GROUP VELOCITY OF GAUSSIAN BEAMS

M.V. VASNETSOV, V.A. PAS'KO

Institute of Physics, Nat. Acad. of Sci. of Ukraine
(46, Nauky Prosp., Kyiv 03039, Ukraine)

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The group velocity of a short light pulse shaped in space in a form of a Gaussian beam is analyzed. The pulse radiation is represented by the spectral decomposition, and the integration of monochromatic components demonstrates the retardation of a Gaussian-beam pulse with respect to a plane-wave reference pulse. The resulting pulse velocity is not uniform and attains c at the far field, thus providing some excess over c at the part of its optical path. The effect is caused by the influence of a wavelength-dependent Gouy phase shift.

The subject of the group velocity of an electromagnetic (light) wave is usually related to the propagation of the polychromatic optical radiation in transparent media. In material media with dispersion, the refractive index depends on the wavelength. Therefore, the phase velocity of an optical wave varies with the wavelength. A light pulse with a broad spectrum will propagate with the velocity determined by pulse parameters and the material dispersion. Since a wave is carrying some information (signal), the group velocity is usually considered as a measure of the signal velocity and, therefore, attracts a considerable attention, especially from the point of view of the causality principles.

It seems that the phase and group velocities must coincide in the case of the free-space propagation. This statement is obviously valid for the plane-wave polychromatic light (e.g., a short pulse) but requires an inspection for real light sources like laser beams. The role of dispersion in the free-space propagation is accepted by the diffraction of a spatially compressed beam which does depend on the wavelength. It is well known that, for a beam with the Gaussian distribution of the intensity in a cross-section, the effect of diffraction spread-out results in a gradual growth of the transversal dimension of the beam (which is minimal at the beam waist), decrease of the wave amplitude, and some difference of the phase velocity with respect to a plane wave velocity c. This difference is rather small and affects only the phase shift with respect to a plane wave which amounts to $\pi/2$ on a propagation distance from the Gaussian beam waist to the far field. For the first time, the phase shift for focused light beams was observed by M. Gouy in 1890, much before the laser era [1].

Whilst the dependence of the phase velocity for spatially compressed beams (passed through a circular aperture or Laguerre–Gauss modes) is analyzed nowadays in details [2,3], an important question on their group velocity remains without clear response. In the present work, we perform a calculation of the group velocity for a short pulse of Gauss and Laguerre–Gauss beams which represent themselves the laser oscillation modes [4]. We have found that, by depending on the distance from the beam waist, the group velocity of a short pulse can attain the values less or higher than c at the beam axis. We emphasize that the excess of the group velocity over c does not contradict the causality principles [5, 6].

The formula describing the propagation of a Gaussian beam in free space is as follows [4]:

$$E(x,y,z,t) = E_0\left(\frac{w_0}{w}\right)\exp\left(-\frac{x^2+y^2}{w^2}\right)\times$$
$$\exp\left[ik\left(z+\frac{x^2+y^2}{2R(z)}-ct\right)-i\arctan\left(\frac{z}{z_R}\right)\right]. \quad (1)$$

Here, k is the wave number of a monochromatic radiation, $R(z) = z+\frac{z_R^2}{z}$ (wave front curvature radius), $w = w_0\sqrt{1+\frac{z^2}{z_R^2}}$ (beam radius in a cross-section), $z_R = \frac{kw_0^2}{2}$ (Rayleigh range), w_0 is the beam radius at the waist, and c is the light velocity. The last term in the block brackets is called the Gouy phase shift.

The radiation pulse shape in the time domain (we suppose a Gaussian pulse with duration τ, and k_0 stands for the wave number of the carrier frequency)

$$F_{\text{pulse}}(t) = \exp(-t^2/\tau^2)\exp(-ik_0ct) \quad (2)$$

can be expressed via the Fourier transformation as a superposition of plane waves with weights (amplitudes)

$$F(k) = \frac{\tau}{\sqrt{2}}\exp\left[-c^2\tau^2\frac{(k-k_0)^2}{4}\right]. \quad (3)$$

Each monochromatic component $E_k(x,y,z)$ propagates according to expression (1). By integrating over all

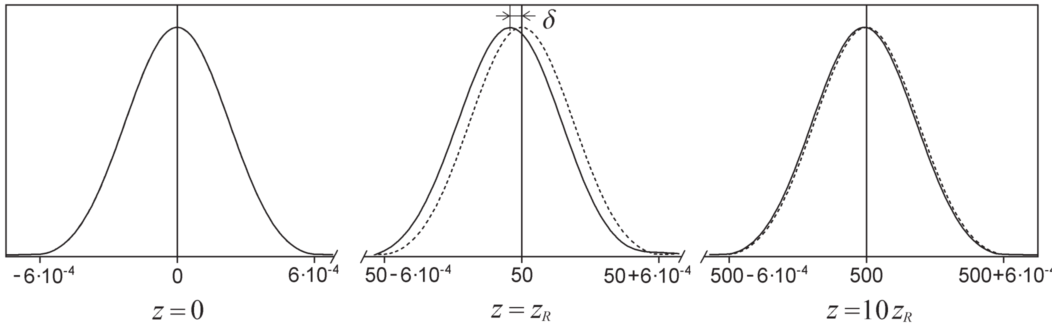

Fig. 1. Calculated pulse shape with normalized amplitude along the propagation direction at different distances: (*a*) just at the waist, (*b*) at the Rayleigh distance, and (*c*) at 10 Rayleigh distances. The reference pulse is shown by a dashed line. The delay of a Gaussian-beam pulse is indicated as δ. The longitudinal scale is in mm

spectral components, we have

$$E_{\text{pulse}}\left(x,y,z,t\right)=\frac{c}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}E_k\left(x,y,z\right)F\left(k\right)\times$$

$$\times\exp\left[ik\left(z-ct\right)\right]dk \tag{4}$$

(the amplitude parameter E_0 in Eq. (1) is unity).

To calculate the motion of the Gaussian pulse along the beam axis Z, we set $x=0$ and $y=0$ in Eq. (4) and combine it with Eq. (3):

$$E_{\text{pulse}}\left(z,t\right)=\frac{c\tau}{2\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\frac{dk}{\sqrt{1+\left(\frac{2z}{kw_0^2}\right)^2}}\times$$

$$\times\exp[-c^2\tau^2\frac{\left(k-k_0\right)^2}{4}-i\arctan\frac{2z}{kw_0^2}+ik\left(z-ct\right)]. \tag{5}$$

A similar expression for the pulse, which is emitted by a plane-wave source with the same frequency and taken as a reference one, has the form

$$E_{ref}\left(z,t\right)=\exp\left[\frac{-\left(z-ct\right)^2}{c^2\tau^2}\right]\exp\left[ik_0\left(z-ct\right)\right]. \tag{6}$$

The choice of the pulse parameters for calculations was determined by the condition to obtain a visual difference between it and the reference pulse. We took $\tau=10^{-15}$ s (this time parameter is close to just one period of the oscillation at the central frequency). We took the wave number of the carrier frequency $k_0=10\ \mu\text{m}^{-1}$, and the waist parameter $w_0=100\ \mu$m. The Rayleigh length for the carrier wavelength amounts to 50 mm. The extension of a pulse in space along the propagation direction amounts to $2\tau c\approx 0.6\ \mu$m ($1/e$ full width).

Figure 1 represents a pulse at different distances. For a better comparison with the reference plane-wave pulse, the amplitude is normalized to unity.

Whilst any monochromatic component has the phase velocity slightly exceeding c (due to the Gouy phase shift), the resulting Gaussian-beam pulse is delayed with respect to the reference (plane-wave) pulse. The explanation is the interference between the monochromatic components, with account of their relative phases. Both beams start at the same time and position (Fig. 1,*a*). Then the whole shape of the pulses is not altered, but a Gaussian-beam pulse appears at the distance δ behind the reference one (Fig. 1,*b*). Just after the waist plane and approximately to the Rayleigh distance, the delay δ increases up to its maximum value. Then, to the far field, the delay diminishes (Fig. 1,*c*). The dependence of the delay on the propagation distance is shown in Fig. 2.

An interesting peculiarity is, as seen, the variation of the pulse top velocity (group velocity v_g) along the propagation path. To find the value of v_g and compare it with c, the differentiation of the dependence in Fig. 2 can be performed. Following the procedure, we obtain

$$\frac{d\delta}{dz}=\frac{d\delta}{dt}\frac{dt}{dz}=\frac{v_g-c}{c}. \tag{7}$$

The corresponding dependence is plotted in Fig. 3. The group velocity of a Gaussian-beam pulse has minimum value at the beam waist and gradually grows, reaches c at the distance, where the delay δ has the maximum, and then overtakes c. Then, after reaching the maximum, the group velocity of the Gaussian beam asymptotically

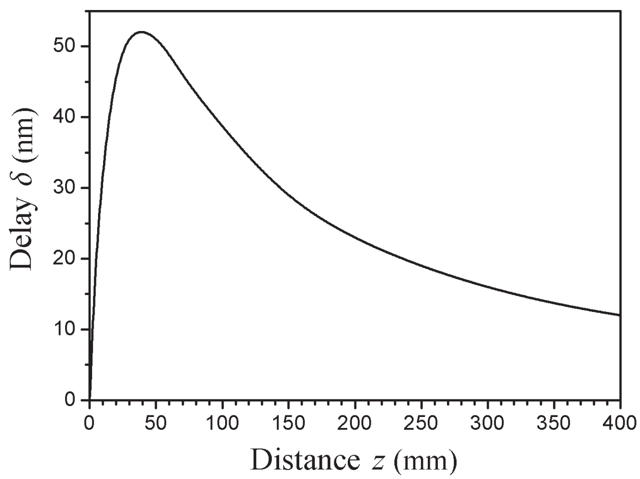

Fig. 2. Calculated delay δ of a Gaussian-beam pulse with respect to the reference plane-beam pulse along the propagation path from the waist

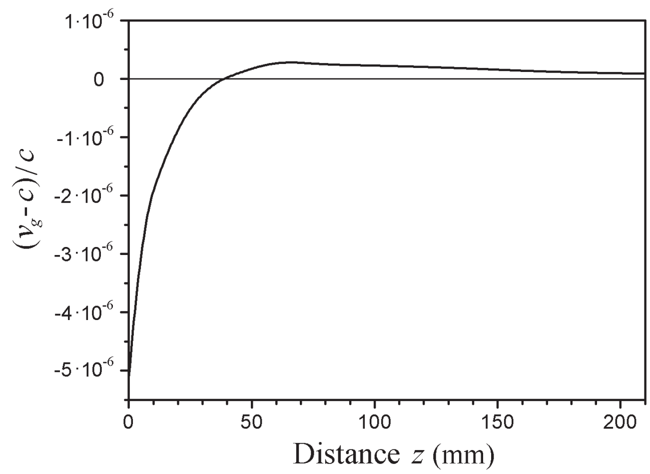

Fig. 3. Calculated difference between the velocity of the pulse top v_g and the light velocity c

tends to c. At the far field, both pulses, the Gaussian-beam and plane-wave ones, have the same group velocity equal to c.

For high-order Laguerre–Gauss modes LG_{pl}, the Gouy phase shift appears in the propagation law with the coefficient $Q = 2p + |l| + 1$, where p is the radial index and l is the azimuthal index [4]. An increase of the phase shift results in an increase of the phase velocity and simultaneously in an increase of the pulse delay.

The revealed excess of the group velocity over c does not come to any contradiction with the causality. The Gaussian-beam pulse is always delayed with respect to the plane-wave reference. This means that a Gaussian-beam signal cannot come to a recipient faster than a plane-wave one. Moreover, the gate generating the signal can be placed not necessarily at the waist of a beam, but at any distance from the waist.

In summary, we have demonstrated by the example of the pulse radiation of a Gaussian-shape beam that its group velocity can slightly exceed c, without contradiction with causality principle. It was pointed out by A. Sommerfeld and L. Brillouin that causality only requires the speed of the signal carried out by light be limited by c, rather than a light pulse itself which travels at its group velocity [5]. This general requirement is absolutely satisfied in the analyzed situation: the time needed for a pulse to reach a recipient is always higher than for a plane wave traveling with the velocity c.

In practical laboratory investigations, the specific properties of the propagation of Gaussian beams must be taken into account in high-precision experiments with ultrashort light pulses.

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ГРУПОВА ШВИДКІСТЬ ГАУСОВИХ ПУЧКІВ

М.В. Васнєцов, В.А. Пасько

Р е з ю м е

Досліджено групову швидкість короткого світлового імпульсу, що має просторову форму у вигляді гаусового пучка. Імпульсне випромінювання представлено у вигляді спектрального розподілу, і інтегрування монохроматичних компонент демонструє запізнювання імпульсу гаусового пучка по відношенню до референтного імпульсу з просторовою формою плоскої хвилі. В результаті виявляється, що швидкість імпульсу є нерівномірною та досягає у дальній зоні значення c, при цьому дещо перевищуючи c на певній частині свого оптичного шляху. Ефект пов'язаний з впливом фазового зміщення Гуі, який залежить від довжини хвилі.