ON A SUBTRACTION FORMALISM FOR THE MULTIPLICATION OF CAUSAL SINGULAR FUNCTIONS

Translated and reprinted from Dokl. Akad. Nauk SSSR **100**, No. 3, pp. 429–432 (1955)

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Starting from representation (2) in [1] for the causal singular functions, one can easily derive the momentum representation of a product of regularized causal functions corresponding to a connected graph $G(x_1, x_2, \dots, x_n)$ in the form

$$I_M(\dots k \dots) = \delta\left(\sum k\right) \int\limits_0^\infty \dots d\alpha_l \dots f(\dots k \dots \alpha) \times$$

$$\times \exp\left[-i \sum \alpha_l m_l^2 - \varepsilon \sum \alpha_l\right] \prod I(\alpha_l), \qquad (1)$$

where

$$f(\dots k \dots \alpha) = F(\dots k \dots \alpha) \exp\left[i \sum_{a,b} A_{a,b}(\alpha) k_a k_b\right];$$

$\sum A_{ab} k_a k_b$ is a quadratic form, $F(\dots k \dots \alpha)$ is a rational function of α_l which possesses poles at $\alpha_l = 0$. These poles are cancelled, however, by zeros of the functions $I(\alpha_l)$. We have $I(\alpha_l) \to 1$ as $M \to \infty$, and the integrals over α_l diverge as $\alpha_l \to 0$, which corresponds to the divergence for large momenta.

As follows from the theorems given below, the expression $\overline{R(G)}$ is free of this disadvantage.

Theorem 1. *The expression* $\overline{R(G_1 G_2 \dots G_s)}$ *is a function of the form*

$$\overline{R(G_1 G_2 \dots G_s)} =$$

$$= \delta\left(\sum k\right) \int\limits_0^\infty \dots d\alpha_l \dots \exp\left[-i \sum \alpha_l m_l^2 - \varepsilon \sum \alpha_l\right] \times$$

$$\times \prod_l I(\alpha_l) \frac{f(\dots \sqrt{\bar\alpha} k \dots \sqrt{\bar\alpha} \omega \dots \sqrt{\bar\alpha})}{\Phi_r(\dots \sqrt{\alpha_l} \dots)} \times$$

$$\times \exp\left[i \sum A_{G_a G_b} p_a p_b\right], \qquad (2)$$

where $p_a = \sum\limits_{G_a} k$ *or* $p_a = \sum\limits_{G_a} \omega$, $\bar\alpha$ *is a homogeneous rational function of the first power in* α_l, $f(\dots \sqrt{\bar\alpha} k \dots \sqrt{\bar\alpha} \omega \dots \sqrt{\bar\alpha})$ *is a polynomial in all its arguments,* $\Phi_r(\dots \sqrt{\alpha_l} \dots)$ *is a homogeneous rational function of* $\sqrt{\alpha_l}$ *of the power* r,

$$r = \sum r[\Delta(G_a)] + \sum r_l + 4L - 4(s-1),$$

r_l *is the power of a polynomial of the propagator corresponding to a line* l, $r[\Delta(G_a)]$ *is the power of the polynomial* $\Delta(G_a)$, L *is the total number of lines connecting the basis vertices,* s *is the total number of basis vertices, and* $\sum A_{G_a G_b}(\alpha) x_a x_b$ *is a homogeneous rational function of the first power in* α_l *that possesses the following properties: a) if all* $\alpha_l > 0$, *the quadratic form in* x *is positive definite and b)* $\sum A_{G_a G_b}(\alpha) x_a x_b \leqslant \sum\limits_l \alpha_l (\sum\limits_a |x_a|)^2$.

Theorem 2. *Let* $G = G_1 G_2 \dots G_s$ *be a connected graph, let* $r[\Delta(G_i)]$ *be the power of the polynomial* $\Delta(G_i)$, *and let* ν_l *be the power of a polynomial of the propagator corresponding to the line* l *connecting* G_i *and* G_k.

Consider the numbers

$$\bar r\,[\Delta(G_i)] \geqslant r[\Delta(G_i)]; \quad \bar\nu_l \geqslant \nu_l.$$

For the generalized vertices Γ_α, *we set*

$$\nu\,(\Gamma_\alpha) = \sum \bar r\,[\Delta(G_k)] + \sum \bar r_l + 2L - 4(s-1), \qquad (3)$$

Here, the summation is executed over those G_k *which constitute* Γ_α *and those* l *which connect these* G_k, s *is the number of elementary vertices in* Γ_α, *and* L *is the number of lines in* Γ_α.

With $\nu(\Gamma_\alpha)$ defined in this manner, we introduce the operation

$$R(G_1G_2\dots G_s|\dots\Omega_\Gamma) =$$

$$= \delta\left(\sum k\right)\int\limits_0^\infty \dots d\alpha_l \dots \exp\left[-i\sum \alpha_l m_l^2 - \varepsilon\sum\alpha_l\right]\times$$

$$\times \prod I(\alpha_l)F(\dots k\dots\omega\dots\alpha)\exp\left[i\sum A_{G_aG_b}p_ap_b\right]. \quad (4)$$

Then the function $F(\dots k\dots\omega\dots\alpha)$, which is a polynomial in k and ω and a rational function of α_l, satisfies the inequality

$$|F(\dots k\dots\omega\dots\alpha)| \leqslant \frac{C(\dots k\dots\omega\dots\alpha)}{\prod\limits_l \alpha_l^{1-1/2L}}, \quad (5)$$

where $C(\dots k\dots\omega\dots\alpha)$ is the polynomially bounded expression, and L is the total number of lines of the graph $G = G_1\times G_2\times\dots\times G_s$.

Theorems 1 and 2 can be proved by induction, basing on the iteration formula (7) in [1]. Here, we give the main points of the proof of Theorem 2.

Proof. The case of a graph consisting of two basis vertices and m lines can be verified directly. Let the theorem be valid for all connected graphs with the numbers of lines and vertices not greater than $m \geqslant s-1$ and s, respectively. We now prove it for a graph with the number of lines $L = m+1$.

Consider a graph with s vertices and $m+1$ lines. Two cases are possible: either removing one line violates the connectedness of $G = G_1G_2\dots G_s$ or does not. The first case is simpler, and we do not consider it for brevity; we just mention that considering this case automatically realizes the induction on the number of vertices.

To consider the second case, we employ the iteration formula (7) in [1] by assuming that just the influence of line 1–2 is studied. One can easily show that the induction assumptions are valid not only for $R(G_1G_2\dots G_s)$ but also for $R_{12}(G_1G_2\dots G_s)$. This yields the inequality

$$|F^*(\dots k\dots\omega\dots\alpha)| \leqslant \frac{K}{\prod\limits_l^* \alpha_l^{1-1/2L}}\frac{1}{\alpha_{12}^d}, \quad (6)$$

where d is a positive number, and the asterisk means that F is taken with regard for the new added line.

Since we could remove any line without violating the connectedness, we remove the line with the largest α_l. For this line, we have

$$\alpha_{ab} \geqslant \frac{1}{L}\sum_l \alpha_l.$$

So we obtain

$$|F^*(\dots k\dots\omega\dots\alpha)|\prod_l{}^*\alpha_l^{1-1/2L} \leqslant \frac{K^*(\dots k\dots\omega\dots\alpha)}{(\sum\limits_l \alpha_l)^d}. \quad (7)$$

We now note that

$$R^*(G_1G_2\dots G_s) = [1-M(G)]\overline{R^*(G_1G_2\dots G_s)}, \quad (8)$$

i.e., R^* is the remainder of the Taylor series of $\bar{R}^*$ in powers of $(k-\omega)$ of the $\nu(G)$-order:

$$R^*(G_1G_2\dots G_s) = \frac{1}{\nu!}\int\limits_0^1 (1-\tau)^\nu \frac{\partial^{\nu+1}}{\partial\tau^{\nu+1}}\times$$

$$\times\overline{R^*(G_1^\tau G_2^\tau\dots G_s^\tau)}\,d\tau, \quad (9)$$

where the superscript τ indicates that $\omega+\tau(k-\omega)$ should be substituted for k.

By virtue of Theorem 1, this obviously yields

$$F^*(\dots k\dots\omega\dots\alpha) = \frac{f^*(\dots\sqrt{\bar\alpha}k\dots\sqrt{\bar\alpha}\omega\dots\sqrt{\bar\alpha})}{\Phi^*_{r-(\nu+1)}(\dots\sqrt{\alpha_l}\dots)}. \quad (10)$$

We now set

$$t = \sum\alpha_l; \quad \alpha'_l = \frac{1}{t}\alpha_l; \quad \sum\alpha'_l = 1.$$

By virtue of properties of the functions f^* and Φ^*, relation (7) yields

$$\left|\frac{f^*(\dots\sqrt{\bar\alpha'}k\dots\sqrt{\bar\alpha'}\omega\dots t\sqrt{\bar\alpha'})}{\Phi^*_{r-(\nu+1)}(\dots\sqrt{\alpha'_l}\dots)}\right|\prod_i{}^*{\alpha'_l}^{1-1/2L} \leqslant$$

$$\leqslant \frac{K^*(\dots k\dots\omega\dots\alpha)}{t^{d-s}}, \quad (11)$$

where $s = \frac{\nu+1-r+2L-1}{2} > 0$.

We note that the left-hand side is a polynomial in t which is bounded by some number N in the interval, say, from 1 to 2. Therefore, all coefficients of the polynomial are bounded by the numbers depending only on N and the power of the polynomial. Whence we can easily prove that

$$|F^*(\ldots k \ldots \omega \ldots \alpha)| \prod_i {}^* \alpha_l^{1-1/2l} \leqslant C^*(\ldots k \ldots \omega \ldots \alpha). \tag{12}$$

By virtue of the proved estimate (5), we can pass to the limit $M \to \infty$ and $I(\alpha_l) \to 1$ in the expression for $R(G)$, i.e., the regularization can be removed. Then we obtain

$$R(G) = \delta\left(\sum k\right) \int\limits_0^\infty \ldots d\alpha_l \ldots \times$$

$$\times \exp\left[-i \sum \alpha_l m_l^2 - \varepsilon \sum \alpha_l\right] f(\ldots k \ldots \omega \ldots \alpha) \equiv$$

$$= \delta\left(\sum k\right) I(k), \tag{13}$$

where

$$f(\ldots k \ldots \omega \ldots \alpha) = F(\ldots k \ldots \omega \ldots \alpha) \times$$

$$\times \exp\left[i \sum A_{ab}(\alpha) k_a k_b\right]. \tag{14}$$

Taking for simplicity that all masses m_l are positive, we can easily show that the function I_k is analytic for sufficiently small k. This fact confirms the possibility of Taylor expansions needed to define the operations $\Delta(\Gamma)$. For reasons involving the Lorentz invariance, these expansions should be performed near the points $\omega = 0$.

As $\varepsilon \to 0$, the limit of $R(G)$ exists in the usual sense for small k and only in the improper sense for large k.

The above-obtained results together with more profound causality-based reasons allow one to consider the functional $R = \lim\limits_{\varepsilon \to 0} R(G)$, where the limit should be understood in the improper sense, as the momentum image of the product of causal singular functions which corresponds to a connected graph G.

At last, we note that it is expedient to set $\Delta(\Gamma)$ only to within certain polynomials; after that, R will depend on arbitrary constants.

1. N.N. Bogolyubov and O.S. Parasiuk, Dokl. Akad. Nauk SSSR **100**, 25 (1954).

Received October 29, 1954

BOGOLYUBOV MYKOLA MYKOLAIOVYCH (21.08.1909–13.02.1992)

M.M. Bogolyubov was the outstanding physicist-theorist and mathematician. From 1928, he worked at the Nat. Acad. of Sci. of Ukraine, 1936-1959 – Professor of Kyiv State University, from 1950 – Professor of Moscow State University, from 1956 – Head of the Laboratory of Theoretical Physics of JINR (Dubna), and from 1965 – Director of JINR, 1965–1973 – the founder and the first Director of Institute for Theoretical Physics of the NAS of Ukraine. Academician of the Acad. of Sci. of UkrSSR (1948), Academician of the Acad. of Sci. of SSSR, twice Hero of Socialist Labor (1969, 1979), winner of the Lenin's Prize (1958) and three State prizes of SSSR (1947, 1953, 1984), awarded by M.V. Lomonosov Gold (1985).

His studies are related to statistical physics, quantum field theory, theory of elementary particles, and mathematical physics. Together with M.M. Krylov, M.M. Bogolyubov developed (1932–1937) the asymptotic theory of nonlinear oscillations, proposed the methods of asymptotic integration of nonlinear equations describing various oscillatory processes and gave their mathematical substantiation. He advanced the idea (1945) of the hierarchy of relaxation times, which has important meaning in the statistical theory of irreversible; proposed (1946) the efficient method of a chain of equations for the distribution functions of complexes of particles; and constructed (1946) the microscopic theory of superfluidity which was based on the model of weakly nonideal Bose-gas. Already in 10 years, by using the quantum-mechanical model of electron gas interacting with the ion lattice of a metal, M.M. Bogolyubov generalized the own apparatus of canonical transformations used in the theory of superfluidity and developed the microscopic theory of superconductivity. Turning to the problems of quantum field theory, he gave (1954–1955) the first version of an axiomatic construction of the scattering matrix based on the original condition for causality; proposed a mathematically correct version of the theory of renormalization with the use of the apparatus of distributions and introduced the so-called "R-operation" (1955, together with O.S. Parasiuk); developed the regular method of refinement of quantum-field solutions – the method of renormalization group (1965, together with D.V. Shirkov); and gave a strong proof of the dispersion relations in the theory of strong interactions (1955–1956); proposed a method of description of the systems with spontaneously broken symmetry which was named the method of quasiaverages (1960–1961); and, by studying the problems of symmetry and dynamics within the quark model of hadrons, introduced (1965, together with B.V. Struminsky and A.N. Tavkhelidze) the notion of a new quantum number "color". His main results are presented in the monographs [1–5].

1. N.N. Bogolyubov, *Problems of the Dynamical Theory in Statistical Physics* (Gostekhizdat, Moscow, 1946) (in Russian).
2. N.N. Bogolyubov, B.V. Medvedev, and M.K. Polivanov, *Questions of the Theory of Dispersion Relations* (Fizmatgiz, Moscow, 1958) (in Russian).
3. N.N. Bogolyubov, V.V. Tolmachev, and D.V. Shirkov, *A New Method in the Theory of Superconductivity* (Izd. AN SSSR, Moscow, 1958) (in Russian).
4. N.N. Bogolyubov, A.A. Logunov, and I.T. Todorov, *Foundations of the Axiomatic Approach in Quantum Field Theory* (Nauka, Moscow, 1969) (in Russian).
5. N.N. Bogolyubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Nauka, Moscow, 1976) (in Russian).

PARASIUK OSTAP STEPANOVYCH (20.12.1921–20.11.2007)

O.S. Parasiuk was the outstanding scientist in the field of theoretical and mathematical physics, Academician of the NAS of Ukraine (1964), Honored Worker in Science and Technique of Ukraine (1992), winner of Krylov's (1982) and Bogoliubov's (1996) prizes of the NAS of Ukraine. He graduated from L'viv University in 1947. From 1956 till 1966, O.S. Parasiuk worked at the Institute of Mathematics of the NAS of Ukraine, and then at Bogoliubov Institute for Theoretical Physics in 1966–2007.

The scientific works of O.S. Parasiuk are devoted to mathematical physics, in particular, to problems of the theory of elasticity and plasticity, the theory of dynamical systems, and probability theory. His most fundamental results concern the applications of methods of functional analysis to the theory of quantized fields (multiplication of causal functions, regularization of divergent integrals, and analytic continuation of distributions) and the analytic properties of scattering amplitudes. In 1955-1960, O.S. Parasiuk together with Academician M.M. Bogoliubov solved the problem of the elimination of ultraviolet divergences in quantum field theory and proved the theorem on the renormalizability of quantum elecrodynamics in any order of perturbation theory. Based on the R-operation and the Bogoliubov-Parasiuk theorem, these results became classical and are the foundation of contemporary quantum field theory. They led to the conception of renormalizability which is the principle playing the important role in quantum field theory. Since the appearance of the theory of R-operation, the theory of renormalizations obtained a new trend of development. As a result, new nontrivial efficient schemes of renormalizations in quantum field theory have been developed. Recently, there arises a new modern approach based on the combination of the combinatorial analysis of a structure of Feynman's diagrams and the methods of the theory of Hopf algebras.

The most important publications:

1. O.S. Parasiuk, Dokl. Akad. Nauk SSSR **63**, 367 (1948).
2. O.S. Parasiuk, Uspekhi Mat. Nauk **3(55)**, 125 (1953).
3. N.N. Bogoliubov and O.S. Parasiuk, Dokl. Akad. Nauk SSSR **109**, 717 (1955).
4. O.S. Parasiuk, Dokl. Akad. Nauk SSSR **110**, 957 (1956).
5. N.N. Bogoliubov and O.S. Parasiuk, Izv. AN SSSR, Ser. Mat. **20**, 585 (1956).
6. O.S. Parasiuk, Izv. AN SSSR, Ser. Mat. **20**, 843 (1956).
7. O.S. Parasiuk, Distributions in field theory, in Proceedings of the 3-rd All-Union Mathematical Congress (Izd. AN SSSR, Moscow, 1956), vol. 2, p. 558 (in Russian).
8. N.N. Bogoliubov and O.S. Parasiuk, Acta Mathematica **97**, 227 (1957).
9. O.S. Parasiuk, Ukr. Mat. Zh. **12**, 287 (1960).
10. O.S. Parasiuk, Ukr. Mat. Zh. **14**, 215 (1962).