MEAN-FIELD APPROACH TO NONEQUILIBRIUM PHASE TRANSITIONS IN SYSTEMS WITH INTERNAL AND EXTERNAL MULTIPLICATIVE NOISES

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Noise-induced phase transitions in systems with conserved and nonconserved dynamics with both internal and external multiplicative fluctuations are considered. On the basis of the mean-field analysis, the reversible course of the ordering on a change of the internal noise intensity is revealed. With increase in the external noise intensity, a system moves to an ordered state. It is shown that internal and external fluctuations render opposite statistical actions.

1. Introduction

The contemporary development of the statistical theory of complicated systems requires the comprehensive investigation of the influence of a nonequilibrium medium, in which the system under study is positioned. For two last decades, a wide circle of nonequilibrium mechanisms, which can essentially change the states of physical systems and induce the processes of ordering of various types, is revealed [1]. Basically important is the problem to clarify the role of fluctuating sources which are generated not only by the internal processes running in the system itself (internal noises), but also to study the action of the nonequilibrium fluctuating medium (the external noise). Such problems are urgent not only in statistical physics [2]. They arise naturally in the physics of lasers and electronics [3–5], the investigations of the action of radiation on the structure of materials [6], the solid-state physics on the description of the reconstruction of a defective structure [7, 8], chemistry, biology [9], etc.

After the discovery of the governing role of a fluctuating medium, the powerful progress in the statistical physics of nonequilibrium processes stimulated the reevaluation of fluctuations as a desorganizing factor. Beginning from the 1980s, it has been proved that, by controlling the properties of stochastic sources, the system can be transferred to a state unattainable in the deterministic case [2,3]. Despite the nondecreasing interest in the study of a contribution of fluctuating forces to the processes of self-organization, till now there exists a large growing circle of problems, where the fluctuations basically change the behavior of nonlinear systems. In the present work, we consider one of such problems, where the internal noises, which were not even considered at all, can significantly vary the states of a system. We will consider both the influence of spatial correlations of two above-mentioned types of stochastic sources and their joint action.

It is known that, from the theoretical viewpoint, the physical systems are separated into relaxation-involved systems (models of classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$), liquid ($\mathcal{D}$), symmetric and asymmetric planar magnetic systems ($\mathcal{E}$ and $\mathcal{F}$, respectively), and isotropic antiferromagnetic and ferromagnetic ($\mathcal{G}$ and $\mathcal{H}$) ones [10]. In the present work, we will separately analyze two classes of stochastic models, where the physical field $x(\mathbf{r}, t)$ is not conserved (model $\mathcal{A}$: $\int \mathrm{d}\mathbf{r} x(\mathbf{r}, t) \neq \mathrm{const}$) and is conserved (model $\mathcal{B}$: $\int \mathrm{d}\mathbf{r} x(\mathbf{r}, t) = \mathrm{const}$). The first class of models concerns systems of the magnetic type. The effect of external fluctuations on the course of ordering in such systems is quite known (see [2] and references therein). However, the contribution of the internal noise, whose intensity depends on the field variable (multiplicative noise), as well as the joint action of such two noises, remains insufficiently understandable. As for model $\mathcal{B}$ which describes the processes of phase stratification, it was comprehensively analyzed in the deterministic and stochastic cases only for the additive (with field-independent intensity) internal noise. We will show that the role of the internal multiplicative noise is critical for such a class of models.

The purpose of this work is the detailed study of the influence of two multiplicative (internal and external) noises which are realized in the presence of nonvanishing fluctuations on the transition to the ordered state and under the fluctuation character of the nonequilibrium medium. We will also consider the influence of the correlation characteristics of fluctuations on the character of reversible phase transitions. First of all, the mentioned situation and the posed problems arise in the study of polymers and their compounds [11]. Moreover, in view of the experimental data on the behavior of Seignette salt, we believe that the results of the present work will find applications to magnetic systems [12]. In addition, the proposed model can be used in the description of the processes of phase stratification [13], decay of solid solutions at the radiation-involved processing of materials, *etc.*

The analysis carried out in the present work is based on the mean-field theory which allows one to adequately predict the main modes of a behavior of the system. The method of our studies is standard. First, we develop a necessary formalism; then we will perform the linear analysis of the stability, apply the mean-field theory for finite values of the parameter of spatial interaction, and finally pass to the macroscopic approximation. We will show that the internal multiplicative noise can lead to the reversible pattern of ordering and, moreover, that the joint action of noncorrelated noises of two types causes also the reversible behavior of the order parameter, despite the fact that stochastic sources act oppositely to each other. Separately, we will study the influence of spatial autocorrelations of the external noise on the character of the ordering.

The structure of the present work is as follows. In Section 2, we present the models of stochastic systems and basic approximations. Section 3 gives the mean-field theory and its development for systems of the classes $\mathcal{A}$ and $\mathcal{B}$. The results of studies of the influence of the internal noise and two stochastic sources are presented in Section 4. Section 5 includes the results based on the macroscopic approximation. The conclusions concerning the results obtained are presented in Section 6.

2. Stochastic Models with Relaxation Flows

The dynamics of a system is determined by a behavior of the collective variable/variables $x(\mathbf{r},t)$ in the presence of the Lyapunov functional which sets, in many cases, the functional of free energy $\mathcal{F}[x]$ for statistical/thermodynamical systems [7]. On the relaxation of the system to the state of a thermodynamical equilibrium, the evolution equations for the field $x(\mathbf{r},t)$ take the form $\partial x/\partial t = -\Gamma\delta\mathcal{F}[x]/\delta x$, where the kinetic coefficient Γ is related in the simplest cases to the dissipation at Γ =const (model $\mathcal{A}$) and to the diffusion at $\Gamma = -\mathrm{const}\nabla^2$ (model $\mathcal{B}$). The equilibrium state is set by the global minimum of the functional of free energy; the corresponding condition is $\delta\mathcal{F}[x]/\delta x = 0$. It is worth noting that, for such systems, the relaxation occurs along the steepest descent lines of a local potential. Such dynamical equation describes, in fact, the relaxation to local minima of the free energy $\mathcal{F}[x]$, where the system can remain forever, if it does not undergo the action of forces which can transfer it to a global minimum. In order to prevent the presence of the system in a metastable state, we consider the fluctuations $\xi(\mathbf{r},t)$ which are Gaussian in the simplest case, describe the thermostat, and transfer the system in the equilibrium state for a certain time. Such fluctuations are added to the evolution equations for the field, and, instead of the deterministic equation, we have the stochastic one $\partial x/\partial t = -\Gamma\delta\mathcal{F}[x]/\delta x + \xi(\mathbf{r},t)$. Statistical properties of the noise are as follows: $\langle\xi(\mathbf{r},t)\rangle = 0$, $\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle = 2\Gamma\sigma^2\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, where the presence of the delta-singularity in the correlator means that the noise $\xi(\mathbf{r},t)$ is white in space and time; and σ^2 is the intensity. The presence of the kinetic coefficient in the correlator indicates that the noise is internal, which is related to the fulfillment of the fluctuation-dissipation theorem. As follows from the standard approach, the equilibrium distribution of the field x has the Boltzmann form $\mathcal{P}[x] \propto \exp(-\mathcal{F}[x]/\sigma^2)$ and is independent of the choice of the kinetic coefficient [10].

A separate place in the theory of dynamical systems, synergetics, and statistical physics is occupied by the systems with relaxation along the trajectories which do not coincide with steepest descent lines. In such systems, it is possible to form temporal, spatial, and spatio-temporal structures. Such models are characterized by the functional dependence of the kinetic coefficient on the field $x(\mathbf{r},t)$ or, in the general case, on the spatial derivatives of this field. In the frame of the standard formalism, the kinetic coefficient can be considered a function or an operator of the type $\Gamma = \mathcal{M}(x,\nabla)$ and be determined as the mobility in different phases corresponding to different values of the field $x(\mathbf{r},t\to\infty)$. For such systems, basically important is the fact that the fluctuation-dissipation theorem allows one to identify the corresponding fluctuations as the internal multiplicative noise.

Despite the fact that such a noise is internal and multiplicative, its influence on the statistical peculiarities of a behavior of systems is studied insufficiently. For the first time, the attention to the problem of the influence of internal fluctuations on a behavior of distributed systems was paid in work [11].

There, it was shown that the action of the internal noise is a source of variability of the entropy which promotes, in its turn, the running of the processes of ordering in the system, where the local potential does not allow a similar behavior. Such phase transitions in systems of the class $\mathcal{A}$ known as entropy-driven phase transitions are a generalization of noise-induced transitions in the zero-dimensional systems [3] to the case of distributed systems. For systems of class $\mathcal{B}$, the influence of the internal multiplicative noise was not studied sufficiently, except for the numerical modeling of the initial and late stages of the phase stratification [14]. The statistical approaches to the description of such a mechanism of the ordering are based on the use of the mean-field theory which gives a qualitatively proper result. This theory was sufficiently developed on the description of both equilibrium deterministic systems of class $\mathcal{A}$ and nonequilibrium ones with external fluctuations [2,15,16], was generalized to the case of a correlated action of two stochastic sources [17,18], and was further elaborated for the systems with internal multiplicative noise [11,19,20]. For systems of class $\mathcal{B}$, it was modified with regard for the action of the external multiplicative and internal additive noises [21].

In the classes of models considered in the present work, the functional of free energy corresponds to the Ginzburg–Landau model

$$\mathcal{F}[x] = \int \left(V(x) + \frac{D}{4d}(\nabla x)^2 \right) \mathrm{d}\mathbf{r}, \quad D = \mathrm{const}, \tag{1}$$

with a local potential

$$V(x) = -\frac{\varepsilon}{2}x^2 + \frac{x^4}{4}, \tag{2}$$

where ε is the controlling parameter characterizing the influence of the medium, and d is the dimension of the space. We assume that the mobility which determines the field-dependent kinetic coefficient is given as

$$M(x) = \frac{1}{1 + \alpha x^2}, \quad \alpha \geq 0. \tag{3}$$

Such a choice follows the well-known mathematical models of mobility [23] which meet the following physical conditions: in the disordered state ($x = 0$), the fluctuations are great, whereas they are small in an ordered state ($x \neq 0$). A variation of the parameter α allows one to separately consider the influence of an additive noise at $\alpha = 0$ and that of a multiplicative one at $\alpha \neq 0$.

We assume the presence of the nonequilibrium medium which sets external fluctuations. Since the influence intensity of the medium is determined by the controlling parameter, we may consider the assumption about its fluctuations to be suitable for the description of real situations: $\varepsilon \to \varepsilon_0 + \zeta(\mathbf{r}, t)$. We endow the Langevin source $\zeta(\mathbf{r}, t)$ by the Gauss properties

$$\langle \zeta(\mathbf{r}, t) \rangle = 0, \quad \langle \zeta(\mathbf{r}, t)\zeta(\mathbf{r}', t') \rangle = \tilde{\sigma}^2 C(\mathbf{r} - \mathbf{r}')\delta(t - t'), \tag{4}$$

with the spatial correlation function

$$C(\mathbf{r} - \mathbf{r}') = \left(\lambda\sqrt{2\pi}\right)^{-d} \exp\left(-\frac{|\mathbf{r} - \mathbf{r}'|^2}{2\lambda^2}\right), \tag{5}$$

where λ is the correlation length of the external noise; and $\tilde{\sigma}^2$ is the intensity.

3. Mean-Field Theory

Consider the principles of the use of the mean-field theory on the basis of a stationary distribution function, the evolution equation for which must take the corresponding peculiarities of the classes of models into account.

3.1. Model $\mathcal{A}$

The general form of the initial model with internal multiplicative noise is as follows:

$$\frac{\partial x}{\partial t} = -M(x)\frac{\delta \mathcal{F}[x]}{\delta x} + \sqrt{M}\xi(\mathbf{r}, t). \tag{6}$$

In the simplest case, the internal fluctuations $\xi(\mathbf{r}, t)$ are assumed to be Gaussian

$$\langle \xi(\mathbf{r}, t) \rangle = 0,$$

$$\langle \xi(\mathbf{r}, t)\xi(\mathbf{r}', t') \rangle = 2\sigma^2\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{7}$$

The external fluctuations are set by correlator (4). For the further analysis, we pass to the lattice representation, by considering the dynamics on a lattice with the spatial scale $\ell = 1$, where the continual field $x(\mathbf{r}, t)$ is replaced by the collection $\{x_i\}_{i=1}^{N^d}$ from N^d dynamical variables $x_i(t)$, and i enumerates elements of the lattice. The functional of free energy takes the form

$$F = \sum_{i=1}^{N^d} \left[V_i + \frac{D}{4d} \sum_{j \in nn^+(i)} (x_j - x_i)^2 \right], \tag{8}$$

where $nn^+(i)$ denotes the nearest neighbors in the positive direction of each of the axes. Finally, the Langevin continual equation with two noises (internal ξ and external ζ ones) is replaced by the collection of equations for each element of the lattice

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -M_i \left(\frac{\partial V}{\partial x_i} - \frac{D}{2d} \sum_j \Delta_{ij} x_j \right) + m_i \xi_i(t) + g_i \zeta_i(t), \tag{9}$$

where $x_i(t) \equiv x(\mathbf{r}_i, t)$, $m_i^2 = M_i \equiv M(x_i)$, $g_i = x_i M_i$; and the discrete Laplacian is set by the standard formula

$$\Delta x \to \sum_j \Delta_{ij} x_j = \sum_j (\delta_{nn(i),j} - 2d\delta_{i,j}) x_j \tag{10}$$

with the summation over the nearest neighbors of the i-th element. The corresponding correlators

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\sigma^2 \delta_{ij} \delta(t - t'),$$

$$\langle \zeta_i(t) \zeta_j(t') \rangle = 2\tilde{\sigma}^2 C_{|i-j|} \delta(t - t') \tag{11}$$

are written with the help of the discrete representation of the spatial correlation functions δ_{ij}, $C_{|i-j|}$. It is obvious that, at $r_c = 0$, we have a white noise in the space with $C_0 = 1$, $C_1 = 0$. But, at $\lambda \gg 1$, we get $C_0 \propto \lambda^{-d}$.

In order to perform the statistical analysis, we use the standard positions of stochastic dynamics [4,22]. By interpreting the corresponding Langevin equations by Stratonovich, we get an evolution equation for the total probability density $\mathcal{P} = P(\{x_i\}, t)$ in the form

$$\frac{\partial \mathcal{P}}{\partial t} = \sum_i \frac{\partial}{\partial x_i} \left[M_i \left(\frac{\partial V}{\partial x_i} - \frac{D}{2d} \sum_j \Delta_{ij} x_j \right) + \right.$$

$$\left. + \sigma^2 m_i \frac{\partial}{\partial x_i} m_i + \tilde{\sigma}^2 g_i \sum_j C_{|i-j|} \frac{\partial}{\partial x_j} g_j \right] \mathcal{P}. \tag{12}$$

The evolution equation for the one-point probability density $P_i(t) = \int \left[\prod_{m \neq i} \mathrm{d}x_m \right] \mathcal{P}$ under the condition that the distribution tends to zero in the limit $\pm\infty$, which gives

$$\int \left[\prod_{m \neq i} \mathrm{d}x_m \right] \frac{\partial}{\partial x_j} g_j \mathcal{P} = 0, \quad i \neq j, \tag{13}$$

can be obtained by the direct integration over all variables, except for x_i. Then, in view of both the definition of a conditional mean

$$\sum_{j \in nn(i)} \int \left[\prod_{m \neq i} \mathrm{d}x_m \right] \mathcal{P} x_j =$$

$$= \left[\sum_{j \in nn(i)} \int \mathrm{d}x_j P(x_j | x_i, t) x_j \right] P_i(t) = 2d \langle x \rangle P_i(t) \tag{14}$$

and the positions of the mean-field theory

$$\sum_j \Delta_{ij} x_j \to 2d(\langle x \rangle - x), \tag{15}$$

we get an evolution equation for the one-point probability density in the form

$$\frac{\partial P_i}{\partial t} = \frac{\partial}{\partial x_i} \left[M_i \left(\frac{\partial V}{\partial x_i} - D(\langle x \rangle - x) \right) + \right.$$

$$\left. + \sigma^2 m_i \frac{\partial}{\partial x_i} m_i + \tilde{\sigma}^2 C_0 g_i \frac{\partial}{\partial x_i} g_i \right] P_i, \tag{16}$$

where the mean-field quantity $\langle x \rangle$ in models of class $\mathcal{A}$ plays the role of the order parameter and can be calculated in a self-consistent way. To obtain it, we need know the stationary probability density.

In the stationary case in the absence of flows, the required distribution takes the form

$$P_s(x; \langle x \rangle) = N \exp \left\{ - \int \mathrm{d}x' \frac{\mathcal{D}_1(x'; \langle x \rangle)}{\mathcal{D}_2(x')} + \frac{1}{2} \ln \mathcal{D}_2(x) \right\}, \tag{17}$$

where N is the normalization constant, and the functions

$$\mathcal{D}_1(x; \langle x \rangle) = M(x) \left[\frac{\partial V(x)}{\partial x} + D(\langle x \rangle - x) \right],$$

$$\mathcal{D}_2(x) = \sigma^2 M(x) + \tilde{\sigma}^2 C_0 g^2(x) \tag{18}$$

are terms of the Kramers–Moyal series and set the effective drift and the effective coefficient of diffusion, respectively [4]. Despite the fact that the obtained solution is formal, because the mean field $\langle x \rangle$ depends on just the probability density, we use the condition of self-consistency

$$\langle x \rangle = \int x P_{st}(x; \langle x \rangle) \mathrm{d}x \equiv \Phi(\langle x \rangle) \tag{19}$$

and determine each of these two quantities in terms of another.

We note that two stochastic sources do not correlate with each other, and the conditions of symmetry $V(x) = V(-x)$, $M(x) = M(-x)$, $g(x) = -g(-x)$ are valid. Then, as follows from [2, 17, 18], the function $\Phi(\langle x\rangle)$ is a monotonously increasing one. Hence, in the given system, we may expect the presence of phase transitions of the second kind [19]. The critical values of parameters of the system, which set the lines of a phase transition, are determined from the condition

$$\left.\frac{\mathrm{d}\Phi(\langle x\rangle)}{\mathrm{d}\langle x\rangle}\right|_{\langle x\rangle=0} = 1. \tag{20}$$

It is considered that the disordered phase is characterized by the trivial value of the mean field $\langle x\rangle = 0$, whereas the ordered one — by $\langle x\rangle \neq 0$. Since $\Phi(\langle x\rangle)$ possesses the central symmetry, it is obvious that the ordered phase is characterized by two identical values of $\langle x\rangle$ which differ by their signs. Since model $\mathcal{A}$ corresponds to systems of the magnetic type, it is expedient to define, in what follows, the order parameter as the "mean magnetic moment" $\eta = |\langle x\rangle|$.

3.2. Model $\mathcal{B}$

On the description of stochastic models with the conserved order parameter, we use the equation of continuity for the field $x(\mathbf{r}, t)$,

$$\frac{\partial x}{\partial t} = -\nabla J, \tag{21}$$

where J is the flow. This equation is exact and cannot be modified directly. However, assuming the flow to be dynamical, we can use and modify the Fick law as

$$J \simeq -M\nabla\frac{\delta\mathcal{F}[x]}{\delta x} + \xi(x;\mathbf{r},t), \tag{22}$$

where we introduced the fluctuations of the flow ξ which are assumed Gaussian and can be, in the general case, a function of the field x. For the x-dependent kinetic coefficient $M = M(x)$, the fluctuation-dissipation theorem yields

$$\langle\xi(x;\mathbf{r},t)\rangle = 0,$$

$$\langle\xi(x;\mathbf{r},t)\xi(x;\mathbf{r}',t')\rangle = 2\sigma^2 M(x)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{23}$$

Then the substitution of expression (22) in (21) leads to the stochastic equation of continuity. Assuming the presence of fluctuations of the controlling parameter, the Langevin equation takes the form

$$\frac{\partial x}{\partial t} = \nabla\cdot\left(M(x)\nabla\left[\frac{\delta\mathcal{F}[x]}{\delta x} + x\zeta(\mathbf{r},t)\right]\right) + \nabla m(x)\xi(\mathbf{r},t), \tag{24}$$

where the autocorrelators of the sources ξ and ζ are given in (7) and (4), respectively. The presented model is a stochastic generalization of the well-known model of the phase stratification of binary systems [24].

In what follows, we again pass to a discrete space, by presenting the continual equation (24) in the form

$$\frac{\mathrm{d}x_i}{\mathrm{dt}} = (\nabla_L)_{ij}M_j(\nabla_R)_{jl}\left[\frac{\partial F}{\partial x_l} + x_l\zeta_l(t)\right] + (\nabla_L)_{ij}m_j\xi_j(t), \tag{25}$$

where we introduced the left- ($(\nabla_L)_{ij} = \delta_{i,j} - \delta_{i-1,j}$) and right-side ($(\nabla_R)_{ij} = \delta_{i+1,j} - \delta_{i,j}$) gradient operators with the following properties: $(\nabla_L)_{ij} = -(\nabla_R)_{ji}$, $(\nabla_L)_{ij}(\nabla_R)_{jl} = \Delta_{il}$.

The construction of the evolution equation for the distribution density is based on certain peculiarities of the operation with discrete gradient operators. In the frame of the standard positions, the total probability density $\mathcal{P}([x], t)$ satisfies the Fokker–Planck equation in the form [2, 4, 21, 22]

$$\frac{\partial\mathcal{P}}{\partial t} = \sum_{ij}\frac{\partial}{\partial x_i}\Delta_{ij}\Bigg(M_j\left[-\frac{\partial V}{\partial x_j} + \frac{D}{2d}\sum_r\Delta_{jr}x_r\right] -$$

$$-\sigma^2 m_j\frac{\partial}{\partial x_j}m_j + \tilde{\sigma}^2 g_j\sum_{m,n}\frac{\partial}{\partial x_n}\Delta_{mn}C_{|j-n|}g_n\Bigg)\mathcal{P}. \tag{26}$$

The evolution equation for the one-point probability density is obtained by the integration of (26) over all variables except for x_i. As a result, we get the equation

$$\frac{\partial P_i(t)}{\partial t} = \frac{\partial}{\partial x_i}\sum_j\Delta_{ij}\langle\widehat{M_j}\rangle P_i(t), \tag{27}$$

where we used the notation

$$\widehat{M_j} = M_j\left[-\frac{\partial V}{\partial x_j} + \frac{D}{2d}\sum_r\Delta_{jr}x_r\right] -$$

$$-\sigma^2 m_j\frac{\partial}{\partial x_j}m_j + \tilde{\sigma}^2 g_j\sum_{m,n}\frac{\partial}{\partial x_n}\Delta_{mn}C_{|j-n|}g_n. \tag{28}$$

Assuming the stationary distribution to be flowless, $\langle\widehat{M_j}\rangle$ should satisfy the equation

$$\sum_j\Delta_{ij}\langle\tilde{M_j}\rangle P_s(x_i) = 0. \tag{29}$$

For the further consideration, we take the deterministic evolution equation for the field $x(\mathbf{r},t)$ in the form $\partial x/\partial t = \nabla M\nabla\delta\mathcal{F}/\delta x$. For such systems, the important point is the limitation imposed by the conservation law $x_0 = \int \mathrm{d}\mathbf{r}x(\mathbf{r},t)$, where x_0 is the initial value given by the initial conditions. Just the latter influence essentially the character of the phase stratification in the system. By positions of the theory of phase stratification for such systems, we can introduce a transition point $\varepsilon_T(x_0)$: at $\varepsilon < \varepsilon_T(x_0)$, the homogeneous state x_0 is stable; at $\varepsilon > \varepsilon_T(x_0)$, the system is stratified into two phases with x_1 and x_2. The transition point will coincide with the critical one only at $x_0 = 0$, i.e. $\varepsilon_T(x_0 = 0) = \varepsilon_c$. It is known that, in the deterministic case, the kinetic coefficient affects only the dynamics of phase transitions, not changing the stationary states of the system. Hence, the stationary states can be calculated by solving the reduced equation $\nabla\delta\mathcal{F}/\delta x = 0$. Therefore, the restricted solution will be $\delta\mathcal{F}/\delta x = h$, where h is a constant which represents generally the effective field. In equilibrium systems, this field is reduced to the difference of the chemical potentials of two phases. For a homogeneous system, the field h depends on the initial conditions x_0. Above the transition point, the homogeneous state is unstable, and the system is stratified into two phases with the values of the field x_1 and x_2, and the corresponding share u is given by the rule $ux_1 + (1-u)x_2 = x_0$. Since the specific potential of the free energy is symmetric, $x_1 = -x_2$. Therefore, we have $h = 0$, i.e. the two phases have the identical chemical potentials [21].

The presented consideration can be used also in the stochastic case. Assuming $\langle\widehat{M}_j\rangle = -h$, we can set $i = j$ and carry out the averaging by rule (14). As a result, we get the equation

$$-hP_s(x) = \left(M(x)\left[-\frac{\partial V}{\partial x} + D(\langle x\rangle - x)\right] - \right.$$

$$-\sigma^2 m(x)\frac{\partial}{\partial x}m(x) + +2d\tilde{\sigma}^2 g(x)\times$$

$$\left.\times\left[C_1 g(\langle x\rangle)\frac{\partial}{\partial x} - C_0\frac{\partial}{\partial x}g(x)\right]\right)P_s(x), \tag{30}$$

where we took $\langle g(x)\rangle \simeq g(\langle x\rangle)$ following [21] as a result of the mean-field averaging of the function $g(x)$ over the nearest neighbors. This equation has the solution

$$P_s(x,\langle x\rangle,h) = N\exp\left(\int \mathrm{d}x'\frac{\Omega(x';\langle x\rangle;h)}{\Theta(x';\langle x\rangle)}\right), \tag{31}$$

where

$$\Omega(x;\langle x\rangle;h) = M(x)\left[-\frac{\partial V(x)}{\partial x} + D(\langle x\rangle - x)\right] -$$

$$-\frac{\sigma^2}{2}\frac{\partial M(x)}{\partial x} - d\tilde{\sigma}^2 C_0\frac{\partial g^2(x)}{\partial x} + h,$$

$$\Theta(x;\langle x\rangle) = \sigma^2 M(x) + 2d\tilde{\sigma}^2 g(x)(C_0 g(x) - C_1 g(\langle x\rangle)). \tag{32}$$

It should be noted that the stationary distribution depends now on two parameters, namely: on the mean field $\langle x\rangle$ and on the effective field h which are determined, in their turn, through the stationary distribution in a self-consistent way.

In order to calculate these unknown parameters, we note that the presented mean-field theory is truly local and leads to the determination of the distribution function in terms of h and $\langle x\rangle$ only in the vicinity of a given element of the spatial lattice. Hence, in the homogeneous case, the mean field is the same over the whole system and coincides with the initial value, i.e. $\langle x\rangle = x_0$. Then, due to the substitution of the given value x_0 for $\langle x\rangle$ into the stationary distribution (31), the quantity h will be calculated by solving the equation

$$\langle x\rangle = \int xP_s(x,\langle x\rangle,h)\mathrm{d}x \tag{33}$$

at $P_s = P_s(x;x_0;h)$. Behind the transition point, where the system is separated into two phases with $\langle x_1\rangle$ and $\langle x_2\rangle$, the equality $\langle x_1\rangle = -\langle x_2\rangle$ holds due to the symmetry of the specific potential $V(x)$. Therefore, h is now the same for both phases, and we can set $h = 0$ in such ordered state. Hence, the distribution function becomes dependent on a single parameter $\langle x\rangle$ which is determined by solving the equation of self-consistency (33) at $P_s = P_s(x;\langle x\rangle;0)$.

4. Results

4.1. Influence of the internal multiplicative noise

Model $\mathcal{A}$. The fact that the stationary distribution can be obtained exactly is basically important for systems with internal multiplicative noise. Indeed, assuming that $\tilde{\sigma}^2 = 0$, we get a stationary distribution in the continual form

$$\mathcal{P}_{\mathrm{st}}[x] \propto \exp\left(-\frac{\mathcal{U}_{\mathrm{eff}}[x]}{\sigma^2}\right), \tag{34}$$

where the effective functional of free energy $\mathcal{U}_{\text{eff}}[x]$ can be written as

$$\mathcal{U}_{\text{eff}}[x] = \mathcal{F}[x] + \sigma^2 S_{\text{eff}}[x] \tag{35}$$

and is set by the effective entropy

$$S_{\text{eff}}[x] = \frac{1}{2}\int \mathrm{d}\mathbf{r} \ln M(x) \tag{36}$$

defined in terms of the mobility $M(x)$. It is characteristic that, despite the invariability of the functional of free energy, the states of the system will be determined by the variability of the entropy caused by the internal multiplicative noise.

In addition, while analyzing the stability of the first moment, the corresponding Langevin equation implies that the internal multiplicative noise does not lead to the loss of the stability of a homogeneous state. Indeed, we have in the linear approximation for the first moment:

$$\frac{\partial\langle x\rangle}{\partial t} = (\varepsilon - \alpha\sigma^2)\langle x\rangle + \frac{D}{2d}\Delta\langle x\rangle. \tag{37}$$

As seen, the loss of the stability in the linear approximation is influenced only by the controlling parameter, whereas the noise stabilizes the disordered state.

These two signs (the variability of the entropy at the invariable free energy and the stabilization of the disordered state by the noise) are those of the so-called entropy-driven phase transitions. For their detailed study in the frame of the mean-field theory, we introduce the effective-energy function

$$U_{\text{eff}}(x;\eta) = V(x) + \frac{D}{2}(\eta - x)^2 + \frac{\sigma^2}{2}\ln M(x) \tag{38}$$

which allows us to trace a behavior of the function $\Phi(\langle x\rangle)$. In this case, the analysis of its properties is simplified. For example, the differentiation with respect to the argument gives

$$\frac{\mathrm{d}\Phi(\langle x\rangle)}{\mathrm{d}\langle x\rangle} = \frac{2D}{\sigma^2}\left(\langle x^2\rangle - \langle x\rangle^2\right), \tag{39}$$

which is the definition of generalized susceptibility χ, whose divergence must indicate the points of phase transitions. By applying the Schwartz inequality to the obtained derivative, $\langle\psi^2(x)\rangle\langle\phi^2(x)\rangle \geq |\langle\psi(x)\phi(x)\rangle|^2$, and taking $\psi(x) = x$ and $\phi(x) = 1$, we get the result indicating the monotonous behavior of $\Phi(\langle x\rangle)$. In this case, $\lim_{\langle x\rangle\to\pm\infty}\Phi(\langle x^n\rangle) \to \Phi^n(\langle x\rangle)$ and, respectively, $\lim_{\langle x\rangle\to\pm\infty}\mathrm{d}\Phi(\langle x\rangle)/\mathrm{d}\langle x\rangle \to 0$, where $\Phi(\langle x^n\rangle) \equiv \langle x^n\rangle$. Thus, we arrive at the conclusion [19]

$$\lim_{\langle x\rangle\to\infty}\Phi(\langle x\rangle) < \langle x\rangle, \quad \lim_{\langle x\rangle\to-\infty}\Phi(\langle x\rangle) > \langle x\rangle. \tag{40}$$

Because $\Phi(\langle x\rangle) = -\Phi(-\langle x\rangle)$, the even derivatives will be trivial. In this case, the additional condition for the transition between the disordered and ordered states is $\mathrm{d}^3\Phi(\langle x\rangle)/\mathrm{d}^3\langle x\rangle\big|_{\langle x\rangle=0} < 0$. The obtained relations for the derivatives can be rewritten in terms of the cumulants (semiinvariants) $\mu_n = \langle(x-\langle x\rangle)^n\rangle$, by using the expansion in the Taylor series

$$\Phi(\langle x\rangle) = \sum_{n=0}^{\infty}\frac{\langle x\rangle^{2n+1}}{(2n+1)!}\int \frac{\partial^{2n+1}P_s(x,\langle x\rangle)}{\partial\langle x\rangle^{2n+1}}\bigg|_{\langle x\rangle=0} x\mathrm{d}x. \tag{41}$$

The integration in quadratures yields the conditions for a realization of phase transitions of the second kind in the form [19]

$$\frac{\mathrm{d}\Phi(\langle x\rangle)}{\mathrm{d}\langle x\rangle} = \frac{2D}{\sigma^2}\mu_2, \quad \mu_2 = \frac{\sigma^2}{D};$$

$$\frac{\mathrm{d}^3\Phi(\langle x\rangle)}{\mathrm{d}^3\langle x\rangle} = \left(\frac{2D}{\sigma^2}\right)^3\mu_4, \quad \mu_4 < 0. \tag{42}$$

In order to carry out the quantitative analysis, we determine the region of variations of the main parameters of the system. The controlling parameter is related to the dimensionless temperature counted from the critical one. Therefore, we take $\varepsilon \in [-1,1]$, the coefficient $D > 0$, and the noise intensity $\sigma^2 \geq 0$. As for the parameter α, we note that it sets a character of the relaxation of the system to a certain state x_s, in the vicinity of which the fluctuations are infinitely small, but nonvanishing. Therefore, the function $M(x)$ can be written in dimensionless units for model $\mathcal{A}$ in the form $M(x) = 1/(1+(x/x_s)^2)$, where $\alpha = x_s^{-2}$. Since the quantity x_s can vary from $x_s^{\min} \to 0$ to $x_s^{\max} = \text{const}$, we set $\alpha \geq 0$ without any loss of generality.

We now consider the pattern of the ordering on the basis of solutions of the equation of self-consistency (19) and the definition of susceptibility (39). The mean-field values of the order parameter $|\langle x\rangle|$ and the susceptibility $\chi = \mathrm{d}\Phi/\mathrm{d}\langle x\rangle|_{\langle x\rangle=0}$ are given in Fig. 1. It is seen from Fig. 1,*a* that, in the case of a two-well initial potential $V(x)$ with $\varepsilon > 0$, an increase of the multiplicative noise intensity causes the decrease of the order parameter to zero (the transition to the disordered state). The basic peculiarity of the behavior of the order parameter as a function of the noise intensity consists in the realization

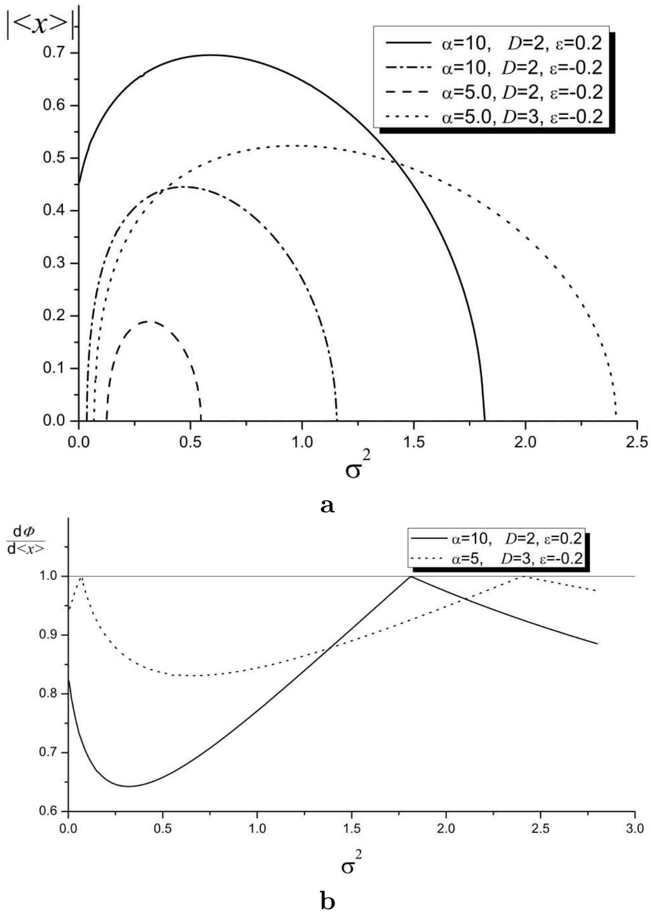

Fig. 1. Mean-field values of the order parameter $|\langle x\rangle|$ (a) and the susceptibility $\chi = \mathrm{d}\Phi/\mathrm{d}\langle x\rangle|_{\langle x\rangle=0}$ (b) versus the internal noise intensity σ^2 at various values of α, D, and ε. Curves in part (b) correspond to the curves in part (a) at the same values of the parameters

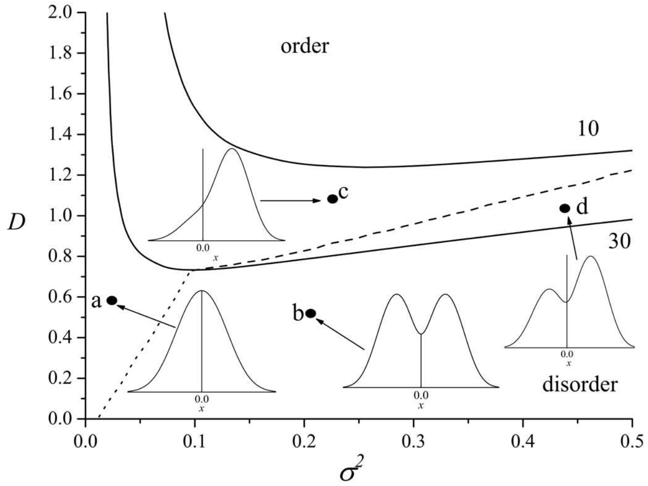

Fig. 2. Phase diagram of the system at α=10 and 30. The corresponding solid lines bound the regions of the ordered and disordered states. The dashed line separates the unimodal and bimodal stochastic distributions $P_s(x; \langle x\rangle > 0)$, and the dotted line separates the stochastic distributions at $\langle x\rangle = 0$

of a reversible phase transition at $\varepsilon < 0$, which corresponds to a one-well potential $V(x)$. Here, the system is disordered at small intensities of the internal noise σ^2. On the transition through the critical value of σ^2_{1c}, the order parameter becomes nontrivial, and the ordered state is conserved to the value of σ^2_{2c} such that the internal noise suppresses the processes of ordering on the passage through this value. The corresponding dependences of the generalized susceptibility χ demonstrate the specific features of the behavior of the order parameter: for the two-well local potential, we observe one peak on the corresponding curve (solid line), which testifies to one critical value of the noise intensity; but, for the one-well local potential (dotted line), we see two characteristic peaks, which indicates the existence of two critical points σ^2_{1c} and σ^2_{2c}. The comparison of the mentioned dependences allows us to conclude that the increase of α (a decrease of fluctuations at the deviation of x from zero) reduces significantly the region of ordering; whereas the increase of the interaction parameter D extends the interval of the intensities $\sigma^2 \in [\sigma^2_{1c}, \sigma^2_{2c}]$, where the ordered state exists.

We obtained the phase diagram by solving Eq. (20). Because the disorganizing role of the noise is not of interest, we present only a diagram illustrating the reversible course of a phase transition in the plane (σ^2, D) (see Fig. 2). It is seen that, in the case of a one-well local potential $V(x)$ and great α, the reversible course of the process of ordering is possible on a large interval of the intensities σ^2; whereas the region of ordering narrows at smaller α (curves with the notations 30 and 10, respectively). The increase in α causes the increase in the critical values of D, by conserving the topology of the phase diagram.

To illustrate the running of entropy-driven phase transitions in the given system, we show the behavior of a stochastic stationary probability density $P_s(x; \langle x\rangle)$ at $\langle x\rangle = 0$ (at points a and b) and at $\langle x\rangle > 0$ (at points c and d) in the inserts in Fig. 2. First, we consider the disordered state, $\langle x\rangle = 0$. In the region containing point a, we have the unimodal probability density centered at zero. On the passage through the dotted line, we have the noise-induced transition related to the entropy contribution to the effective potential $\mathcal{U}_{\text{eff}}[x]$.

On the line, there occurs the threefold degeneration of the extremum of the distribution of the stochastic field x. Behind the line (point b), the distribution becomes bimodal. By moving from point a into the region containing point c, we see the appearance of a nonzero value of the mean field $\langle x\rangle$ on the passage through the solid line, which causes the loss of a symmetry of the distribution function. On the passage through the dashed line (we fall into the region containing point d), there appears the additional maximum of the distribution, which is the sign of a noise-induced transition. With the subsequent growth of the noise intensity, the ordered state is destroyed (the symmetry of the distribution is renewed), and we fall into the region containing point b, where the distribution is bimodal and symmetric, because $\langle x\rangle = 0$ there. Thus, we possess the following pattern of the ordering: with increase of the noise intensity, the unimodal symmetric distribution becomes asymmetric, then the asymmetric distribution is transformed into a bimodal one, and, finally, the bimodal distribution becomes symmetric.

Model $\mathcal{B}$. By considering the model with a conserved dynamics, we note that the parameter α is bounded by the values $\alpha \in [0, 1]$, because it is now related to the coefficients of surface diffusion D_s and bulk diffusion D_b by the formula $\alpha \approx 1 - D_b/D_s$ [26]. In the case where $\alpha \ll 1$, we get the approximation formula $M(x) \approx 1-\alpha x^2$ which is frequently used in the descriptions of the processes of phase stratification with a bounded value of the field $x(\mathbf{r}, t) \in [-1, 1]$.

Like the previous case, we consider the pattern of the ordering for positive and negative values of the controlling parameter ε. First, we will show that the internal multiplicative noise within model $\mathcal{B}$ also does not lead to the loss of the stability of a disordered state. Since the condition $\int x(\mathbf{r}, t)\mathrm{d}t$=const is satisfied in the given model, we consider the dynamics of the structural factor $S(k, t) = \langle x_{\mathbf{k}}(t)x_{-\mathbf{k}}(t)\rangle$, where $x_{\mathbf{k}}(t) = \int x(\mathbf{r}, t)e^{i\mathbf{kr}}\mathrm{d}\mathbf{r}$. In the linear approximation, the evolution equation for the spherically averaged structural factor looks as [27]

$$\frac{\mathrm{d}S(k,t)}{\mathrm{d}t} = -\omega(k)S(k,t)+$$

$$+2\sigma^2k^2 - 2\alpha\sigma^2k^2\frac{1}{(2\pi)^d}\int \mathrm{d}\mathbf{q}S(q,t), \quad (43)$$

where the dispersion relation is as follows:

$$\omega(k) = k^2\left(\frac{D}{2d}k^2 - \varepsilon + \alpha\sigma^2\right). \quad (44)$$

The exponential solution implies that, on the early stage, the unstable modes with the wave vector $k < k_c = \sqrt{2d(\varepsilon - \alpha\sigma^2)/D}$ grow. With increase in α and σ^2, the size of the spinodal region with $k < k_c$ decreases. The unstable modes which promote the development of structures are not realized at $\varepsilon < \alpha\sigma^2$. Therefore, the increase of the parameters α and σ^2 suppresses the creation of spatial structures and promotes the stabilization of the disordered state.

First, we note that model $\mathcal{B}$ with an internal noise inherits certain peculiarities of the statistical representation of the corresponding model $\mathcal{A}$. Indeed, under the condition of the absence of flows, the formal stationary distribution at $h = 0$ is set by relations (34)–(36) and (38). However, the calculation of mean-field values and the phase transition points must use the additional condition that different forms of a behavior of the system are set by the initial conditions.

For this purpose, we will determine, first of all, the behavior of the effective field h by solving the equation of self-consistency at the initial concentration $x_0 = 0.2$ and the mean field $\langle x\rangle$. The corresponding dependences on the noise intensity are given in Fig. 3. Figure 3,a shows that, at $\varepsilon > 0$, the field h increases from zero value starting from the point, where $\sigma^2 = \sigma_T^2$. Prior to this point, the field $h = 0$. This allows us to calculate the mean-field value of $\langle x\rangle$ from the equation of self-consistency. The noise-induced effect is clearly seen from the dependence $h(\sigma^2)$ at $\varepsilon < 0$. Here, the effective field behaves itself nonmonotonously. For small σ^2, it drops to zero and remains trivial in the interval $\sigma^2 \in [\sigma_{1T}^2, \sigma_{2T}^2]$. The last fact means that, in the given interval, the phase stratification with $\langle x_1\rangle = -\langle x_2\rangle$ occurs. At $\sigma^2 > \sigma_{2T}^2$, we have a growth of h, which indicates that the system is now homogeneous with $\langle x\rangle = x_0$. The dependence of the mean field $\langle x\rangle$ on the noise intensity under the condition $h = 0$ is presented in Fig. 3,b. Since the quantity $\langle x\rangle$ is determined from the solution of the equation of self-consistency at $h = 0$ and $x_0 = 0$, the bifurcation points are the critical ones σ_{1c}^2 and σ_{2c}^2. It is seen from Fig. 3 that the value of $\langle x\rangle$ drops to zero for the two-well local potential $V(x)$. But, for the one-well local potential, the reversible behavior of $\langle x\rangle$ is realized. Thus, the above-presented results imply that, like the previous case, a reversible phase transition caused by the action of the internal multiplicative noise for the negative values of the controlling parameter occurs in the system.

The phase diagrams were calculated under the condition that the effective field h takes the zero value.

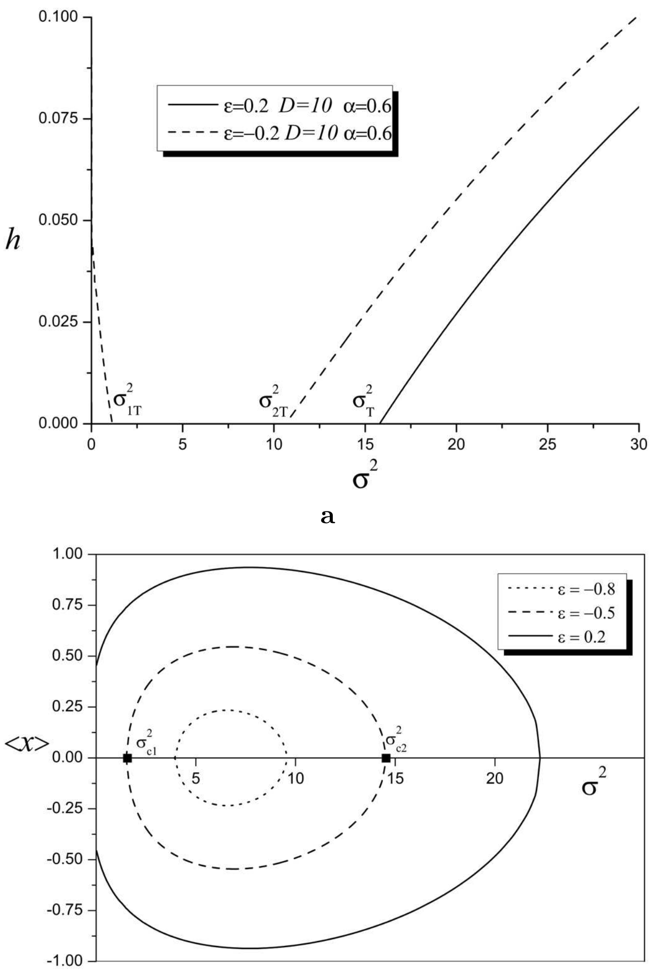

Fig. 3. Effective field h at a fixed initial value $x_0 = 0.2$ (a) and the mean field $\langle x \rangle$ at $D = 10$ and $\alpha = 0.8$ obtained by solving the equation of self-consistency (33) at $h = 0$ (b) versus the internal noise intensity σ^2

In the plane (σ^2, D), the phase diagram has the form shown in Fig. 4 (dash-dotted line — the line of transition points, solid line — the line of critical points). First, let us consider the dash-dotted line obtained at the fixed intial value $x_0 = 0.6$. As above, at $\varepsilon < 0$ and low noise intensities $\sigma^2 < \sigma_T^2$, we have the condition $\langle x \rangle = x_0$ which is satisfied up to $\sigma^2 > \sigma_T^2$. In the region $\sigma^2 \in [\sigma_{1T}^2, \sigma_{2T}^2]$, we have $h = 0$ and $\langle x \rangle \neq x_0$. The growth of the parameter α decreases the value of ε, at which the state with $\langle x \rangle \neq x_0$ is realized, and extends the region of existence of such ordered state. The line of critical points is positioned at less values of the parameter of inhomogeneity D and shifts along the direction to low intensities of the noise. The presented result indicates that, for a given nonzero initial concentration x_0, the phase transition is possible at enhanced values of the parameter of inhomogeneity D, whereas the region of existence of the heterogeneous phase along the axis of the noise intensity decreases. The influence of the parameter α on the position of the transition points and the critical points is analogous to that seen in Fig. 2. The dashed line sets the points, where the topology of the corresponding stationary stochastic distribution $P_s(x; \langle x \rangle > 0, h = 0)$ is changed, i.e. we have the noise-induced transition. Thus, on the phase stratification, we are also faced with a generalization of noise-induced transitions (with a changing topology) to the case of distributed systems.

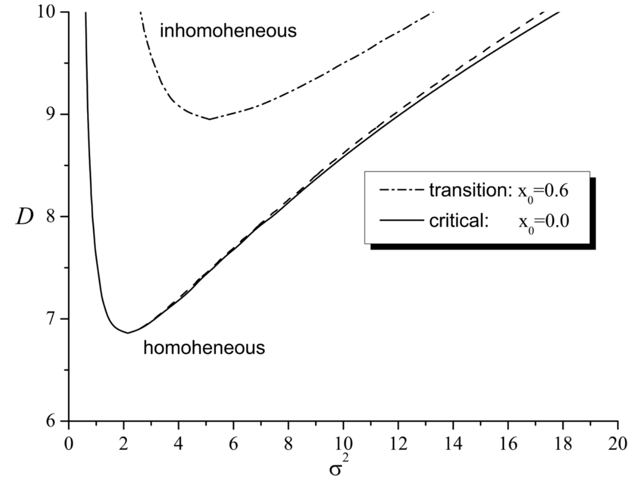

Fig. 4. Phase diagram at $\varepsilon = -0.2$, $\lambda = 0.0$: solid line — the line of critical points ($x_0 = \langle x \rangle = 0.0$); dotted line — the line of transition points ($x_0 = \langle x \rangle = 0.6$). Dashed line — the line where the modality of the stationary distribution $P_s(x; \langle x \rangle, h = 0)$ is changed

4.2. Influence of the internal and external noises

We now consider the case of two noises and clarify the character of an influence of the external noise with intensity $\tilde{\sigma}^2$ on the course of reversible phase transitions.

Model $\mathcal{A}$. In the case of the nonconserved order parameter, the presence of an external noise with intensity $\tilde{\sigma}^2$ leads to the destabilization of the disordered phase in the linear approximation. Indeed, the linear evolution equation for the first moment looks as

$$\frac{\partial \langle x \rangle}{\partial t} = (\varepsilon - \alpha\sigma^2 + C_0\tilde{\sigma}^2)\langle x \rangle + \frac{D}{2d}\Delta\langle x \rangle. \quad (45)$$

It is seen that the internal and external noises become competitive in the processes, where the stability of a disordered state is lost.

The solutions of the equation of self-consistency are shown in Fig. 5,*a*. It is seen that, at a fixed value of the internal noise intensity σ^2, the system becomes ordered with increase of the intensity of external fluctuations. Thus, the external noise is a source of the ordering in the system. We note that the noise also favors the extension of the region of intensities of the internal noise, at which the order parameter is nonzero, by suppressing processes of the reversible type. For example, at great $\tilde{\sigma}^2$, the reversible behavior of the order parameter as a function of the intensity σ^2 disappears. As a result, we arrive at the known pattern of disordering by the internal noise.

The corresponding phase diagram is given in Fig. 5,*b*. It is seen that the reversible course of the phase transition without external noise is possible at high intensities of the spatial interaction D (solid curve). A decrease in the quantity D leads to that the ordering becomes possible only in the presence of two noises (dotted line). An increase in the spatial correlation length of fluctuations λ enhances the critical values of the external noise intensity, by narrowing the region of existence of the ordered phase along the σ^2 axis.

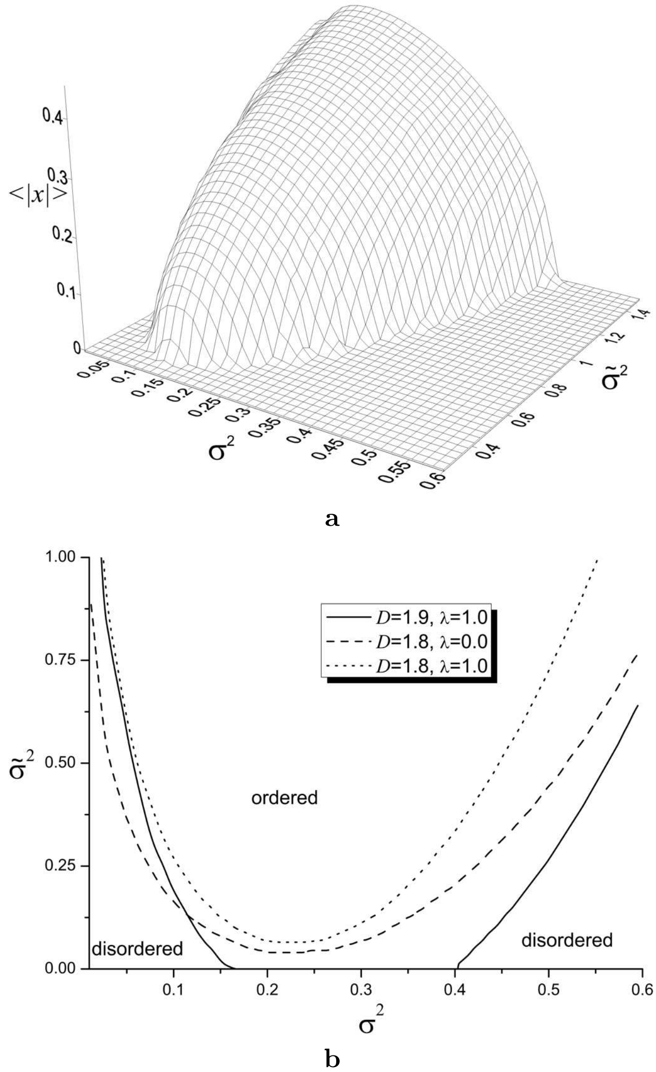

Fig. 5. Order parameter versus the intensities of the internal and external noises at $\alpha = 5.0$, $D = 1.5$, $\varepsilon = -0.2$, $\lambda = 0.0$ (a) and the phase diagram of model $\mathcal{A}$ in the presence of two noises at $\alpha = 5.0$, $\varepsilon = -0.2$ and various values of D and λ (b)

Model $\mathcal{B}$. Within the model with a conserved order parameter, the linear analysis of the stability of the structural factor allows us to write the equation in the form

$$\frac{\mathrm{d}S(k,t)}{\mathrm{d}t} = -\omega(k)S(k,t) + 2\sigma^2 k^2 -$$

$$-\frac{2\alpha\sigma^2 k^2}{(2\pi)^d}\int \mathrm{d}\mathbf{q}S(q,t) + \frac{2k^2\tilde{\sigma}^2}{(2\pi)^d}\int \mathrm{d}\mathbf{q}G(q)S(q,t), \qquad (46)$$

where $G(q)$ is the Fourier transform of the correlation function $C(\mathbf{r}-\mathbf{r}')$. The dispersion law is given by the formula [27]

$$\omega(k) = k^2\left(\left[\frac{D}{2d} - \tilde{\sigma}^2 C_1\right]k^2 - \varepsilon + +\alpha\sigma^2 - \tilde{\sigma}^2 2d(C_0 - C_1)\right). \qquad (47)$$

The above-presented consideration implies that the critical value for the point, where the stability of the homogeneous phase is lost, is renormalized due to the spatial correlations of the external noise.

The dependence of the mean field on the intensities of the internal and external noises is presented in Fig. 6,*a* at $h = 0$. It is seen that the increase of the external noise intensity at negative values of the controlling parameter suppresses the reversible course of the phase transition along the axis of the internal noise intensity σ^2. The first critical point σ^2_{c1} shifts to the left, and, without internal noise, the system is ordered due to the external fluctuations on the passage through the critical value. The critical point σ^2_{c2} moves to the right, so that the increase of $\tilde{\sigma}^2$ extends the region of the internal noise intensity, where two equivalent phases of the system with $\langle x_1\rangle = -\langle x_2\rangle$ are realized. Let us consider the influence of spatial correlations of the external noise λ on positions of the critical points. The corresponding phase diagram is shown in Fig. 6,*b* for various values of the parameter of inhomogeneity D, the parameter α (curves

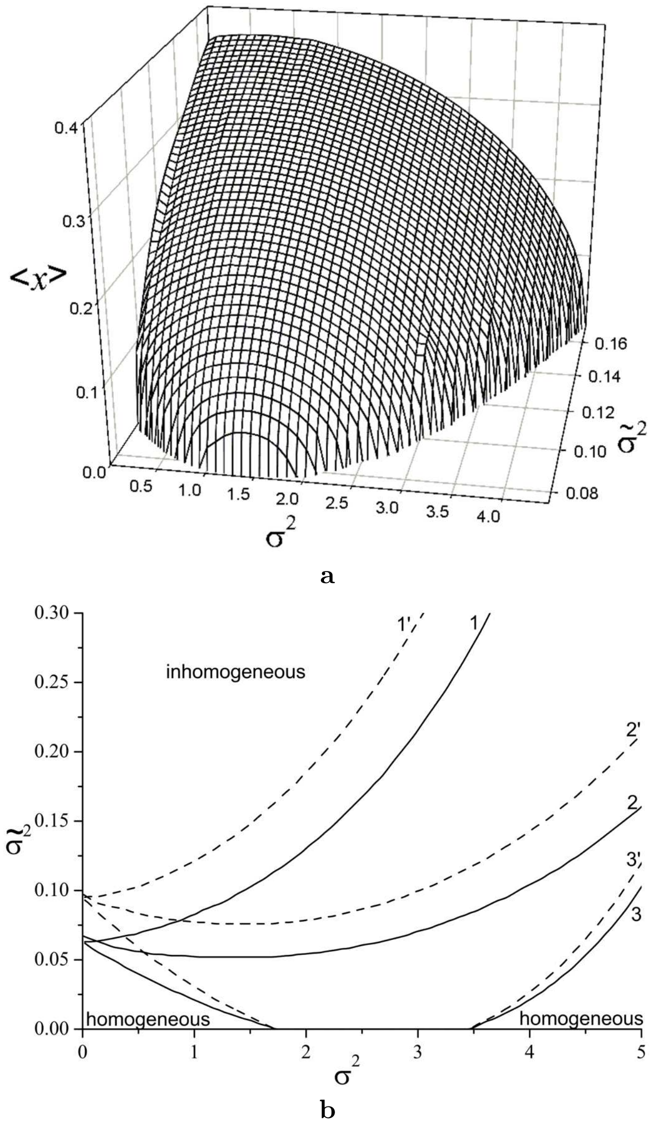

Fig. 6. Dependence of the mean field on the intensity of noises at $\alpha = 0.4$, $\varepsilon = -0.2$, and $D = 10$ (a) and the phase diagram (b) for various values of the correlation length λ, the parameter of spatial inhomogeneity D, and the parameter α (curves *1, 2, 3* correspond to $\lambda = 0.0$, and *1′, 2′, 3′* — $\lambda = 1.0$: *1, 1′* — $D = 8.3$, $\alpha = 0.4$; *2, 2′* — $D = 10.0$, $\alpha = 0.4$, *3, 3′* — $D = 8.3$, $\alpha = 0.6$)

1, 2, 3), and the lengths of spatial correlation of the noise λ (for comparison of the influence of λ at the corresponding D and α, we present the curves with primes). It is seen that the increase in D leads to both a decrease of the critical intensities $\tilde{\sigma}^2$ and the realization of the reversible course of the phase transition (compare curves *1* and *2*). An analogous situation is observed on the growth of the parameter α (compare curves *1* and *3*). The increase of the correlation length of the external noise induces the increase of the critical values of its intensity (compare curves *1′, 2′, 3′*). In this case, the region of reversible behavior of the mean field along the intensity axis of the internal noise is narrowed.

5. Macroscopic Approximation

Let us consider, finally, the macroscopic approximation, by setting $D \to \infty$. This allows us to neglect the correlations, by representing the averaging in the form $\langle\phi(x)\rangle \simeq \phi(\langle x\rangle)$. In such a case, the stationary distribution takes the form $P_s(x, \langle x\rangle) = \delta(x - \langle x\rangle)$. This allows us to write the stationary equation used for the determination of the critical values of parameters of the system for models of class $\mathcal{A}$ in the form

$$-M(\langle x\rangle)V'(\langle x\rangle) + \frac{\sigma^2}{2}M'(\langle x\rangle) + \frac{C_0\tilde{\sigma}^2}{2}(\langle x\rangle^2 M^2(\langle x\rangle))' = 0. \tag{48}$$

For class $\mathcal{B}$, we have

$$h = M(\langle x\rangle)V'(\langle x\rangle) - \frac{\sigma^2}{2}M'(\langle x\rangle) + \frac{2d(C_1 - C_0)\tilde{\sigma}^2}{2}(\langle x\rangle^2 M^2(\langle x\rangle))', \tag{49}$$

where the prime means the differentiation with respect to the argument. The presented equations were obtained by the integration of Eq. (16) in the stationary case and Eq. (30), respectively.

For class $\mathcal{A}$, the solution of Eq. (48) gives the root $\langle x\rangle = 0$ which exists always, whereas the nontrivial roots $\langle x\rangle_{1,2} \neq 0$ set the critical value of the controlling parameter ε_c determined by the formula

$$\varepsilon_c = \alpha\sigma^2 - C_0\tilde{\sigma}^2. \tag{50}$$

Thus, at $\varepsilon > \varepsilon_c$, the ordered phase is formed, which follows from the analysis of the stability. Thus, the fluctuating forces are competitive: the internal noise leads to the growth of the critical value of the controlling parameter, whereas the external noise decreases it.

For models of class $\mathcal{B}$, we should consider again the influence of initial conditions which set a value of the effective field h. At $\varepsilon < \varepsilon_T$, the field is homogeneous. Therefore, we should set $\langle x\rangle = x_0$ in Eq. (49), which gives h as a function of x_0. At $\varepsilon > \varepsilon_T$, we have $h = 0$.

Then Eq. (49) is solvable relative to $\langle x\rangle$. The transition line corresponds to the condition $\langle x\rangle_1 = x_0$ and, in a relevant manner, determines the transition point ε_T. The critical point $\varepsilon_c = \varepsilon_T(x_0 = 0)$ is set now by the relation

$$\varepsilon_c = \alpha\sigma^2 - 2d\tilde{\sigma}^2(C_0 - C_1). \quad (51)$$

The above consideration implies that we are faced, as earlier, with the competition between the internal and external noises. However, as distinct from (50), we have the displacement of the critical point with a factor of $2d$, which is related to not only the noise intensity $\tilde{\sigma}^2 C_0$ but to spatial correlations (the term C_1) of the first neighbors. It is characteristic that, in model $\mathcal{A}$, the correlation contribution from the adjacent sites of the lattice disappears, whereas it becomes significant for models of class $\mathcal{B}$. In addition, it is worth to note that, in the case of the influence only of the internal noise, the critical value of the controlling parameter is the same for both classes of models. Finally, we note that, the controlling parameter has only one critical value in the macroscopic approximation, which corresponds to a single point of the phase transition. This is related to the fact that we used the assumption $D \to \infty$. The two points of the phase transition and hence the reversibility of a behavior of the order parameter are possible only at finite values of the intensity of a spatial interaction [16–18].

6. Conclusions

We have carried out the theoretical study of the processes of ordering in systems of classes $\mathcal{A}$ and $\mathcal{B}$ in the presence of internal and external multiplicative noises. In the frame of the developed mean-field theory, we have shown that the action of the internal multiplicative noise leads to the reversible course of phase transitions on changing its intensity. The obtained bifurcation and phase diagrams illustrate the presence of two points of the phase transition only at finite values of the parameter of spatial interaction.

While considering the joint influence of internal and external noises, we have established that they reveal the opposite statistical actions: the internal noise stabilizes the disordered state, and the external one promotes its destabilization. It is fiound that, at a fixed value of the external noise intensity, an increase of the intensity of internal fluctuations leads to a realization of a transition of the order–disorder type, whereas the external fluctuations at a fixed intensity of the internal noise promote the appearance of an ordered state. The obtained results are in agreement with both the linear analysis of the stability and the macroscopic approximation. We have revealed that the points of phase transitions within the models under study in the presence of the internal noise coincide, whereas they differ from one another in the presence of an external noise, which is related to the influence of the spatial correlation length.

The obtained results can be used in the further theoretical study of systems of the magnetic type, polymers, and the processes of phase stratification and decay under the active interaction of the system and the external medium. The use of the proposed theoretical approaches will allow one to predict the behavior of solid solutions surrounded by a nonequilibrium medium.

Here, we have considered the case of a noncorrelated action of two stochastic sources; therefore, the perspective of further studies is the analysis of the role of temporal correlations of two processes. Moreover, while considering models of class $\mathcal{B}$, we have used the approximation of a thermodynamical equilibrium, whereas the study of the processes of spinodal decay in the limit case of a dynamical flow, and not just the study of the concentration-related problems, seems to be urgent and promising.

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СЕРЕДНЬОПОЛЬОВИЙ ПІДХІД ДО НЕРІВНОВАЖНИХ ФАЗОВИХ ПЕРЕХОДІВ У СИСТЕМАХ ІЗ ВНУТРІШНІМ ТА ЗОВНІШНІМ МУЛЬТИПЛІКАТИВНИМИ ШУМАМИ

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Р е з ю м е

Розглянуто індуковані шумом фазові переходи за наявності внутрішніх та зовнішніх флуктуацій мультиплікативного характеру у системах із динамікою, яка зберігається та не зберігається. На основі середньопольового аналізу виявлено реверсивний хід упорядкування при зміні інтенсивності внутрішнього шуму. Із зростанням інтенсивності зовнішнього шуму система переходить до упорядкованого стану. Встановлено, що внутрішні та зовнішні флуктуації мають протилежну статистичну дію.