MATHEMATICAL DESCRIPTION OF THE EQUILIBRIUM STATE OF CLASSICAL SYSTEMS ON THE BASIS OF THE CANONICAL ENSEMBLE FORMALISM

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This paper[1] gives a rigorous mathematical description of the equilibrium state of an infinite system of particles on the basis of canonical ensemble theory. The proof of the existence and uniqueness of limiting distribution functions and their analytical dependence on the density is given. The results have been obtained by using the methods developed by two of the authors in 1949 and are based on the application of the theory of Banach spaces to the study of the equations for distribution functions.

1. Introduction

In order to obtain thermodynamic relations on the basis of statistical mechanics, one requires to study systems with an infinite number of degrees of freedom. Such systems are derived from finite systems when there is an infinite increase in the number of particles N accompanied by a proportional increase in the volume V_N ($V_N = vN$, v =const). Here, difficult problems arise associated with the rigorous mathematical substantiation for the limiting transition as $N \to \infty$. To solve these problems, we use the formalism of canonical ensemble and involve the apparatus of distribution functions.

The considerable progress in this direction has been made in the last two decades.

A monograph [1] described the methods of giving a rigorous mathematical substantiation for the limiting transition in statistical mechanics, using the formalism of the Gibbs canonical ensemble, and developed a general method for seeking for the limiting distribution functions in the form of formal series in powers of the density $1/v$.

In 1949 in [2], the foundations for a rigorous mathematical description of infinite systems in statistical mechanics were developed. The detailed presentation of the results was published in 1956 in [3]. References [2,3] gave the full solution to the mathematical problems arising during the consideration of the limiting transition $N \to \infty$ in systems described by a canonical ensemble, for the case of a positive binary interaction potential of particles and sufficiently small densities. In this case, the system of equations for the distribution functions was treated in essence as an operator equation in the Banach space.

However, the methods developed in these papers evidently escaped the attention of investigators. In 1963, Ruelle [4] again suggested a similar approach to the study of the systems of equations for distribution functions. There, Ruelle used the formalism of large canonical ensemble which led him, in our opinion, to simpler tasks in formulating a basis for the limiting transition. At the same time, Ruelle was able to enlarge the class of potential functions under consideration by using the very ingenious idea of making the original equations for the distribution functions symmetric.

The objective of the present paper is a rigorous mathematical description, based on the theory of canonical ensemble, of the equilibrium state (at low densities) of infinite systems of particles, whose interaction potential is free from the restriction of positiveness and satisfies the Ruelle condition [4]. Here, we make use of the methods developed by two of the authors in [2,3] and the Ruelle method of symmetrization.

In the second section, we formulate the problem and derive the relations between the distribution functions in a finite volume, which become, on the limiting transition, the well-known Kirkwood–Salzburg equations. In contrast to the case of a large canonical ensemble, for a Gibbs ensemble in a finite volume, there are generally no equations for the distribution functions: the appropriate equations appear only after the limit transition to the infinite volume. This leads to new problems, in comparison with the case of a large canonical ensemble.

[1]The content of this paper was reported by one of the authors (N.N. Bogolyubov) at the College de France in April of 1969.

The third section proves a theorem on the existence and uniqueness of a solution of the Kirkwood–Salzburg equations for the potentials satisfying the Ruelle condition. In this case, we give the explicit estimate the densities, for which the solution is a series of iterations, and prove a theorem concerning the analytical nature of the dependence of the limiting distribution functions on the density is established.

The fourth section deals with the proof of the existence of limiting distribution functions when the number of particles in the system tends to infinity. Finally, in the fifth section, the uniqueness of these limiting functions is proved.

2. Statement of the Problem

1. We consider a system of N identical particles enclosed in a three-dimensional macroscopic volume V_N and interacting with one another via central forces characterized by an interaction potential $\Phi(q)$. We assume that the position of each particle is completely determined by its three Cartesian coordinates $q^\alpha (\alpha = 1, 2, 3)$, $q = (q^1, q^2, q^3)$.

We start out from the ordinary theory of equilibrium states based on the canonical Gibbs distribution; in the presentation, we follow [1] and [2, 3].

We introduce the probability distribution function for the positions of all the particles with a density

$$D_N = D_N(q_1 \dots, q_N) = Q^1(N, V_n) \exp\left\{-\frac{U_N}{\theta}\right\},$$

where U_N is the potential energy of the system,

$$U_N = \sum_{1\le i<j\le N} \Phi(q_i - q_j), \quad \Phi(q_i - q_j) = \Phi(|q_i - q_j|);$$

and $Q(N, V_N)$ is the configuration integral,

$$Q(N, V_N) = \int\limits_{V_N} \dots \int\limits_{V_N} \exp\left\{-\frac{U_N}{\theta}\right\} \times$$

$$\times dq_1 \dots dq_N, dq = dq_i^1 dq_i^2 dq_i^3.$$

The physical system under consideration is a canonical Gibbs ensemble. We now introduce a series of distribution functions [1]:

$$F_s^{(N)}(q_1, \dots, q_s;\ V_N) =$$

$$= V_N^s \int\limits_{V_N} \dots \int\limits_{V_N} D_N(q_1, \dots, q_s; q_{s+1}, \dots, q_N) \times$$

$$\times dq_{s+1} \dots dq_N. \qquad (2.1)$$

As usual, we assume that V_N is a ball: we also denote its volume by V_N, $V_N = vN$, where v is the volume per particle, and $1/v$ is the particle density.

The basic object of an investigation in statistical physics is the limiting functions

$$F_s(q_1, \dots, q_s; v) = \lim_{N\to\infty} (F_s^{(N)}(q_1 \dots, q_s; V_N)).$$

On this limiting transition, the density $1/v$ is considered constant.

By studying the questions related to the above-mentioned limiting transition, we used a system of equations for limiting functions $F_s(q_t, \dots, q_s; v)$.

Below, we will derive certain relations required for obtaining the appropriate equations.

2. We consider the expression $\exp\{-U_N/\theta\}$ and transform it as follows:

$$\exp\left\{-\frac{1}{\theta} \sum_{1\le i<j\le N} \Phi(q_i - q_j)\right\} = \exp\left\{-\frac{1}{\theta} \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times$$

$$\times \exp\left\{-\frac{1}{\theta} \sum_{2\le i<j\le N} \Phi(q_i - q_j)\right\} \prod_{i=s+1}^{N} [\varphi_{q_1}(q_i) + 1], \qquad (2.2)$$

where

$$\varphi_q(q_i) = \exp\left\{-\frac{1}{\theta}\Phi(q - q_i)\right\} - 1.$$

Substituting Eq. (2.2) into Eq. (2.1), we obtain

$$F_s^{(N)}(q_1, \dots, q_s; V_N) = V_N^s \frac{Q(N-1, V_N)}{Q(N, V_N)} \times$$

$$\times \exp\left\{-\frac{1}{\theta} \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times$$

$$\times \int\limits_{V_N} \dots \int\limits_{V_N} D_{N-1}(q_2, \dots, q_s; q_1^*, \dots, q_{N-s}^*) \times$$

$$\times \left[1 + \sum_{k=1}^{N-s} \frac{(N-s)(N-s-1)\dots(N-s-k+1)}{k!} \times\right.$$

$$\left.\times \prod_{i=1}^{k} \varphi_{q_1} d(q_i^*)\right] dq_1^* \dots dq_{N-s}^*, \qquad (2.3)$$

where

$$D_M(q_1, \dots, q_M) =$$

$$= Q^{-1}(M,V_N)\exp\left\{-\frac{1}{\theta}\sum_{1\le i<j\le M}\Phi(q_i-q_j)\right\},$$

$$Q(M,V_N) =$$

$$= \int\limits_{V_N}\dots\int\limits_{V_N}\exp\left\{-\frac{1}{\theta}\sum_{1\le i<j\le M}\Phi(q_i-q_j)\right\}dq_1\dots dq_M.$$

We also consider the distribution functions

$$F_k^{(N-l)}(q_1,\dots,q_k;V_N) = V_N^k\int\limits_{V_N}\dots\int\limits_{V_N}D_{N-1}\times$$

$$\times(q_1,\dots,q_k;q_{k+1},\dots,q_{N-l})dq_{k+1}\dots dq_{N-l} \qquad (2.4)$$

and the quantities

$$a_M(V_N) = vM\frac{Q(M-1,V_N)}{Q(M,V_N)}.$$

Here, it is evident that

$$F_s^{(N)}(q_1,\dots,q_s;V_N) = F_s^{(N-0)}(q_1,\dots,q_s;V_N).$$

Using the function $F_k^{(N-l)}$ and the quantities $a_M(V_N)$, Eq. (2.3) can be transformed into the form

$$F_s^{(N)}(q_1,\dots,q_s;V_N) = a_N(V_N)\times$$

$$\times\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\left[F_{s-1}^{(N-1)}(q_2,\dots,q_s;V_N)+\right.$$

$$+\sum_{k=1}^{N-s}\frac{(1-\frac{s}{N})(1-\frac{s+1}{N})\dots(1-\frac{s+k-1}{N})}{k!v^k}\times$$

$$\times\int\limits_{V_N}\dots\int\limits_{V_N}F_{s+k-1}^{N-1}(q_2,\dots,q_s;q_1^*,\dots,q_k^*;V_N)\times$$

$$\left.\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_k^*\right], \qquad (2.5)$$

where

$1<s<N$;

$$F_N^{(N)}(q_1,\dots,q_N;V_N) = a_N(V_N)\times$$

$$\times\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}F_{N-1}^{(N-1)}(q_2,\dots,q_N;V_N),$$

$$F_1^{(N)}(q_1;V_N) =$$

$$= a_N(V_N)\left[1+\sum_{k=1}^{N-1}\frac{(1+\frac{1}{N})(1-\frac{2}{N})\dots(1-\frac{k}{N})}{k!v^k}\times\right.$$

$$\times\int\limits_{V_N}\dots\int\limits_{V_N}F_k^{N-1}(q_1^*,\dots,q_k^*;V_N)\times$$

$$\left.\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_k^*\right]$$

Similarly, for the functions $F_s^{(N-l)}(q_n,\dots,q_s;V_N)$, we obtained the relations

$$F_s^{(N-l)}(q_1,\dots,q_s;V_N) =$$

$$= \frac{N}{N-l}a_{N-l}(V_N)\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\times$$

$$\times\left[F_{s-1}^{(N-l-1)}(q_2,\dots,q_s;V_N)+\right.$$

$$+\sum_{k=1}^{N-l-s}\frac{(1-\frac{l+s}{N})\cdots(1-\frac{l+s+k-1}{N})}{k!v^k}\times$$

$$\times\int\limits_{V_N}\dots\int\limits_{V_N}F_{s+k-1}^{N-l-1}(q_2,\dots,q_s;q_1^*,\dots,q_k^*;V_N)\times$$

$$\left.\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_k^*\right], \qquad (2.6)$$

where

$1<s<N-l$;

$$F_{N-l}^{(N-l)}(q_1,\dots,q_{N-l};V_N) = \frac{N}{N-l}a_{N-l}(V_N)\times$$

$$\times\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{N-l}\Phi(q_1-q_i)\right\}\times$$

$$\times F_{N-l-1}^{(N-l-1)}(q_2,\dots,q_{N-l};V_N);$$

$$F_1^{(N-l)}(q_1;V_N) =$$

$$= \frac{N}{N-l}a_{N-l}(V_N)\left[1+\sum_{k=1}^{N-l-1}\frac{(1-\frac{l+1}{N})\dots(1-\frac{l+k}{N})}{k!v^k}\times\right.$$

$$\times \int\limits_{V_N} \dots \int\limits_{V_N} F_k^{N-l-1}(q_1^*, \dots, q_k^*; V_N) \times$$

$$\times \prod_{i=1}^{k} \varphi_{q_1}(q_i^*) dq_1^* \dots dq_k^* \Bigg].$$

3. We assume temporarily that the limits

$$F_s(q_1, \dots, q_s; v) = \lim_{N\to\infty} F_s^{(N)}(q_1, \dots, q_s; V_N),$$

$$s = 1, 2, \dots,$$

$$F_s^l(q_1, \dots, q_s; v) = \lim_{N\to\infty} F_s^{(N-l)}(q_1, \dots, q_s; V_N),$$

$$s = 1, 2, \dots,$$

$$a(v) = a_0(v) \lim_{N\to\infty} a_N(V_N),$$

$$a_l(v) = \lim_{N\to\infty} a_{N-l}(V_N), \tag{2.7}$$

exist in some sense, and we will carry out a formal limiting transition in relations (2.5) and (2.6).

Relations (2.5) take the form

$$F_s(q_1, \dots, q_s; v) = a(v) \exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times$$

$$\times \Bigg[F_{s-1}^1(q_2, \dots, q_s; v) + \sum_{k=1} \frac{1}{k! v^k} \times$$

$$\times \int \dots \int F_{s+k-1}^1(q_2, \dots, q_s; q_1^*, \dots, q_k^*; v) \times$$

$$\times \prod_{i=1}^{k} \varphi_{q_1}(q_i^*) dq_1^* \dots dq_k^* \Bigg]. \tag{2.8}$$

(We adopt the convention of not showing the integration limits if the integral is taken over the whole three-dimensional space.)

The formal limiting transition in relations (2.6) gives

$$F_s^l(q_1, \dots, q_s; v) = a_l(v) \exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times$$

$$\times \Bigg[F_{s-1}^{l+1}(q_2, \dots, q_s; v) + \sum_{k=1}^{\infty} \frac{1}{k! v^k} \times$$

$$\times \int \dots \int F_{s+k-1}^{l+1}(q_2, \dots, q_s; q_1^*, \dots, q_k^*; v) \times$$

$$\times \prod_{i=1}^{k} \varphi_{q_1}(q_i^*) dq_1^* \dots dq_k^* \Bigg]. \tag{2.9}$$

If the equalities

$$F_s^l(q_1, \dots, q_s; v) = F_s(q_1, \dots, q_s; v); \quad a_l(v) = a(v) \tag{2.10}$$

were to hold for any $l \geq 1$, then relations (2.8) and (2.9) would pass into the well-known Kirkwood–Salzburg equations [5]

$$F_s(q_1, \dots, q_s; v) = a(v) \exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times$$

$$\times \Bigg[F_{s-1}(q_2, \dots, q_s; v) + \sum_{k=1}^{\infty} \frac{1}{k! v^k} \times$$

$$\times \int \dots \int F_{s+k-1}(q_2, \dots, q_s; q_1^*, \dots, q_k^*; v) \times$$

$$\times \prod_{i=1}^{k} \varphi_{q_1}(q_i^*) dq_1^* \dots dq_k^* \Bigg]. \tag{2.11}$$

In order to completely determine (2.11) for $s = 1$, we assume that $F_0 = 1$ here. It follows from the definitions that all $F_s(q_1, \dots, q_s; v)$ are symmetric functions of the variables q_i.

4. We now discuss the problems arising on the mathematical description of the system in an equilibrium state.

In our opinion, it is necessary to solve three following problems in order to give the complete substantiation for the mathematical description of such a system which is based on the Gibbs canonical distribution and uses a sequence of distribution functions:

1) to prove that limits (2.7) exist in a definite sense;

2) to prove that the limiting distribution functions do not depend on the method of going to the limit with regard for the validity of equality (2.10);

3) to prove that, for sufficiently small densities $1/v$, a unique solution exists for the system of Kirkwood–Salzburg equations.

We will solve the above-given problems in the following order: first, the third problem, then the first, and, finally, the second.

3. Theorem on the Existence of a Solution of the System of Kirkwood–Salzburg Equations

In this section, we consider the system of Kirkwood–Salzburg equations (2.11) for the distribution functions

$F_s(q_1,\dots,q_s;v)$ and prove that, for sufficiently small densities $1/v$, the system has the unique solution.

Thus, we have the system

$$F_s(q_1,\dots,q_s;v)=a(v)\exp\left\{-\frac{1}{\theta}\sum_{i=2}\Phi(q_1-q_i)\right\}\times$$

$$\times\left[F_{s-1}(q_2,\dots,q_s;v)+\sum_{k=1}^{\infty}\frac{1}{k!v^k}\times\right.$$

$$\times\int\dots\int F_{s+k-1}(q_2,\dots,q_s;q_1,\dots,q_k;v)\times$$

$$\left.\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_k^*\right],\quad F_0=1. \tag{3.1}$$

We will seek for a solution to this system in some Banach space [2,3,4] introduced as follows.

We consider a linear space, whose elements are the columns of measurable bounded functions

$$j=\begin{pmatrix} f_1(q_1)\\ f_2(q_1,q_2)\\ f_3(q_1,q_2,q_3)\\ \vdots\end{pmatrix}$$

with the usual summation of the columns and the multiplication of them by a number. This linear space becomes a Banach space B, if we introduce the norm of an element

$$\| f \|=\sup_s\left[\frac{1}{A^s}\sup_{q_1,\dots,q_s}|f_s(q_1,\dots,q_s)|\right], \tag{3.2}$$

where A is some positive constant to be determined later on.

In [2,3], the equivalent norm

$$\| f \|=\sup_s\left[\frac{1}{sA^s}\sup_{q_1,\dots,q_s}|f_s(q_1,\dots,q_s)|\right]$$

was introduced. It is more convenient in studying the equations for distribution factions in the Mayer–Montroll form ([1], p. 23).

We define, as yet only formally, an operator K acting in the space B according to the formula

$$(Kf)_s(q_1,\dots,q_s)=\exp\left\{-\frac{1}{\theta}\sum_{i=2}\Phi(q_1-q_i)\right\}\times$$

$$\times\left[f_{s-1}(q_2,\dots,q_s)+\sum_{k=1}^{\infty}\frac{1}{k!v^k}\times\right.$$

$$\times\int\dots\int f_{s+k-1}(q_2,\dots,q_s;q_1^*,\dots,q_k^*)\times$$

$$\left.\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\dots dq_k^*\right],\quad f_0=1. \tag{3.3}$$

With the use of the operator K, the system of equations (3.1) can be represented in the form

$$F=a(v)KF+a(v)F^0, \tag{3.4}$$

where

$$F=\begin{pmatrix} F_1(q_1;v)\\ F_2(q_1,q_2;v)\\ F_3(q_1,q_2,q_3;v)\\ \vdots\end{pmatrix},\quad F^0=\begin{pmatrix}1\\0\\0\\ \vdots\end{pmatrix}. \tag{3.5}$$

Later on, we will establish the existence and uniqueness of the solution of the operator equation (3.4) in the space B under certain assumptions regarding the potential $\Phi(q)$ and for sufficiently small $1/v$.

For Eq. (3.4) to have a unique solution in B, as is known, it is sufficient that $a(v)F^0\in B$, the operator $a(v)K$ be defined throughout the whole space B, and its norm be less than unity. In this case, we represent the solution F as the series

$$F=\sum_{n=1}^{\infty}(a(v)K)^n a(v)F^0 \tag{3.6}$$

which converges in the norm of the space B.

In the case of a nonnegative potential $\Phi(q)$ which, as usual, is assumed to be a real function in E_3 such that

$$J=\int\left|\exp\left\{-\frac{1}{\theta}\Phi(q)\right\}-1\right|dq<\infty, \tag{3.7}$$

it is easy to verify that these conditions are satisfied for sufficiently small $1/v$ [2,3].

First, we postulate that $a(v)$ and v are independent parameters (generally speaking, complex) such that

$$|a(v)|<2, \tag{3.8}$$

$$\left|\frac{1}{v}\right|<\frac{1}{2eJ}. \tag{3.9}$$

We now evaluate the norm of the operator $a(v)K$. We note first that the inequality $\Phi(q)\geq 0$ yields

$$\exp\left\{-\frac{1}{\theta}\sum_{n=2}^{s}\Phi(q_1-q_i)\right\}\leq 1. \tag{3.10}$$

Further, Eq. (3.3) yields

$$\sup_{q_1,\ldots,q_s} |a(v)(Kf)_s(q_1,\ldots,q_s)| \le |a(v)|\times$$

$$\times\Bigg[\sup_{q_2,\ldots,q_s} |f_{s-1}(q_2,\ldots,q_s)|\frac{A^{s-1}}{A^{s-1}} + \sum_{k=1}^{\infty}\frac{1}{k!|v|^k}\times$$

$$\times\int\ldots\int \sup_{q_2,\ldots,q_s;q_1^*,\ldots,q_k^*} |f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)|\times$$

$$\times\frac{A^{s+k-1}}{A^{s+k-1}}\prod_{i=1}^{k}|\varphi_{q_1}(q_i^*)|dq_1^*\ldots dq_k^*\Bigg] \le$$

$$\le |a(v)|e^{\frac{A}{|v|}J}A^{s-1} \parallel f \parallel . \tag{3.11}$$

Hence, we obtain

$$a(v)K_f| = \sup_s\left\{\frac{1}{A^s}\sup_{q_1,\ldots,q_s}|a(v)(K_f)_S(q_1,\ldots,q_s)|\right\} \le$$

$$\le \frac{|a(v)|}{A}e^{\frac{A}{|v|}J} \parallel f \parallel . \tag{3.12}$$

If we now put $A = 2e$ and take inequalities (3.8) and (3.9) into account, we obtain

$$\parallel a(v)K \parallel < \frac{|a(v)|e^{\frac{2eJ}{|v|}}}{2e} < 1.$$

At the same time, we have been convinced that the operator K is defined throughout the whole space B. It remains only to verify that $a(v)F^0 \in B$. But this follows immediately from definitions (3.5) and (3.2), since

$$\parallel a(v)F^0 \parallel = \frac{|a(v)}{A} < 1.$$

Thus, it has been proved that, for $\Phi(q) \ge 0$ and with assumptions (3.7) and (3.8), the system of Kirkwood–Salzburg equations (1.2) has the unique solution in a certain neighborhood of the point $1/v = 0$.

The method described above was developed by two of the authors in [2,3]. We will prove that it can be applied also to the more general case where the requirement that the potential $\Phi(q)$ be nonnegative is replaced by the following Ruelle condition [4].

There exists a positive constant b such that, for all s and for any $q_1,\ldots,q_s \in E_{3s}$, the inequality

$$\frac{1}{\theta}U_s(q_1,\ldots,q_s) \ge -sb \tag{3.13}$$

holds. It follows from this condition that, for any point $q_1,\ldots,q_s$ of E_{3s}, there exists at least one index i such that

$$\frac{1}{\theta}\sum_{j\neq i}\Phi(q_i - q_j) > -2b. \tag{3.14}$$

Using the symmetry property of the functions F_s, we can write Eqs. (3.1), following Ruelle [4], in a symmetric form.

By π_l, we denote an operator acting on the function $f_s(q_1,\ldots,q_s)$ by the formula

$$\pi_l f_s(q_1,q_2,\ldots,q_{l-1},q_l,q_{l+1},\ldots,q_s) =$$

$$= f_s(q_l,q_2,\ldots,q_{l-1},q_1,q_{l+1},\ldots,q_s).$$

It follows from Eq. (3.14) that there exist the measurable functions $v_i(q_1,\ldots,q_s)$ which are invariant relative to the group of rotations, take their values in the interval [0,1], and are such that

$$\sum_{i=1}^{s} v_i(q_1,\ldots,q_s) = 1,$$

$$\pi_k v_1(q_1,\ldots,q_s) = v_k(q_1,\ldots,q_s).$$

In this case, inequality (3.14) is valid if $v_i(q_1,\ldots,q_s) \neq 0$. The collection of the functions v_i is a resolution of unity.

Finally, we define the operator π according to the formula

$$\pi f_s(q_1,\ldots,q_s) = \sum_{l=1}^{s}\pi_l[v_1(q_1,\ldots,q_s)f_s(q_1,\ldots,q_s)].$$

Using the symmetry property of the functions $F_s(q_1,\ldots,q_s;v)$, we represent system (3.1) in such final form

$$F_s(q_1,\ldots,q_s;v) = a(v)\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\times$$

$$\times\Bigg[F_{s-1}(q_2,\ldots,q_s;v) + \sum_{k=1}^{\infty}\frac{1}{k!v^k}\times$$

$$\times\int\ldots\int F_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*;v)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*\Bigg]. \tag{3.15}$$

Assuming that the potential function $\Phi(q)$ satisfies conditions (3.8) and (3.13), we will establish the existence and uniqueness of the solution of system (3.15) for sufficiently small $1/v$. As was done above, we consider, for the time being, that $a(v)$ and v are independent complex parameters, and we assume the validity of inequality (3.8).

System (3.15) can be written in the form of Eq. (3.4) under the condition that the operator K is now defined by the symmetrized formulas (3.3) as

$$(Kf)_s(q_1,\ldots,q_s)=\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{S}\Phi(q_1-q_i)\right\}\times$$

$$\times\Bigg[f_{s-1}(q_2,\ldots,q_s)+\sum_{k=1}^{\infty}\frac{1}{k!v^k}\times$$

$$\times\int\ldots\int f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*\Bigg],\quad f_0=0. \tag{3.16}$$

Theorem I. *The system of equations (3.15) possesses the unique solution in the space B when*

$$|a(v)|<2;\quad \frac{1}{|v|}<\frac{1}{2e^{2b+1}J}. \tag{3.17}$$

This solution is a holomorphic function of $a(v)$ and $1/v$ in region (3.17).

Proof. In order to prove the first assertion of the theorem, it is sufficient to establish that the norm of the operator $a(v)K$ defined by formulas (3.16) is less than 1 under conditions (3.17).

Taking into account that estimate (3.10) is now replaced by the expression

$$\pi\exp\left\{-\frac{1}{\theta}\sum_{i=1}^{s}\Phi(q_1-q_i)\right\}<e^{2b},$$

following from (3.14), we obtain from Eq. (3.15), in complete analogy with (3.11) and (3.12), that

$$\sup_{q_1,\ldots,q_s}|a(v)(Kf)_s(q_1,\ldots,q_s)|\le$$

$$\le|a(v)|e^{2b}e^{\frac{A}{|v|}J}A^{s-1}\parallel f\parallel, \tag{3.18}$$

$$\parallel a(v)Kf\parallel\le\frac{|a(v)|}{A}e^{2b}e^{\frac{A}{|v|}J}\parallel f\parallel .$$

Setting $A=2e^{2b+1}$ and taking Eq. (3.17) into account, we obtain, from Eq. (3.18), the required estimate

$$\parallel a(v)K\parallel<\frac{|a(v)|}{2e^{2b+1}}\exp\left\{2b+\frac{2e^{2b+1}J}{|v|}\right\}\le k<1.$$

Thus, we have proved the existence and uniqueness of the solution of system (3.15) or, which is the same, Eq. (3.4) in region (3.17). The holomorphic nature of the solution as a function of $a(v)$ and v in the indicated region follows from the possibility to represent it by series (3.6) which converges uniformly with respect to $a(v)$ and v in any closed region contained in (3.17).

Returning to the case of limiting distribution functions $F_s(q_1,\ldots,q_s)$; v, we must regard v as a real (positive) variable and $a(v)$ as a function of v. We will show that, in this case, the condition $|a(v)|<2$ in Eq. (3.17) can be dropped.

To this end, we will find the estimate for the numbers $a_l(v)$.

Lemma 1. *The numbers $a_l(v)$ satisfy the inequality*

$$a_l(v)\le\frac{1}{1-J/v}. \tag{3.19}$$

Proof. Indeed, let us consider the quantity [3]

$$\frac{Q(M,V_N)}{Q(M-1,V_N)}=\frac{1}{Q(M-1,V_N)}\times$$

$$\times\int_{V_N}\ldots\int_{V_N}\exp\left\{-\frac{1}{\theta}\sum_{1\le i<j\le M}\Phi(q_i-q_j)\right\}\times$$

$$\times dq_1\ldots dq_M=\int_{V_N}dq\Bigg[\int_{V_N}\ldots\int_{V_N}\frac{1}{Q(M-1,V_N)}\times$$

$$\times\exp\left\{-\frac{1}{\theta}\sum_{1\le i<j\le M-1}\Phi(q_i-q_j)\right\}\times$$

$$\times\prod_{i=1}^{M-1}\{\varphi_q(q_i)+1\}\,dq_1\ldots dq_{M-1}\Bigg]. \tag{3.20}$$

We use the elementary inequality

$$\prod_{i=1}^{M-1}[\varphi_q(q_i)+1]\ge1-\sum_{i=1}^{M-1}|\varphi_q(q_i)| \tag{3.21}$$

which is valid not only for our specific functions $\varphi_q(q_i)$, but also generally for arbitrary quantities a_i such that $1+a_i>0$.

Using (21.2), we obtain, from Eq. (20.2), the estimate

$$\frac{Q(M,V_N)}{Q(M-1,V_N)}\ge\int_{V_N}dq\Bigg[\int_{V_N}\ldots\int_{V_N}\left(1-\sum_{i=1}^{M-1}|\varphi_q(q_i)|\right)\times$$

$$\times \frac{1}{Q(M-1,V_N)} \exp\left\{-\frac{1}{\theta}\sum_{1\le i<j\le M-1}\Phi(q_i-q_j)\right\}\times$$

$$\times dq_1\ldots dq_{M-1} \ge V_N-(M-1)J.$$

Whence we get that, for any $M\le N$,

$$a_M(V_N)=$$

$$= v\frac{MQ(M-1,V_N)}{Q(M,V_N)} \le vM\frac{1}{V_N-(M-1)J} < \frac{1}{1-J/v}.$$

Since, for any l, $a_l(v) = \lim\limits_{N\to\infty} a_{N-l}(V_N)$ by definition, the quantity $a_l(v)$ also satisfies inequality (3.19).

Lemma 1 and the inequalities $\frac{1}{v} < \frac{1}{2e^{2b+1}J} < \frac{1}{2J}$ yield the inequality $a_l(v)<2$. That is, the second condition in (3.17) yields the first one.

We consider, finally, the question about the character of the dependence of the limiting distribution functions on the density.

THEOREM II. *The distribution functions $F_s(q_1,\ldots,q_s;v)$ are holomorphic functions of $1/v$ in a certain neighborhood of zero.*

Proof [3]. By Theorem I, $F_s(q_1,\ldots,q_s;v)$ are holomorphic functions of $a(v)$ and $1/v$ in region (3.17), i.e. (by virtue of Lemma 1) at $\frac{1}{v}<\frac{1}{2e^{2b+1}J}$. Therefore, it is sufficient to prove that, in some neighborhood of the point $1/v=0$, $a(v)$ is a holomorphic function of $1/v$.

The solution of Eq. (3.4) is translation-invariant, since each term of series (3.6) possesses this property. Therefore, $F_1(q_1;v)=$const.

As will be proved in Section 5,

$$\lim_{N\to\infty}\frac{1}{V_N}\int\limits_{V_N} F_1^{(N)}(q_1;V_N)dq_1 =$$

$$= \lim_{N\to\infty}\frac{1}{V_N}\int\limits_{V_N} F_1(q;v)dq_1 = 1, \tag{3.22}$$

whence it follows that $F_1(q_1;v)=1$.

Now, using the Kirkwood–Salzburg equation, we obtain

$$1 = F_1(q_1;v) =$$

$$= a(v)\Bigg[1+\sum_{k=1}^{\infty}\frac{1}{k!v^k}\int\ldots\int F_k(q_1^*,\ldots,q_k^*;v)\times$$

$$\times \prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*\Bigg] \equiv \chi(a,v), \tag{3.23}$$

where the function $\chi(a,v)$ is holomorphic with respect to a and $1/v$ in region (17.2).

We will prove that, in some neighborhood of the point $1/v=0$ for $|a|<2$, the partial derivative $\partial\chi/\partial a\neq 0$. Whence it will follow that equality (23.2) can be solved for a, and the function $a(v)$ is a holomorphic function of l/v in a neighborhood of the point $1/v=0$.

For $\partial\chi/\partial a$, we obtain

$$\frac{\partial\chi}{\partial a} = 1+\sum_{k=1}^{\infty}\frac{1}{k!v^k}\int\ldots\int F_k(q_1^*,\ldots,q_k^*;v)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^* + a\frac{1}{v}\frac{\partial}{\partial a}\Bigg[\sum_{k=1}^{\infty}\frac{1}{k!v^{k-1}}\times$$

$$\times\int\ldots\int F_k(q_1^*,\ldots,q_k^*;v)\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*\Bigg]. \tag{3.24}$$

It follows from (28.2) that $\partial\chi/\partial a=1$ at the point $1/v=0$, and, hence, $\partial\chi/\partial a\neq 0$ in some neighborhood of zero.

The theorem has been proved.

4. Existence of the Limiting Distribution Functions

1. In this section, we will investigate Problem 1 formulated in Section 2. In solving this problem, it is convenient to use relations (2.5) and (2.6) which are a basis for obtaining the Kirkwood–Salzburg equations.

Thus, we consider the relations

$$F_s^{(N-l)}(q_1,\ldots,q_s;V_N) = \frac{N}{N-l}a_{N-l}(V_N)\pi\times$$

$$\times\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}[F_{s-1}^{(N-l-1)}(q_2,\ldots,q_s;V_N)+$$

$$+\sum_{k=1}^{N-l-s}\frac{\left(1-\frac{l+s}{N}\right)\cdots\left(1-\frac{l+s+k-1}{N}\right)}{k!v^k}\times$$

$$\times\int\limits_{V_N}\ldots\int\limits_{V_N} F_{s+k-1}^{(N-l-1)}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*;V_N)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dg_1^*\ldots dq_k^*], \tag{4.1}$$

$$F_{N-l}^{(N-l)}(q_1,\ldots,q_{N-l};V_N) = \frac{N}{N-l}a_{N-l}(V_N)\times$$

$$\pi \exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\} F_{N-l-1}^{(N-l-1)}(q_2,\ldots,q_{N-l};V_N),$$

$$F_0^{(N-l-1)}=1$$

which follow from Eq. (6.1), if we make use of the symmetry property of the functions $F_s^{(N-l)}(q_i,\ldots,q_s;V_N)$ [see Section 3, Eq.(3.15)].

We consider that these relations are valid throughout the entire $3s$-dimensional Euclidean space E_{3s}. We define the functions $F_s^{(N-l)}(q_1,\ldots,q_s;V_N)$ in E_{3s} according to formulas (2.4) and (2.10).

We introduce the notation

$$F^{(N-l)}=\begin{pmatrix} F_1^{(N-l)}(q_1;V_N)\\ F_2^{(N-l)}(q_1,q_2;V_N)\\ \vdots\\ F_{N-l}^{(N-l)}(q_1,q_2,\ldots,q_{N-l};V_N)\\ 0\\ \vdots\end{pmatrix},$$

$$F^{0(N-l)}=\begin{pmatrix}\frac{N}{N-l}a_{N-l}(V_N)\\ 0\\ \vdots\end{pmatrix}. \quad (4.2)$$

By $K_s^{(N-l)}$, we denote the operator acting in the Banach space B on an arbitrary column f by the formulas

$$(K_s^{(N-l)}f)_s(q_1,\ldots,q_s)=\frac{N}{N-l}a_{N-l}(V_N)\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\Bigg[f_{s-1}(q_2,\ldots,q_s)+$$

$$+\sum_{k=1}^{N-l-s}\frac{\left(1-\frac{l+s}{N}\right)\cdots\left(1-\frac{l+s+k-1}{N}\right)}{k!v^k}\times$$

$$\times\int\limits_{V_N}\cdots\int\limits_{V_N} f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*\Bigg],$$

$$(K_s^{(N-l)}f)_{i\neq s}(q_1,\ldots,q_i)=0,\quad s\neq N-l,$$

$$(K_{N-l}^{(N-l)}f)_{N-l}(q_1,\ldots,q_{N-l})=\frac{N}{N-l}a_{N-l}(V_N)\times$$

$$\times\pi\exp\Big\{-\frac{1}{\theta}\sum_{i=2}^{N-l}\Phi(q_1-q_i)\Big\}f_{N-l-1}(q_2,\ldots,q_{N-l}),$$

$$(K_{N-l}^{(N-l)}f)_{i\neq N-l}(q_1,\ldots,q_i)=0,\quad f_0=0. \quad (4.3)$$

Here, the expressions $(K_s^{(N-l)}f)_s(q_1,\ldots,q_s)$ are defined also throughout the whole space E_{3s}. We denote the operator

$$K^{(N-l)}=K_1^{(N-l)}+K_2^{(N-l)}+\ldots+K_{N-l}^{(N-l)}$$

by K^{N-l}.

Using the operators K^{N-l} and the columns $F^{(N-l)}$, we can represent relations (4.1) in the compact form:

$$F^{(N-l)}=K^{(N-l)}F^{(N-l-1)}+F^{0(N-l)}.$$

2. We now examine the properties of the operator $K^{(N-l)}$. It can be shown that the norms of the operators $K^{(N-l)}$ and $K_s^{(N-l)}(1\le s\le N-l)$ for $\frac{1}{v}<\frac{1}{2e^{2b+1}J}$ are less than unity, $\parallel K^{(N-l)}\parallel\le k<1$, $\parallel K_s^{(N-l)}\parallel\le k<1$. To do this, it is necessary to use literally the same arguments as in the proof of Theorem I and to take into account that the numbers $\frac{N}{N-l}a_{N-l}(V_N)$ satisfy, for $1/v<1/2J$, the inequality

$$\frac{N}{N-l}a_{N-l}(V_N)<2 \text{ and}$$

$$\left(1-\frac{l+s}{N}\right)\ldots\left(1-\frac{l+s+k-1}{N}\right)<1.$$

Here and in what follows, we consider that the number A appearing in the definition of the norm is subject to the same restriction as that in Section 3.

The positive sequences $a_N(V_N)$, $\frac{N}{N-1}a_{N-1}(V_N),\ldots,\frac{N}{N-l}a_{N-l}(V_N)$; $N=3,4,\ldots$, for $1/v<1/2J$ are bounded from above by number two.[2] Therefore, using a diagonal process, we can choose any finite number $l+1$ of convergent subsequences

$$a_{N_i}(V_{N_i}),\quad \frac{N_i}{N_i-1}a_{N_i-1}(V_{N_i}),\ldots,\frac{N_i}{N_i-l}a_{N_i-l}(V_{N_i}),$$

whose limits for $N_i\to\infty$ are denoted by $a(v),a_1(v),\ldots,a_l(v)$:

$$\lim_{N_i\to\infty}a_{N_i}(V_{N_i})=a(v);\quad \lim_{N_i\to\infty}\frac{N_i}{N_i-1}a_{N_i-1}(V_{N_i})=$$

$$=a_1(v),\ldots,\lim_{N_i\to\infty}\frac{N_i}{N_i-l}a_{N_i-l}(V_{N_i})=a_l(v).$$

[2]We put $a_{N-l}(V_N)=0$ for $l>N-3$.

The further analysis shows (see Section 4) that, for each sequence $a_N(V_N)$, $\frac{N}{N-1}a_{N-1}(V_N),\ldots,$ $\frac{N}{N-l}a_{N-l}(V_N)$, only one limiting point exists. That is, indeed, the sequences $a_N(V_N)$, $\frac{N}{N-1}a_{N-1}(V_N),\ldots,\frac{N}{N-l}a_{N-l}(V_N)$ converge as $N\to\infty$. In addition, it will be shown that $a(v)=a_1(v)=\ldots=a_l(v)$. In what follows, we accept the sequences $a_N(V_N)$, $\frac{Na_{N-1}}{N-1}(V_N),\ldots,\frac{N}{N-l}a_{N-l}(V_N)$ to be some convergent subsequences. Moreover, when we say that we carry out the transition to the limit as $N\to\infty$, we mean that we carry out the limiting transition with respect to the corresponding subsequence of the indices $N_i\to\infty$.

By K_s, we denote the operator acting on an arbitrary element f of the Banach space B by the formula

$$(K_sf)_s(q_1,\ldots,q_s)=(Kf)_s(q_1,\ldots,q_s),$$

$$(K_sf)_{i\neq s}(q_1,\ldots q_i)=0.$$

By $\psi^{(R)}(q)$, we denote the characteristic function of the ball V_N with the radius $R_N=\sqrt{\frac{3}{4\pi}vN}$ and its center at the coordinate origin. We now introduce the ball V'_N with the radius R_N-r_N and the characteristic function $\psi^{(R-r)}(q)$. We require that the function r_N possess the following properties: $r_N\to\infty$ as $N\to\infty$, and $r_N/R_N\to 0$ as $N\to\infty$

Finally, we denote, by $\Psi^{(R-r)}$, the operator acting on an arbitrary element $f\in B$ by the formula

$$(\Psi^{(R-r)}f)_s(q_1,\ldots,q_s)=$$

$$=\psi^{(R-r)}(q_1)\ldots\psi^{(R-r)}(q_s)f_s(q_1,\ldots,q_s).$$

We formulate one more property of the operator $K_s^{(N-l)}$ in the following lemma.

Lemma 2. *If $\frac{1}{v}<\frac{1}{2e^{2b+1}J}$, the sequence of operators $\Psi^{(R-r)}(a_l(v)K_s-K_s^{N-l})$ converges in its norm to zero as $N\to\infty$ and for fixed l.*

Proof. For any f, we consider the expression

$$\Psi^{(R-r)}(a_l(v)K_s-K_s^{(N-l)})f=a_l(v)\Psi^{(R-r)}\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}[f_{s-1}(q_2,\ldots,q_s)+$$

$$+\sum_{k=1}^{\infty}\frac{1}{k!\,v^k}\int\ldots\int f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=2}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*]-\frac{N}{N-l}a_{N-l}(V_N)\Psi^{(R-r)}\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}[f_{s-1}(q_2,\ldots,q_s)+$$

$$+\sum_{k=1}^{N-l-s}\frac{\left(1-\frac{l+s}{N}\right)\cdots\left(1-\frac{l+s+k-1}{N}\right)}{k!\,v^k}\times$$

$$\times\int\limits_{V_N}\ldots\int\limits_{V_N}f_{s_ik-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*]=a_l(v)\Psi^{(R-r)}\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}[f_{s-1}(q_2,\ldots,q_s)+$$

$$+\sum_{k=1}^{N_0}\frac{1}{k!\,v^k}\int\ldots\int f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*]-\frac{N}{N-l}a_{N-l}(V_N)\Psi^{(R-r)}\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}[f_{s-1}(q_2,\ldots,q_s)+$$

$$+\sum_{k=1}^{N_0}\frac{\left(1-\frac{l+s}{N}\right)\cdots\left(1-\frac{l+s+k-1}{N}\right)}{k!\,v^k}\times$$

$$\times\int\ldots\int f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*]+$$

$$+a_l(v)\Psi^{(R-r)}\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\times$$

$$\times\Big[\sum_{k=N_0+1}^{\infty}\frac{1}{k!\,v^k}\int\ldots\int f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*]-\frac{N}{N-l}a_{N-l}(V_N)\Psi^{(R-r)}\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\times$$

$$\times\left[\sum_{k=N_0+1}^{N-l-s}\frac{\left(1-\frac{l+s}{N}\right)\cdots\left(1-\frac{l+s+k-1}{N}\right)}{k!\,v^k}\times\right.$$

$$\times\int\limits_{V_N}\ldots\int\limits_{V_N} f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots,q_k^*)\times$$

$$\left.\times\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*\right]+\frac{N}{N-l}\,a_{N-l}(V_N)\Psi^{(R-r)}\times$$

$$\times\pi\exp\left\{-\frac{1}{\theta}\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\times$$

$$\times\left[\sum_{k=1}^{N_0}\frac{\left(i-\frac{l+s}{N}\right)\ldots\left(1-\frac{l+s+k-1}{N}\right)}{k!\,v^k}\times\right.$$

$$\times\int\ldots\int f_{s+k-1}(q_2,\ldots,q_s;q_1^*,\ldots q_k^*)\times$$

$$\times[1-\prod_{i=1}^{k}\psi^{(R)}(q_i^*)\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*], \tag{4.4}$$

where N_0 is an arbitrary finite number as yet.

By β,γ, and δ, we denote three last terms, respectively. For those, the estimates

$$|\beta|\leqslant A^{s-1}a(v)e^{2b}\parallel f\parallel\times$$

$$\times\sum_{k=N_0+1}^{\infty}\frac{1}{k!\,v^k}A^kJ^k\leqslant A^s\varepsilon_1(N_0)\parallel f\parallel,$$

$$|\gamma|\leqslant A^{s-1}\frac{N}{N-l}\,a_{N-l}(V_N)e^{2b}\parallel f\parallel\times$$

$$\times\sum_{k=N_0+1}^{\infty}\frac{1}{k!\,v^k}A^kJ^k\leqslant A^s\varepsilon_1(N_0)\parallel f\parallel,$$

$$|\delta|\leqslant A^{s-1}\frac{N}{N-l}\,a_{N-l}(V_N)e^{2b}\parallel f\parallel\times$$

$$\times\sum_{k=1}^{\infty}\frac{1}{k!\,v^k}\varepsilon_2(r_N)kJ^{k-1}A^k\leqslant A^s\varepsilon_2(r_N)\parallel f\parallel,$$

are valid, where

$$\varepsilon_2(r_N)=\int|e^{-(1/\theta)\Phi(q)}-1|(1-\psi^{(r)}(q))dq.$$

The quantity $\varepsilon_1(N_0)$, as the remainder of the absolutely convergent series, can be made as small as possible for sufficiently large but finite N_0, independently of N. The quantity $\varepsilon_2(r_N)$ can be made as small as possible for sufficiently large N by virtue of the absolute convergence of the integral

$$\int|e^{-(1/\theta)\Phi(q)}-1|dq.$$

Denoting the difference between the first two expressions by *alpha*, we obtain the estimate

$$|a|\leqslant A^{s-1}e^{2b}\parallel f\parallel\sum_{k=0}^{N_0}\times$$

$$\times\left|a_l(v)-\frac{N}{N-l}a_{N-l}(V_N)\left(1-\frac{l+s}{N}\right)\ldots\right.$$

$$\left.\ldots\left(1-\frac{l+s+k-1}{N}\right)\right|\frac{A^kJ^k}{k!\,v^k}\leqslant A^s\varepsilon_3(N)\parallel f\parallel,$$

where the quantity $\varepsilon_3(N)$ can be made as small as possible for sufficiently large N, since $(a_{N-l}(V_N)\to a_l(v)$ (N_0 is a fixed number here).

Thus, we obtain

$$\sup_{q_1,\ldots,q_s}|\Psi^{(R-r)}[(a_l(v)K_s-K_s^{(N-l)})f]_s(q_1,\ldots,q_s)|\leqslant$$

$$\leqslant A^s(2\varepsilon_1(N_0)+\varepsilon_2(r_N)+\varepsilon_3(N))\parallel f\parallel,$$

which is equivalent to the estimate

$$\parallel\Psi^{(R-r)}(a_l(v)K_s-K_s^{(N-l)})f\parallel\leqslant$$

$$\leqslant(2\varepsilon_1(N_0)+\varepsilon_2(r_N)+\varepsilon_3(N))\parallel f\parallel.$$

Whence it follows immediately that the estimate

$$\parallel\Psi^{(R-r)}(a_l(v)K_s-K_s^{(N-l)})\parallel\leqslant$$

$$\leqslant 2\varepsilon_1(N_0)+\varepsilon_2(r_N)+\varepsilon_3(N)=\varepsilon(r_N,N),$$

$$\lim_{N\to\infty}\varepsilon(r_N,N)=0 \tag{4.5}$$

is valid, which means that the sequence of operators $\Psi^{(R-r)}(a_l(v)K_s-K_S^{(N-l)})$ converges to zero in norm as $N\to\infty$.

The lemma has been proved.

By B_n, we denote the Banach space composed of columns f such that $f_{n+1}=f_{n+2}=\ldots=0$. Their norm

$$\parallel f\parallel=\sup_{1\leqslant s\leqslant n}\left\{\frac{1}{A^s}\sup_{q_1,\ldots,q_s}|f_s(q_1,\ldots,q_s)|\right\}.$$

We now consider the operators

$$K_{[n]}^{(N-l)} = \sum_{1\leqslant s\leqslant n} K_s^{(N-l)}, \quad K_{[n]} = \sum_{i\leqslant s\leqslant n} K_s.$$

These operators act from the space B into the space B_n, and their norm is less than unity. The operators $\Psi^{(R-r)}(K_{[n]}^{(N-l)} - a_l(v)K_{[n]})$ are the sum of a finite number of operators $\Psi^{(R-r)}(K_s^{(N-l)} - a_l(v)K_s)$ which converge in norm to zero. Therefore, the operators $\Psi^{(R-r)}(K_{[n]}^{(N-l)} - a_l(v)K_{[n]})$ also converge in norm to zero. It is easy to see that, for the norm of the operator $\Psi^{(R-r)}(K_{[n]}^{(N-l)} - a_l(v)K_{[n]})$, the estimate $\| \Psi^{(R-r)}(K_{[n]}^{(N-l)} - a_l(v)K_{[n]}) \| \leq \varepsilon(r_N - N)$ is valid. In what follows, we will use the inequalities

$$\| \Psi^{(R-r+\frac{j}{n}r)}a_l(v)K_{[n]} - \Psi^{(R-r+\frac{j}{n}r)}a_l(v)K_{[n]}\times$$

$$\times\Psi^{(R-r+\frac{j+1}{n}r)} \| \leqslant \varepsilon\left(\frac{1}{n}\,r_N,\, N\right);$$

$$\| \Psi^{(R-r+\frac{j}{n}r)}K_{[n]}^{(N-l)} - \Psi^{(R-r+\frac{j}{n}r)}K_{[n]}^{(N-l)}\times$$

$$\times\Psi^{(R-r+\frac{j+1}{n}r)} \| \leqslant \varepsilon\left(\frac{1}{n}\,r_N,\, N\right), \tag{4.6}$$

where n, j are integers, $j < n$. These inequalities can be established exactly like inequality (4.5) in Lemma 2.

Remark. The sequence of the operators $\Psi^{(R-r)}(K^{(N-l)} - a_l(v)K)$ does not converge in norm to zero. Indeed, the operator $\Psi^{(R-r)}K^{(N-l)}$ acts from B into B_{N-l}, and $\Psi^{(R-r)}K^{(N-l)} = 0$ on the elements $f \in B$, for which $f_s = 0$ as $s \leq N-l$. Therefore, for any N as large as possible, there is always an element f such that $\Psi^{(R-r)}K^{(N-l)}f = 0$, $\Psi^{(R-r)}a_l(v)Kf \neq 0$, and the norm of the element $\Psi^{(R-r)}a_l(v)Kf$ is finite.

3. We now examine relation (4.4). By using it repeatedly, we obtain

$$F^{(N-l)} = K^{(N-l)}K^{(N-l-1)}\dots K^{(3)}F^{(2)} + K^{(N-l)}\times$$

$$\times K^{(N-l-1)}\dots K^{(4)}F^{c(3)}\dots + K^{(N-l)}K^{(N-l-1)}\dots$$

$$\dots K^{(i)}F^{0(i-1)} + \dots + K^{(N-l)}F^{0(N-l-1)} + F^{0(N-l)} =$$

$$= K^{(N-l)}K^{(N-l-1)}\dots K^{(3)}F^{(2)}+$$

$$+\sum_{i=0}^{N-l-4}\Big(\prod_{j=0}^{i} K^{(N-l-j)}\Big)F^{0(N-l-i-1)} + F^{0(N-l)}. \tag{4.7}$$

Here, the operator K^{N-l-j} acts following the operator $K^{N-l-j-1}$. Here, by definition,[3]:

$$F^{(2)} = \begin{pmatrix} F_1^{(2)}(q_1;V_N) \\ F_2^{(2)}(q_1,q_2;V_N) \\ 0 \\ \vdots \end{pmatrix};$$

$$F_1^{(2)}(q_1;V_N) = \frac{V_N^2 \int\limits_{V_N} \exp\left\{-\frac{1}{\theta}\Phi(q_1-q_2)\right\}dq_2}{\int\limits_{V_N}\int\limits_{V_N} \exp\left\{-\frac{1}{\theta}\Phi(q_1-q_2)\right\}dq_1\,dq_2},$$

$$F_2^{(2)}(q_1,q_2;V_N) = \frac{V_N^2 \exp\left\{-\frac{1}{\theta}\Phi(q_1-q_2)\right\}}{\int\limits_{V_N}\int\limits_{V_N} \exp\left\{-\frac{1}{\theta}\Phi(q_1-q_2)\right\}dq_1\,dq_2}. \tag{4.8}$$

Further, we have

$$\| F^{(2)} \| \leqslant \max\left\{\frac{1}{A}\,\frac{(V_N+J_1)V_N}{V_N(V_N+J_2)},\ \frac{V_N\,e^{2b}}{A^2V_N(V_N+J_2)}\right\}.$$

In relations (4.8), the following formulas are used:

$$\int\limits_{V_N} \exp\left\{-\frac{1}{\theta}\Phi(q_1-q_2)\right\}dq_2 =$$

$$= \int\limits_{V_N} \varphi_{q_1}(q_2)dq_2 + V_N = J_1 + V_N,$$

$$|J_1| \leqslant \sup_{q_1}\Big|\int\limits_{V_N} \varphi_{q_1}(q_2)dq_2 \leqslant J,$$

$$\int\limits_{V_N}\int\limits_{V_N} \exp\left\{-\frac{1}{\theta}\Phi(q_1-q_2)\right\}dq_1\,dq_2 =$$

$$= \int\limits_{V_N}\int\limits_{V_N} \varphi_{q_1}(q_2)dq_1dq_2 + V_N^2 = V_NJ_2 + V_N^2; \tag{4.9}$$

$$|J_2| \leqslant J.$$

It follows from relations (4.8) and (4.9) that, for sufficiently large N (what will be assumed below), the inequality $\| F^{(2)} \| < 1$ is valid.

From (4.7) with regard for $\| F^{(2)} \| < 1$, $\| F^{0(N-l)} \| < 1, \dots, \| F^{0(3)} \| < 1$, we obtain the estimate

$$\| F^{(N-l)} \| \leqslant \sum_{i=0}^{N-l-3}\prod_{j=0}^{i} \| K^{(N-l-j)} \| + 1 <$$

[3]The following is pertinent: the columns $F^{(N_1-l_1)}$ and $F^{(N_2-l_2)}$ for $N_1 \neq N_2$, $l_1 \neq l_2$, and $N_1 - l_1 = N_2 - l_2$ are different by definition.

$$< \left[1 - \frac{2e^{2b}\, e^{JA/v}}{A}\right]^{-1} \leqslant \frac{1}{1-k}. \tag{4.10}$$

It follows from Eq. (4.10) that the columns $F^{(N-l)}$ for $\frac{1}{v} < \frac{1}{2e^{2b+1}J}$ belong to the Banach space B, and their norms are bounded uniformly with respect to N and l.

We now examine the columns

$$F^l = \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{i+j}(v) K^i F^0 \tag{4.11}$$

which arise on the iteration of the relations [see (4.9)]

$$F^l = a_l(v) K F^{l+1} + a_l(v) F^0 \tag{4.12}$$

and, according to the estimate

$$\| F^l \| \leqslant \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{i+j}(v) \| K \|^i \| F^0 \| \leqslant \sum_{i=0}^{\infty} k^i =$$

$$= \frac{1}{1-k}, \tag{4.13}$$

belong to the space B for $\frac{1}{v} < \frac{1}{2e^{2b+1}J}$.

THEOREM III. *For any fixed l and $\frac{1}{v} < \frac{1}{2e^{2b+1}J}$, the sequence $\Psi^{(R-r)}(F^{(N-l)} - F^l)$ tends to zero in the space B as $N \to \infty$.*

Proof. We represent the difference $\Psi^{(R-r)}(F^{(N-l)} - F^l)$ in the form

$$\Psi^{(R-r)}(F^{(N-l)} - F^l) = \Psi^{(R-r)} \Big[K^{(N-l)} K^{(N-l-1)} \dots$$

$$\dots K^{(3)} F^{(2)} + \sum_{i=0}^{N-l-4} \prod_{j=0}^{i} K^{(N-l-j)} F^{0(N-l-i-1)} +$$

$$+ F^{0(N-l)} - \sum_{i=0}^{\infty} \prod_{j=0}^{i} a_{l+j}(v) K^i F^0 \Big] =$$

$$= \Psi^{(R-r)} \Big[\sum_{i=0}^{n} \prod_{j=0}^{i} K^{(N-l-j)} F^{0(N-l-i-1)} + F^{0(N-l)} -$$

$$- \sum_{i=0}^{n+1} \prod_{j=0}^{i} a_{l+j}(v) K^i F^0 \Big] + \eta(n). \tag{4.14}$$

The norm of the column $\eta(n)$ does not exceed $2k^n/1-k$ and can be made as small as possible for sufficiently large n. By the definition of the operators $K^{(N-l)}$ and K and the columns $F^{0(N-l)}$ and F^0, the equalities

$$\sum_{i=0}^{n} \prod_{j=0}^{i} K^{(N-l-j)} F^{0(N-l-i-1)} + F^{0(N-l)} =$$

$$= \sum_{i=0}^{n+1} \prod_{j=0}^{i} K^{(N-l-j)}_{[n+2]} F^{0(N-l-i-1)} + F^{0(N-l)},$$

$$\sum_{i=0}^{n} \prod_{j=0}^{i} a_{l+j}(v) K^i F^0 = \sum_{i=0}^{n+1} \prod_{j=0}^{i} a_{l+j}(v) K^i_{[n+2]} F^0 \tag{4.15}$$

are valid.

Inequalities (4.5) and (4.6) yield

$$\| \Psi^{(R-r)} \prod_{j=0}^{i-1} K^{(N-l-j)}_{[n+2]} - \Psi^{(R-r)} K^{(N-l)}_{[n+2]} \Psi^{(R-r+\frac{1}{n}r)} \times$$

$$\times K^{(N-l-1)}_{[n+2]} \dots \Psi^{(R-r+\frac{i-1}{n}r)} K^{(N-l-i+1)}_{[n+2]} \| \leqslant$$

$$\leqslant (i-1) k^{i-1} \varepsilon \left(\frac{1}{n} r_N,\, N \right),$$

$$\| \Psi^{(R-r)} \prod_{j=0}^{i-1} a_{l+j}(v) K^i_{[n+2]} -$$

$$- \Psi^{(R-r)} a_l(v) K_{[n+2]} \Psi^{(R-r+\frac{1}{n}r)} a_{l+1}(v) \times$$

$$\times K_{[n+2]} \dots \Psi^{(R-r+\frac{i-1}{n}r)} a_{l+i-1}(v) K_{[n+2]} \| \leqslant$$

$$\leqslant (i-1) k^{i-1} \varepsilon \left(\frac{1}{n} r_N, N \right),$$

$$\| \Psi^{R-r)} K^{(N-l)}_{[n+2]} \Psi^{(R-r+\frac{1}{n}r)} K^{(N-l-1)}_{[n+2]} \dots$$

$$\dots \Psi^{(R-r+\frac{i-1}{n}r)} K^{(N-l-i+1)}_{[n+2]} -$$

$$- \Psi^{(R-r)} a_l(v) K_{[n+2]} \Psi^{(R-r+\frac{1}{n}r)} a_{l+1}(v) K_{[n+2]} \dots$$

$$\dots \Psi^{(R-r+\frac{i-1}{n}r)} a_{l+i-1}(v) K_{[n+2]} \| \leqslant$$

$$\leqslant i k^{i-1} \varepsilon \left(\frac{1}{n} r_N, N \right). \tag{4.16}$$

From (4.16), we obtain the estimate

$$\| \Psi^{(R-r)} \Big(\prod_{j=0}^{i-1} a_{l+j}(v) K^i_{[n+2]} - \prod_{j=0}^{i-1} K^{(N-l-j)}_{[n+2]} \Big) \| \leqslant$$

$$\leqslant 3 i k^{i-1} \varepsilon \left(\frac{1}{n} r_N, N \right). \tag{4.17}$$

It is easy to verify the validity of the inequality

$$\| F^{0(N-l-j)} - a_{i+j}(v) F^0 \| \leqslant \varepsilon \left(\frac{1}{n} r_N, N \right). \tag{4.18}$$

On the basis of these inequalities, we obtain the estimate

$$\| \Psi^{(R-r)}\Big[\sum_{i=0}^{n}\prod_{j=0}^{i}K_{[n+2]}^{(N-l-j)}F^{0(N-l-i-1)}+F^{0(N-l)}-$$

$$-\sum_{i=0}^{n+1}\prod_{j=0}a_{i+j}(v)K_{[n+2]}^{i}F^{0}\Big] \|\leqslant 3\varepsilon\left(\frac{1}{n}r_N, N\right)\times$$

$$\times\sum_{i=0}^{n}(i+1)k^i+\varepsilon\left(\frac{1}{n}r_N, N\right)\sum_{i=0}^{n+1}k^i\leqslant 3\varepsilon\left(\frac{1}{n}r_N, N\right)\times$$

$$\times\frac{1}{(1-k)^2}+\varepsilon\left(\frac{1}{n}r_N, N\right)\frac{1}{1-k}\leqslant$$

$$\leqslant 4\varepsilon\left(\frac{1}{n}r_N, N\right)\frac{1}{(1-k)^2}. \tag{4.19}$$

Finally, we obtain the estimate

$$\| \Psi^{(R-r)}(F^{(N-l)}-F^l) \|\leqslant 4\varepsilon\left(\frac{1}{n}r_N, N\right)\times$$

$$\times\frac{1}{(1-k)^2}+2\,\frac{k^n}{1-k}=\varepsilon(r_N\, N). \tag{4.20}$$

By choosing sufficiently large n and N, we can make the right-hand side of inequality (4.20) as small as possible. This means that

$$\lim_{N\to\infty}\| \Psi^{(R-r)}(F^{(N-l)}-F^{\,l}) \|=0.$$

The theorem has been proved.

As a consequence of Theorem III, we obtain

$$\lim_{N\to\infty}\| \Psi^{(R-r)}(K^{(N-l)}F^{(N-l-1)}+F^{0(N-l)}-$$

$$-a_l(v)KF^{l+1}+a_l(v)F^0) \|=0.$$

5. Uniqueness of the Limiting Distribution Functions

1. In this section, we will establish the uniqueness of the limiting distribution function, i.e., we will prove that $F^l=F$ and

$$a_l(v)=a(v),\quad l=1,2,\ldots \tag{5.1}$$

THEOREM IV. *For sufficiently small $1/v$, the limiting distribution functions coincide, and equalities (5.1) are valid.*

Proof. According to (4.12), we have

$$F^l=a_l(v)KF^{l+1}+a_l(v)F^0,$$

$$F^{l+1}=a_{l+1}(v)KF^{l+2}+a_{l+1}(v)F^0. \tag{5.2}$$

We now consider the difference

$$F^l-F^{l+1}=a_l(v)K(F^{l+1}-F^{l+2})-(a_{l+1}(v)-$$

$$-a_l(v))KF^{l+2}+(a_l(v)-a_{l+1}(v))F^0. \tag{5.3}$$

This yields the inequality

$$|F^l-F^{l+1} \|\leqslant a_l(v)\| K \|\| F^{l+1}-F^{l+2} \|+$$

$$+|a_{l+1}(v)-a_l(v)|\ \| K \|\| F^{l+2} \|+|a_{l+1}(v)-a_l(v)|\times$$

$$\times\| F^0 \|\leqslant\frac{a_l(v)}{A}\,e^{2b}\,e^{(A/v)J}\,\| F^{l+1}-F^{l+2} \|+$$

$$+|a_{l+1}(v)-a_l(v)|\frac{1}{A}e^{2b}\,e^{(A/v)J}\,\| F^{l+2} \|+$$

$$+\frac{|a_{l+1}(v)-a_l(v)|}{A}. \tag{5.4}$$

Below, we will prove the estimate of such a kind for the difference $a_l(v)-a_{l+1}(v)$:

$$|a_l(v)-a_{l+1}(v)|\leqslant a_l(v)a_{l+1}(v)e^{(A/v)J}\,\| F^{l+1}-F^{l+2} \|. \tag{5.5}$$

With regard for (5.5), inequality (5.4) yields

$$\| F^l-F^{l+1} \|\leqslant\left[\frac{a_l(v)}{A}e^{2b}\,e^{(A/v)J}+a_l(v)a_{l+1}(v)\times\right.$$

$$\left.\times e^{(2A/v)J}\frac{1}{A}e^{2b}\,\| F^{l+2} \|+\frac{a_l(v)a_{l+1}(v)}{A}e^{(A/v)J}\right]\times$$

$$\times\| F^{l+1}-F^{l+2} \|=\delta(v)\,\| F^{l+1}-F^{l+2} \|. \tag{5.6}$$

For $\frac{1}{v}<\frac{1}{2J}$, the quantities $a_l(v)$ satisfy, by Lemma 1, the inequality

$$a_l(v)<2.$$

It follows from Eq. (4.13) that, for $\frac{1}{v}<\frac{1}{2Je^{2b+1}}$, the estimate

$$\| F^{l+2} \|\leqslant\frac{1}{1-\frac{2e^{2b}\,e^{(A/v)J}}{A}},\quad A>2e^{2b+1} \tag{5.7}$$

is valid.

By choosing a sufficiently large A, it is easy to prove that, for sufficiently small $1/v$, the quantity $\delta(v)$ in inequality (5.6) can be made less than unity. Inequality (5.6) yields

$$\| F^l-F^{l+1} \|\leqslant\delta^i(v)\,\| F^{l+1}-$$

$$-F^{l+i+1} \| \leqslant \delta^i(v) \frac{2}{1-\frac{2e^{2b}\,e^{(A/v)J}}{A}}. \quad (5.8)$$

Since $\delta(v) < 1$ and inequality (5.8) is valid for any i, the norm $\| F^l - F^{l+1} \|$ is as small as possible, i.e. $F^l = F^{l+1}$, and, in general, $F = F^1 = F^2 = \ldots = F^{l+1} = \ldots$.

It now follows from inequalities (5.5) that, for sufficiently small $1/v$, the values of $a_l(v)$ and $a_{l+1}(v)$ coincide; in general, we have that $a(v) = a_1(v) = a_2(v) = \ldots a_l(v) = \ldots$.

The theorem is proved.

It follows from Theorem IV that a column F satisfies the Kirkwood–Salzburg equation

$$F = a(v)KF + a(v)F^0.$$

2. By definition, the functions $F_s^{(N-l)}(q_1,\ldots,q_s;V_N)$ satisfy the relations

$$\frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N} F_s^{(N-l)}(q_1,\ldots,q_s;V_N)dq_1\ldots dq_s = 1. \quad (5.9)$$

We will go to the limit in this relation and prove that the equality

$$\lim_{N\to\infty}\frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N} F_s^l(q_1,\ldots,q_s;v)dq_1\ldots dq_s =$$

$$= \lim_{N\to\infty}\frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N} F_s^{(N-l)}(q_1,\ldots,s_s;V_N)\times$$

$$\times dq_1\ldots dq_s = 1 \quad (5.10)$$

is valid. To this end, we will examine the identity

$$\frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N}(F_s^l(q_1,\ldots,q_s;v)-$$

$$-F^{(N-l)}\times(q_1,\ldots,q_s;V_N))dq_1\ldots dq_s =$$

$$= \frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N}\Big[\Psi^{(R-r)}(F_s^l(q_1,\ldots,q_s;v)-$$

$$-F_s^{(N-l)}(q_1,\ldots,q_s;V_N)+(1-\Psi^{(R-r)})\times$$

$$\times(F_s^l(q_1,\ldots,q_s;v)-F_s^{(N-l)}(q_1,\ldots,q_s;V_N))\Big]\times$$

$$\times dq_1\ldots dq_s \quad (5.11)$$

and estimate separately the first and second terms on the right-hand side. For the second term on the right-hand side of Eq. (5.11), the estimate

$$\Big|\frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N}(1-\Psi^{(R-r)})(F_s^l(q_1,\ldots,q_s;v)-$$

$$-F_s^{(N-l)}(q_1,\ldots,q_s;V_N))dq_1\ldots dq_s\Big| \leqslant \frac{2}{1-k}\times$$

$$\times A^s s\frac{R_N^3-(R_N-r_N)^3}{R_N^3} = \frac{2}{1-k}A^s s\left[1-\left(1-\frac{r_N}{R_N}\right)^3\right]$$

is valid. It follows from this inequality that, as $N\to\infty$, the second term in (5.11) tends to zero. For the first term, the estimate

$$\Big|\frac{1}{V_N^s}\int\limits_{V_N}\ldots\int\limits_{V_N}\Psi^{(R-r)}(F_s^l(q_1,\ldots,q_s;v)-$$

$$-F_s^{(N-l)}(q_1,\ldots,q_s;V_N))dq_1\ldots dq_s\Big| \leqslant$$

$$\leqslant \frac{(R_N-r_N)^{3s}}{R_N^{3s}}A^s\tilde\varepsilon(r_N,N) =$$

$$= \left(1-\frac{r_N}{R_N}\right)^{3s}A^s\tilde\varepsilon(r_N,N); \quad \lim_{N\to\infty}\tilde\varepsilon(r_N,N)=0,$$

is valid.

This proves formula (5.10).

It is easy to see that the functions $F_s^l(q_1,\ldots,q_s;v)$ are translation-invariant. Therefore, the functions $F_1^l(q_1;v)$ are invariable.

It follows from relation (5.10) that $F_1^l(q_1;v)=1,\ l\geqslant 0$.

We now pass to the proof of inequalities (5.5). It follows from (4.12) that the formula

$$\frac{1}{a_l(v)} = 1+\sum_{k=1}^{\infty}\frac{1}{k!v^k}\times$$

$$\times\int\ldots\int F_k^{l+1}(q_1^*,\ldots,q_k^*;v)\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*$$

is valid.

We now use the formula

$$a_l(v)-a_{l+1}(v) = a_l(v)a_{l+1}(v)\left(\frac{1}{a_{l+1}(v)}-\frac{1}{a_l(v)}\right) =$$

$$= a_l(v)a_{l+1}(v)\sum_{k=1}^{\infty}\frac{1}{k!\,v^k}\int\ldots\int[F_k^{l+2}(q_1^*,\ldots,g_k^*;v)-$$

$$-F_k^{l+1}(q_1^*,\ldots,q_k^*;v)]\prod_{i=1}^{k}\varphi_{q_1}(q_i^*)dq_1^*\ldots dq_k^*.$$

Whence we obtain estimate (5.5):

$$|a_l(v)-a_{l+1}(v)| \leqslant a_l(v)a_{l+1}(v)e^{J(A/v)}\,\|\,F^{l+1}-F^{l+2}\,\|\,.$$

3. We now prove that, for the sequence $a_N(V_N)$, there exists a unique limiting point. Indeed, if there would exist two convergent subsequences with limits $a(v)$ and $a^1(v)$, the corresponding columns would be F and F^1 which would satisfy the relations

$$F = a(v)KF + a(v)F^0, \quad F^1 = a^1(v)KF^1 + a^1(v)F^0$$

because of Theorem IV. Taking into account that $F_1(q; v) = 1$ and $F_1^1(q; v) = 1$, we can establish the estimate

$$|a(v) - a^1(v)| \leqslant a(v)a^1(v)e^{JA/r} \parallel F - F^1 \parallel .$$

Hence, like the case of Theorem IV, we prove the validity of the inequality

$$\parallel F - F^1 \parallel \leqslant \delta(v) \parallel F - F^1 \parallel,$$

where $\delta(v) < 1$ for sufficiently small $1/v$. It follows from relations (5.13) and (5.12) that $F = F^1$ and $a(v) = a^1(v)$, respectively. Taking this result into account, we obtain finally from (5.10) that

$$\lim_{N\to\infty} \frac{1}{V_N^s} \int\limits_{V_N} \ldots \int\limits_{V_N} F_s^{(N-l)}(q_1, \ldots, q_s; V_N) dq_1 \ldots dq_s =$$

$$\lim_{N\to\infty} \frac{1}{V_N^s} \int\limits_{V_N} \ldots \int\limits_{V_N} F_s(q_1, \ldots, q_s; v) dq_1 \ldots dq_s = 1.$$

In view of the uniqueness of the limiting distribution functions, Theorem III can now be formulated as follows. For $\frac{1}{v} < \frac{1}{2e^{2b+1}}$ at any fixed $l \geqslant 0$, the sequence of the functions $\psi^{(R-r)}(q_1) \ldots \psi^{(R-r)}(q_s)$ $(F_s^{(N-l)}(q_1, \ldots, q_s; V_N) - F_s(q_1, \ldots q_s, v))$ tends to zero uniformly with respect to $q_1, \ldots q_s$ as $N \to \infty$, and the estimate

$$|\psi^{(R-r)}(q_1) \ldots \psi^{(R-r)}(q_s)(F_s^{(N-l)}(q_1, \ldots, q_s; V_N) -$$

$$-F_s(q_1, \ldots q_s; v))| \leqslant A^s \tilde{\varepsilon}(r_N, N), \quad A = 2e^{2b+1}, \quad s \geqslant 1,$$

is valid. Moreover, the quantity $\tilde{\varepsilon}(r_N, N)$ depends on l, and $\tilde{\varepsilon}(r_N, N) \to 0$ as $N \to \infty$, generally speaking, nonuniformly with respect to l. It follows from the results of Section 2 that the limiting distribution functions $F_s(q_1, \ldots, q_s; v)$, $s \geqslant 1$ are holomorphic functions of the density $1/v$ in some neighborhood of zero.

1. N.N. Bogolyubov, *Problems of the Dynamical Theory in Statistical Physics* (Gostekhizdat, Moscow, 1946) (in Russian).
2. N.N. Bogolyubov and B.I. Khatset, Dokl. Akad. Nauk USSR **66**, 321 (1949).
3. B.I. Khatset, Nauk. Zapysk. Zhytom. Pedag. Inst., Ser. Fiz.-Mat. **3**, 113, 139 (1956).
4. D. Ruelle, Ann. of Phys. **25**, 209 (1963).
5. T.L. Hill, Statistical Mechanics (McGraw-Hill, New York, 1956).

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BOGOLYUBOV MYKOLA MYKOLAIOVYCH (21.08.1909–13.02.1992)

M.M. Bogolyubov was the outstanding physicist-theorist and mathematician. From 1928, he worked at the Nat. Acad. of Sci. of Ukraine, 1936-1959 – Professor of Kyiv State University, from 1950 – Professor of Moscow State University, from 1956 – Head of the Laboratory of Theoretical Physics of JINR (Dubna), and from 1965 – Director of JINR, 1965–1973 – the founder and the first Director of Institute for Theoretical Physics of the NAS of Ukraine. Academician of the Acad. of Sci. of UkrSSR (1948), Academician of the Acad. of Sci. of SSSR, twice Hero of Socialist Labor (1969, 1979), winner of the Lenin's Prize (1958) and three State prizes of SSSR (1947, 1953, 1984), awarded by M.V. Lomonosov Gold (1985).

His studies are related to statistical physics, quantum field theory, theory of elementary particles, and mathematical physics. Together with M.M. Krylov, M.M. Bogolyubov developed (1932–1937) the asymptotic theory of nonlinear oscillations, proposed the methods of asymptotic integration of nonlinear equations describing various oscillatory processes and gave their mathematical substantiation. He advanced the idea (1945) of the hierarchy of relaxation times, which has important meaning in the statistical theory of irreversible; proposed (1946) the efficient method of a chain of equations for the distribution functions of complexes of particles; and constructed (1946) the microscopic theory of superfluidity which was based on the model of weakly nonideal Bose-gas. Already in 10 years, by using the quantum-mechanical model of electron gas interacting with the ion lattice of a metal, M.M. Bogolyubov generalized the own apparatus of canonical transformations used in the theory of superfluidity and developed the microscopic theory of superconductivity. Turning to the problems of quantum field theory, he gave (1954–1955) the first version of an axiomatic construction of the scattering matrix based on the original condition for causality; proposed a mathematically correct version of the theory of renormalization with the use of the apparatus of distributions and introduced the so-called "R-operation" (1955, together with O.S. Parasiuk); developed the regular method of refinement of quantum-field solutions – the method of renormalization group (1965, together with D.V. Shirkov); and gave a strong proof of the dispersion relations in the theory of strong interactions (1955–1956); proposed a method of description of the systems with spontaneously broken symmetry which was named the method of quasiaverages (1960–1961); and, by studying the problems of symmetry and dynamics within the quark model of hadrons, introduced (1965, together with B.V. Struminsky and A.N. Tavkhelidze) the notion of a new quantum number "color". His main results are presented in the monographs [1–5].

1. N.N. Bogolyubov, *Problems of the Dynamical Theory in Statistical Physics* (Gostekhizdat, Moscow, 1946) (in Russian).

2. N.N. Bogolyubov, B.V. Medvedev, and M.K. Polivanov, *Questions of the Theory of Dispersion Relations* (Fizmatgiz, Moscow, 1958) (in Russian).
3. N.N. Bogolyubov, V.V. Tolmachev, and D.V. Shirkov, *A New Method in the Theory of Superconductivity* (Izd. AN SSSR, Moscow, 1958) (in Russian).
4. N.N. Bogolyubov, A.A. Logunov, and I.T. Todorov, *Foundations of the Axiomatic Approach in Quantum Field Theory* (Nauka, Moscow, 1969) (in Russian).
5. N.N. Bogolyubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Nauka, Moscow, 1976) (in Russian).

PETRINA DMYTRO YAKOVYCH (23.03.1934–20.06.2006)

Professor Dmytro Yakovych Petrina, Academician of the National Academy of Sciences of the Ukraine, Doctor of Physical and Mathematical Sciences, and Head of the Department of Mathematical Methods in Statistical Mechanics at the Institute of Mathematics of the National Academy of Sciences of the Ukraine.

D.Ya. Petrina carried out intensive investigations in many fields of contemporary mathematical physics. He obtained many profound results in constructive quantum field theory, the theory of analytic scattering matrix, classical and quantum statistical mechanics, the theory of boundary-value problems in domains with complicated structure and its applications to the theory of membranes, and in solving different models in statistical mechanics.

Among his fundamental results, one should especially mention the theorem on the impossibility of nonlocal quantum field theory with a positive spectrum of the energy-momentum operator and the criteria of the validity of spectral representations of scattering amplitudes.

D.Ya. Petrina deduced and studied equations for the coefficient functions of the scattering matrix and suggested to use the methods of equilibrium statistical mechanics in the Euclidean field theory. In statistical mechanics, together with N.N. Bogolyubov, D.Ya. Petrina established the fundamental theorem on the existence of the thermodynamic limit for equilibrium states in the canonical ensemble. He developed a new approach to the Bogolyubov equations as evolutionary equations, constructed the evolution operator in the explicit form, proved the existence of the thermodynamic limit for nonequilibrium states, and gave a mathematically rigorous derivation of the Boltzmann equation in the Boltzmann-Grad limit.

N.M. Krylov's prize in mathematics was awarded to a series of Prof. Petrina's works in 1984.