz. \tag{18}$$

These expressions can be simply integrated, by assuming the deviation in the direction to the center of the string to be positive. We get

$$\alpha_x = \frac{4G\mu}{c^2} \left(\arctan\frac{l+x}{y} + \arctan\frac{l-x}{y} \right), \tag{19}$$

$$\alpha_y = \frac{2G\mu}{c^2} \ln \frac{(l+x)^2 + y^2}{(l-x)^2 + y^2}. \tag{20}$$

The limiting cases are the point-like mass (a) and an infinite string (b):
a)

$$\alpha_x = \frac{4G(2l\mu)}{c^2} \frac{x}{x^2 + y^2}, \tag{21}$$

$$\alpha_y = \frac{4G(2l\mu)}{c^2} \frac{y}{x^2 + y^2}; \tag{22}$$

thus, we obtain the well-known formula for a deviation of rays in the gravitational field written in components;
b)

$$\alpha_x = 0, \tag{23}$$

$$\alpha_y = \frac{4GM}{c^2}. \tag{24}$$

The last formula is of significant importance. In fact, it means that, in the gravitational field of an infinite string, all rays are deviated by the same angle which is independent of the impact parameter. This implies that such a string will create two absolutely identical images of an object.

We now calculate the magnification on a string. The lens formula reads [11]

$$\vec{R} = \frac{a+b}{a}\vec{r} - b\vec{\alpha}(\vec{r}), \tag{25}$$

where a is the distance between the object and the lens, b is the distance between the observer and the lens. We now give the relation between the areas of the cross-sections of a beam of rays which passes through the lens ($d\Pi_G$ – area of the cross-section of a beam in the plane of the lens, $d\Pi_0$ – area of the cross-section of a beam in the plane of the observer):

$$d\Pi_0 = \det\left|\frac{\partial R_i}{\partial r_j}\right| d\Pi_G. \tag{26}$$

The ratio of the intensities is the inverse ratio of the illuminated areas (the light flow is the same). Therefore,

$$I = \left(\det\left|\frac{\partial R_i}{\partial r_j}\right|\right)_{M=0} \left(\det\left|\frac{\partial R_i}{\partial r_j}\right|\right)^{-1}_M. \tag{27}$$

By using the lens formula and by introducing a new notation

$$D = \frac{ab}{a+b}, \tag{28}$$

we get

$$I = \left(\det\left|\delta^i_j - D\frac{\partial \alpha_i}{\partial r_j}\right|\right)^{-1}. \tag{29}$$

Into this formula, we introduce the expression for α and obtain

$$I^{-1} = 1 - \frac{(8G\mu l D/c^2)^2}{((l+x)^2 + y^2)((l-x)^2 + y^2)}. \tag{30}$$

Let

$$R_1 = \sqrt{(l+x)^2 + y^2}, \tag{31}$$

$$R_2 = \sqrt{(l-x)^2 + y^2} \tag{32}$$

be the distances to the ends of the string. Introducing the square of the Einstein's radius

$$\frac{8G\mu l D}{c^2} = \frac{4GMD}{c^2} = R_E^2, \tag{33}$$

we get

$$I = \frac{1}{\left|1 - \left(\frac{R_E}{\sqrt{R_1 R_2}}\right)^4\right|}. \tag{34}$$

Let us consider some partial cases: if $l \to 0$, then $R_1 \to R$; $R_2 \to R$,

$$I = \frac{1}{\left|1 - \left(\frac{R_E}{R}\right)^4\right|}, \tag{35}$$

and we obtain the well-known formula for the magnification by a point-like mass. But if $l \to \infty$, then $I = 1$. That is, an infinitely long string gives no magnification.

The formula implies that its denominator becomes zero under some conditions, i.e., the magnification tends to infinity. However, this is possible only in the ideal case. We now qualitatively describe an approximate shape of the critical curve. Let us consider a partial case where $R_E \ll 2l$, i.e. the string is positioned quite close. The critical curve is described by the equation $R_1 R_2 = R_E^2$, where R_i are the distances to the ends of the string. By our condition, one of the distances must be close to $2l$ and be almost invariable for the whole curve (otherwise, the inequality $R_E \ll 2l$ is not satisfied).

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The second distance will be small (since the distance between the ends of the string is equal to $2l$). We obtain $R_1 = 2l, R_2 = \frac{R_E^2}{2l}$, or $R_1 = \frac{R_E^2}{2l}, R_2 = 2l$. As seen, the second distance is really small. That is, the assumptions have no contradictions. The critical curve is two circles with the radii

$$R = \frac{R_E^2}{2l} \tag{36}$$

around the ends of the string. For another limiting case where a string is positioned very far and, therefore, $R_E \gg 2l$, the critical curve has form of a circle with radius R_E.

Consider the most symmetric configuration of the loop of a string, i.e. an ordinary circle.

We introduce a coordinate system, where the ring lies in the $x0y$ plane. We will characterize the position of a point on the ring by an angle φ which is reckoned from the x axis. Then $x = r\cos\varphi$, $y = r\sin\varphi$, and $z = 0$. We will seek a value of the field at the point $(R, 0, z_k)$. The relation $g_y = 0$ holds by the symmetry reasoning. We are not interested in g_z, since it does not influence the lensing in the first approximation. That is, we will seek only g_x. By the formula for the gravitational field ($\vec{g} = -mG\frac{\vec{R_1}-\vec{R_2}}{|\vec{R_1}-\vec{R_2}|^3}$), we obtain the x-component of the field at our point which is created by a small piece of the loop with size $d\varphi$:

$$dg_x = -G\mu d\varphi \frac{R - r\cos\varphi}{(z_k^2 + (R - r\cos\varphi)^2 + r^2\sin^2\varphi)^{3/2}}. \tag{37}$$

It can be written as

$$dg_x = -G\mu d\varphi \frac{R - r\cos\varphi}{(z_k^2 + R^2 + r^2 - 2rR\cos\varphi)^{3/2}}. \tag{38}$$

By integrating over z_k, we obtain

$$\int\limits_{-\infty}^{+\infty} dg_x dz_k = -\frac{G\mu r}{R}\left(1 + \frac{R^2 - r^2}{R^2 + r^2}\frac{1}{1 - \frac{2rR}{R^2+r^2}\cos\varphi}d\varphi\right). \tag{39}$$

We now integrate the last expression over all angles:

$$\int\limits_{-\infty}^{+\infty} g_x dz_k = -\frac{GM}{R}(1 + \mathrm{sgn}(R^2 - r^2)). \tag{40}$$

By using formula (22), we get

$$\alpha = \frac{2}{c^2}\int\limits_{-\infty}^{+\infty} g_x dz_k \tag{41}$$

and

$$\alpha = -\frac{2GM}{c^2 R}(1 - \mathrm{sgn}(R - r)). \tag{42}$$

This yields the final result:

$$\alpha = 0, \text{ if } R < r; \tag{43}$$

$$\alpha = -\frac{4GM}{c^2 R}, \text{ if } R > r. \tag{44}$$

The second formula can be easily identified as that for a deviation of a ray in the gravitational field of a point-like mass. That is, the rays which pass outside the ring are deviated in the same manner as those in the case of a point-like mass. But the rays passing inside the ring are not deviated at all. The formula for the magnification in the case of the lensing on a point-like mass was derived in the previous section. Thus, we have

$$I = 1, \text{ if } R < r; \tag{45}$$

$$I = \frac{1}{\left|1 - \left(\frac{R_E}{R}\right)^4\right|}, \text{ if } R > r. \tag{46}$$

The form of this formula implies that the critical curve is a circle with radius R_E. If it will turn out that $r > R_E$, then this will mean that the critical curve is absent. We now determine the conditions, under which it is possible to observe the magnification: $R_E > \frac{l}{2\pi}$; $4\pi^2 R_E^2 > l^2$. By using (1), (2), (5), and (33), we obtain the estimate

$$D > \frac{\Gamma p^{-1/3}}{16\pi^2} d_S, \quad D > 3d_S. \tag{47}$$

That is, a very large magnification can be attained at such and greater distances under the lensing on a string.

4. Peculiarities of the Lensing and the Search for It

In two previous sections, we have calculated the numerical parameters of the lensing for two different configurations of the loop of a cosmic string.

If a string is infinite, then two absolutely identical nonmagnified images will be formed under the lensing on the string: the first and second images are formed by rays which have passed, respectively, near one and another sides of the string. The identity of the images is the main basic distinct feature of the lensing on an infinite string [12].

If a string is a very elongated ellipse, then two images can be also formed by it under certain conditions: one image is created by rays which passed very close to the string and, therefore, are deviated by it almost in the same way as by an infinite one (this image will be identical to the object itself), and the second image is formed by rays which passed at a greater distance. The second image will be distorted and magnified. The calculations indicate that such a gravitational lens has a critical curve. That is, the second image can be very strongly magnified under certain conditions (the size and the form of the critical curve depend on the distances to an observer and to the source). Just by this manifestation, one can fix such an object.

If a string has form of an almost ideal circle, it can also induce the formation of two images. The first nondeformed image is formed by rays which passed through the ring and therefore underwent no refraction. The second deformed magnified image is created by rays which passed outside the ring. If the distance to the ring is sufficiently large, the second image can be very strongly magnified.

The above-presented specific features distinguish, to a significant extent, the gravitational lensing by the loops of strings from the lensing on ordinary massive objects and indicate the possibility to search for it. Therefore, we hope for that the topological defects – cosmic strings – will be discovered in the near future by astronomical observations.

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НЕРЕЛЯТИВІСТСЬКИЙ ВИПАДОК ГРАВІТАЦІЙНОГО ЛІНЗУВАННЯ НАЙПРОСТІШИМИ ПЕТЛЯМИ КОСМІЧНИХ СТРУН

П.Г. Гавриленко, Л.В. Задорожна

Р е з ю м е

В роботі проаналізовано властивості космічних струн, оцінено приблизну їх кількість в околі нашої галактики та аналітично розраховано параметри гравітаційної лінзи, утвореної космічною струною.