QUANTUM EVOLUTION OF THE VERY EARLY UNIVERSE

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A consistent quantum cosmology for the Universe right after the Big Bang is proposed. The stationary state spectrum and the wave functions of the quantum Friedman Universe originally filled with a uniform scalar field and a perfect fluid, which defines a reference frame, are calculated. It is shown that the matter-energy in the Universe has a component in the form of a condensate of massive zero-momentum excitation quanta of oscillations of the primordial scalar field. It is demonstrated that the process of nucleation of the Universe from the initial cosmological singularity point can have an exponential (explosive) nature. The evolution of the Universe is described as transitions with non-zero probabilities between the states of the Universe with different masses of the condensate.

§1. It is well known that, in contrast to a gravitational field in a void, the consideration of the gravitational field coupled with matter allows to use matter in order to give an invariant meaning to space-time points [1–3]. Since the times of Einstein [4] and Hilbert [5], the material reference systems in general relativity have been used to specify space-time events. Using such systems, one can address the conceptual problems of not only classical, but also quantum gravity [6].

In [1], a scheme to include reference frames in general relativity by means of the introduction of coordinate conditions was developed. A task at this point is to find a material source in the Einstein equations which determines the reference frame and has no unphysical properties. This approach was applied in [7–10] in order to solve the problems of the quantum theory of gravity in the minisuperspace model with a material source in the form of relativistic matter (radiation) which defines the reference frame. The variables which describe radiation mark space-time events, since the reference frame is considered as a dynamical system. These variables play the role of the canonical coordinates which determine an embedding in the encompassing space-time. At the same time, the new constraints turn out to be linear with respect to the momenta canonically conjugate with them. Such an approach allows us to obtain the time-dependent Schrödinger equation and to define a conserved positive definite inner product.

On the other hand, it is well known that, during the quantization of different model systems in gravity, one can use a perfect fluid as the reference frame [2, 11, 12]. In this case, one deals directly with a physical medium without coordinate conditions as an intermediate under the construction of a material source. This leaves aside problems connected with the necessity to ensure that the action is coordinate invariant and that a material source which determines a reference frame has correct physical properties. Relativistic matter is a special case of the perfect fluid and can be used, as the simplest physical system, to define a reference frame.

In §2, the constraint system quantization in the presence of a medium which defines a reference frame is considered, and the fundamental equations of a quantum model of the Universe are given.

In §3, the stationary states of the quantum Universe are studied. The stationary state spectrum and the wave functions of the quantum Universe filled with the primordial matter in the form of a uniform scalar field and the perfect fluid (radiation) which defines the reference frame are calculated. It is shown that the matter-energy in the Universe has a component in the form of a condensate of massive zero-momentum excitation quanta of oscillations of the primordial scalar field. The Universe with the Planck mass of a condensate in the ground (vacuum) state has the mean value of the scale factor that practically coincides with the Planck length.

In §4, the nucleation rate of the Universe from the initial cosmological singularity point is calculated, and it is demonstrated that the process of nucleation of the Universe can have an explosive (exponential) nature.

In §5, the probabilities of transitions between the states of the Universe with different masses of a condensate are calculated. It is shown that the probability of a transition from the vacuum state of the Universe to another state obeys the Poisson distribution.

In §6, a conclusion is given.

§2. Let us consider a cosmological system (the Universe) with the action

$$S = S_{E-H} + S_M + S_{PF}, \quad (1)$$

where

$$S_{E-H} = -\frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\,R \quad (2)$$

is the Einstein–Hilbert action for the gravitational field, S_M is the action of matter fields,

$$S_{PF} = \frac{1}{c}\int d^4x\sqrt{-g}\,\{-\rho(\rho_0,s) + \lambda[g^{\mu\nu}U_\mu U_\nu - 1]$$
$$+\rho_0 U^\nu[s\,\Theta_{,\,\nu} - \widetilde{\lambda}_{,\,\nu} + \beta_i\,\alpha^i_{\,,\,\nu}]\,\} \quad (3)$$

is the action of the perfect fluid (macroscopic bodies) [13–17] which defines the material reference frame, where ρ is the energy density as a function of the density of the rest mass ρ_0 and the specific entropy s, U^ν is the four-velocity, and λ is a Lagrange multiplier that ensures the normalization of U^ν. The quantities Θ, $\widetilde{\lambda}$, β_i, and α^i are scalar fields. Here, Θ has a meaning of the thermasy or the "potential" for the temperature T, $T = \Theta_{,\,\nu}U^\nu$. The quantity $\widetilde{\lambda}$ is the "potential" for the specific free energy f taken with the inverse sign, $f = -\widetilde{\lambda}_{,\,\nu}U^\nu$. The quantities β_i and α^i are Lagrange multipliers and Lagrangian coordinates for a fluid on a spacelike hypersurface, respectively (one should introduce them into the action in order to describe rotational flows and, generally speaking, the incorporation of one pair of the variables β_i and α^i into the action would be enough, but then their direct physical interpretation will be lost). The thermodynamic variables are related via the first law of thermodynamics

$$d\rho = h\,d\rho_0 + \rho_0\,T\,ds, \quad (4)$$

where $h = \frac{\rho+p}{\rho_0}$ is the specific enthalpy which plays the role of inertial mass, and p is the pressure.

The components of the metric tensor $g^{\mu\nu}$, the matter fields contained in S_M, and the values ρ_0, s, U^ν, Θ, λ, $\widetilde{\lambda}$, β_i, and α^i play the role of generalized variables. All equations of the classical theory of gravity follow from the principle of least action, $\delta S = 0$, where all independent variables are varied.

Let us assume that the Universe is homogeneous, isotropic, closed and described by the Friedman–Robertson–Walker metric with the cosmic scale factor $a(t)$, t is the time variable. We choose the uniform scalar field ϕ with the potential energy density (potential) $V(\phi)$ as a matter. The choice of such a field as a primordial matter seems to be reasonable, since any other fields (vector or spinor, for instance), being non-averaged over all space variables at every instant of time, can destroy the supposed property of homogeneity and isotropy of the Universe.

In the model of the Universe under consideration, action (1) can be reduced to the form [17]

$$S = \int dt\left(\pi_a\frac{da}{dt} + \pi_\phi\frac{d\phi}{dt} + \pi_\Theta\frac{d\Theta}{dt} + \pi_{\widetilde{\lambda}}\frac{d\widetilde{\lambda}}{dt} - H\right), \quad (5)$$

where π_a, π_ϕ, π_Θ, $\pi_{\widetilde{\lambda}}$ are the momenta canonically conjugate with the variables a, ϕ, Θ, $\widetilde{\lambda}$. We take into account that the momenta conjugate with the variables ρ_0 and s vanish identically. Below, they will be omitted in terms of the conversion of second-class constraints into first-class constraints in accordance with Dirac's proposal [18]. In this model, $U^0 = \frac{1}{N}$ and $U^i = 0$, where N is the lapse function that specifies the time reference scale, and the condition $g^{\mu\nu}U_\mu U_\nu = 1$ is contained explicitly in the variational principle. Therefore, the term with λ should be dropped. Since the variables β_i and α^i describe rotational flows which would create a preferential direction in space, the terms with β_i and α^i should be dropped as well, and one should consider irrotational (potential) flows of the perfect fluid only.

In Eq. (5), the Hamiltonian

$$H = N\left(-\frac{3\pi c^4}{4G}\right)\frac{1}{a}\left\{\left(\frac{2G}{3\pi c^3}\right)^2\pi_a^2 + a^2 -\right.$$
$$\left.-\frac{G}{3\pi^3c^2}\,\frac{\pi_\phi^2}{a^2} - \frac{8\pi G}{3c^4}\,a^4[\rho + V(\phi)]\right\}$$
$$+\lambda_1\left\{\pi_\Theta - \frac{1}{c}\,2\pi^2a^3\rho_0\,s\right\} + \lambda_2\left\{\pi_{\widetilde{\lambda}} + \frac{1}{c}\,2\pi^2a^3\rho_0\right\}, \quad (6)$$

where λ_1 and λ_2 are Lagrange multipliers. The lapse function N also plays the role of a Lagrange multiplier (as in the ADM-formalism [19]). The variation of the action (5) with respect to N, λ_1, and λ_2 leads to three constraint equations.

In quantum theory, the constraint equations become, in accordance with a procedure proposed by Dirac [18], constraints on the wave function Ψ. Replacing the momenta by the operators

$$\pi_a \to \hat{\pi}_a = -\,i\,\hbar\partial_a, \quad \pi_\phi \to \hat{\pi}_\phi = -\,i\,\hbar\partial_\phi,$$
$$\pi_\Theta \to \hat{\pi}_\Theta = -\,i\,\hbar\partial_\Theta, \quad \pi_{\widetilde{\lambda}} \to \hat{\pi}_{\widetilde{\lambda}} = -\,i\,\hbar\partial_{\widetilde{\lambda}}$$

which satisfy the commutation relations

$$[a, \hat{\pi}_a] = i\,\hbar, \quad [\phi, \hat{\pi}_\phi] = i\,\hbar,$$
$$[\Theta, \hat{\pi}_\Theta] = i\,\hbar, \quad [\widetilde{\lambda}, \hat{\pi}_{\widetilde{\lambda}}] = i\,\hbar,$$

while all other commutators vanish, we obtain

$$\left\{-\left(\frac{2G\hbar}{3\pi c^3}\right)^2 \partial_a^2 + a^2 + \frac{G\hbar^2}{3\pi^3 c^2}\,\frac{1}{a^2}\,\partial_\phi^2 - \right.$$
$$\left. -\frac{8\pi G}{3c^4}\,a^4[\rho + V(\phi)]\right\}\Psi = 0, \tag{7}$$

$$\left\{-\,i\,\hbar\,\partial_\Theta - \frac{1}{c}\,2\pi^2 a^3 \rho_0\, s\right\}\Psi = 0, \tag{8}$$

$$\left\{-\,i\,\hbar\,\partial_{\widetilde{\lambda}} + \frac{1}{c}\,2\pi^2 a^3 \rho_0\right\}\Psi = 0. \tag{9}$$

In Eq. (7), the factor ordering parameter associated with a possible ambiguity in the choice of an explicit form of the operator $\hat{\pi}_a^2$ (see, e.g., [20]) is assumed to be zero.

It is convenient to pass from the generalized variables Θ and $\widetilde{\lambda}$ to the non-coordinate co-frame

$$h\,d\tau = s\,d\Theta - d\widetilde{\lambda}, \quad h\,dy = s\,d\Theta + d\widetilde{\lambda}, \tag{10}$$

where τ is the proper time at every point of space. It is easy to prove that the corresponding derivatives commute between themselves,

$$[\partial_\tau, \partial_y] = 0.$$

Then Eqs. (8) and (9) reduce to the form

$$\left\{-\,i\,\hbar\,\partial_{\tau_c} - \frac{1}{c}\,2\pi^2 a^3 \rho_0\right\}\Psi = 0, \qquad \partial_y \Psi = 0, \tag{11}$$

where $d\tau_c = h\,d\tau$.

By varying the action with respect to $\widetilde{\lambda}$, we get the conservation law

$$a^3 \rho_0 = \text{const}. \tag{12}$$

It describes a conserved macroscopic value which characterizes the number of particles. For example, if the perfect fluid is composed of baryons, then Eq. (12) describes the conservation of baryon number.

By varying the action with respect to Θ, we get that the specific entropy is conserved:

$$s = \text{const}.$$

From Eq. (7), one can see that it is convenient to take the matter component ρ in the form of relativistic matter (radiation) with the equation of state $p = \frac{1}{3}\rho$. In this case,

$$a^4 \rho = \text{const}. \tag{13}$$

Denoting the constant in Eq. (13) as E, while that in Eq. (12) as E_0, we obtain the equations

$$\left\{-\,i\,\hbar\,\partial_{\tau_c} - \frac{1}{c}\,2\pi^2 E_0\right\}\Psi = 0, \tag{14}$$

$$\left\{-\left(\frac{2G\hbar}{3\pi c^3}\right)^2 \partial_a^2 + a^2 + \frac{G\hbar^2}{3\pi^3 c^2}\,\frac{1}{a^2}\,\partial_\phi^2 - \right.$$
$$\left. -\frac{8\pi G}{3c^4}\,[a^4 V(\phi) + E]\right\}\Psi = 0, \tag{15}$$

where, according to Eq. (11), the wave function Ψ does not depend on y.

Equation (14) describes the evolution of the state Ψ with respect to the time variable τ_c. Equation (15) does not contain τ_c explicitly. At this point, there is a close analogy with properties of closed systems in quantum mechanics [21].

The constants E_0 and E are dimensional quantities, $[E_0]$ = energy, $[E]$ = energy $\times$ length. It is convenient to rewrite Eqs. (14) and (15) for dimensionless quantities. To this end, we replace

$$a \to \frac{a}{l_P}, \quad \phi \to \frac{\phi}{\phi_P}, \quad \tau_c \to \frac{\tau_c}{l_P}, \quad V \to \frac{V}{\rho_P},$$
$$E_0 \to \frac{4\pi^2}{m_P c^2}\,E_0, \quad E \to \frac{4\pi^2}{\hbar c}\,E,$$

where we have dimensionless quantities from the left, and

$$l_P = \sqrt{\frac{2G\hbar}{3\pi c^3}}, \quad \phi_P = \sqrt{\frac{3c^4}{8\pi G}}, \quad t_P = \frac{l_P}{c},$$
$$\rho_P = \frac{3c^4}{8\pi G l_P^2}, \quad m_P = \frac{\hbar}{l_P c}$$

are the Planck values of length, scalar field, time, energy density, and mass, respectively. Then Eqs. (14) and (15) in new dimensionless variables take the form

$$\left\{-\,i\,\partial_{\tau_c} - \frac{1}{2}\,E_0\right\}\Psi = 0, \tag{16}$$

$$\left\{-\,\partial_a^2 + \frac{2}{a^2}\,\partial_\phi^2 + a^2 - a^4 V(\phi) - E\right\}\Psi = 0. \tag{17}$$

Equation (16) has a particular solution in the form

$$\Psi = \mathrm{e}^{\frac{i}{2}\,E_0 \tau_c}\psi, \tag{18}$$

where ψ is a function which depends on a and ϕ only and is determined by Eq. (17). If we pass from the time variable τ_c to $\overline{\tau} = \frac{E_0}{2E}\,\tau_c$, we arrive, as a result, at an analogy with the Schrödinger equation for stationary states in Eqs. (17) and (18). We have obtained Eqs. (17) and (18) previously in [7–10] within the bounds of the scheme for incorporating a reference system in general relativity through the introduction of a coordinate condition.

Let us note that we can obtain Eqs. (16) and (17) even without the introduction of the proper time by means of Eqs. (10). It is possible to build a time variable from the matter variables (e.g., [11, 12]). We can consider, e.g., Θ as a time variable [17]. (On the correspondence between the thermasy and the proper time, see [22].)

The second-order partial differential equation (17) is given on the intervals $0 \le a < \infty$ and $-\infty < \phi < \infty$. It should be supplemented with boundary conditions. We suppose that the wave function vanishes at $a \to \infty$ and $|\phi| \to \infty$ for any smooth function $V(\phi)$. The choice of boundary conditions at the point $a = 0$ should be stipulated by the physical properties of the system under consideration[1]. Choosing the measure on a minisuperspace, one should take the requirement of normability of the wave function into account (for our case, see §3). This question was widely discussed in [1] (see also references therein) for systems with the reference fluid used for constructing a time variable.

§3. Let us study the properties of the quantum Universe described by the steady-state equation (17). Since the Universe is supposed to be closed, then one can introduce the notion of the Universe mass as a product of its matter density and the comoving volume [24]. In the units under consideration, the mass of a scalar field in the Universe with the scale factor a is equal to

$$M_\phi = \frac{1}{2}\,a^3 \rho_\phi, \tag{19}$$

where $\frac{1}{2}\,a^3$ is the comoving volume, and

$$\rho_\phi = \frac{2}{a^6}\,\pi_\phi^2 + V(\phi) \tag{20}$$

is the energy density of a uniform scalar field,

$$\pi_\phi = \frac{1}{2}\,a^3\,\frac{d\phi}{d\tau} \tag{21}$$

is the momentum canonically conjugate with ϕ [10].

The mass of the scalar field (19) can be associated with the operator

$$\hat{H}_\phi = \frac{1}{2}\,a^3\,\hat{\rho}_\phi, \tag{22}$$

where

$$\hat{\rho}_\phi = -\,\frac{2}{a^6}\,\partial_\phi^2 + V(\phi) \tag{23}$$

is the energy density operator for the field ϕ. Then Eq. (17) takes the form

$$(-\,\partial_a^2 + a^2 - 2a\hat{H}_\phi - E)\psi = 0. \tag{24}$$

Further, let us suppose that the potential $V(\phi)$ is a smooth function of ϕ. Let there exist a value of the field $\phi = \sigma$, at which the function $V(\phi)$ has a minimum, while the value σ itself corresponds to the true vacuum of the field ϕ, $V(\sigma) = 0$ (the absolute minimum [25]). Then, near the point $\phi = \sigma$, the following representation is valid:

$$V(\phi) = \frac{m_\sigma^2}{2}\,(\phi - \sigma)^2, \tag{25}$$

where $m_\sigma^2 = [d^2V(\phi)/d\phi^2]_\sigma > 0$.

Let us make a scaling transformation of the field ϕ and introduce a new variable x which describes a deviation of the field ϕ from its vacuum state σ,

$$x = \left(\frac{m_\sigma a^3}{2}\right)^{1/2}(\phi - \sigma). \tag{26}$$

Operator (22) takes the form

$$\hat{H}_\phi = \frac{m_\sigma}{2}\left(-\,\partial_x^2 + x^2\right). \tag{27}$$

From Eqs. (24) and (27), it follows that the singular point $a = 0$ in Eq. (17) is nonphysical. It is removed by a scaling transformation of the scalar field. Operator (27) is the energy operator of the matter in the Universe.

Let us introduce the eigenfunctions of a harmonic oscillator $u_k(x)$ as a solution of the equation

$$(-\,\partial_x^2 + x^2)\,u_k(x) = (2k+1)\,u_k(x), \tag{28}$$

[1] In this connection, see, e.g., [23].

where $k = 0, 1, 2, \ldots$ is the number of a state of the oscillator. Then we find

$$\hat{H}_\phi u_k = M_k u_k, \tag{29}$$

where

$$M_k = m_\sigma \left(k + \frac{1}{2}\right). \tag{30}$$

The quantity M_k can be interpreted as an amount of matter-energy (or mass) in the Universe related to a scalar field. In the second quantization formalism, this energy is represented in the form of a sum of the excitation quanta of spatially coherent oscillations of the field ϕ about the equilibrium state σ, and k is the number of these excitation quanta. Such oscillations correspond to a condensate of zero-momentum ϕ quanta with the mass m_σ. The mass m_σ is determined by the curvature of the potential $V(\phi)$ near $\phi = \sigma$.

Taking (29) into account, we will look for the solution of Eq. (24) in the form of a superposition of the states with all possible values of the quantum number k (and, correspondingly, with all possible masses M_k)

$$\psi(a, \phi) = \sum_k f_k(a)\, u_k(x). \tag{31}$$

Substituting (31) into (24) and using the orthonormality condition for the states $u_k(x)$, $\langle u_k | u_{k'} \rangle = \delta_{kk'}$, we obtain the equation for $f_k(a)$,

$$\left(-\partial_a^2 + a^2 - 2aM_k - E\right) f_k(a) = 0. \tag{32}$$

We choose the boundary conditions $f_k(0) \neq 0$[2] and $f_k(a) \to 0$ as $a \to \infty$. The solution of (32) with these boundary conditions has the form

$$f_k(a) \equiv f_{n,k}(a) = N_{n,k}\, \mathrm{e}^{-\frac{1}{2}(a-M_k)^2} H_n(a - M_k) \tag{33}$$

at

$$E \equiv E_{n,k} = 2n + 1 - M_k^2, \tag{34}$$

where H_n are the Hermitian polynomials, $n = 0, 1, 2, \ldots$ is the number of a state of the quantum Universe at a given k-state of the condensate (with the mass M_k) in the potential well

$$U(a) = a^2 - 2aM_k. \tag{35}$$

For the states $f_{n,k}(a)$ normalized by the condition $\langle f_{n,k} | f_{n,k} \rangle = 1$, where the standard measure on minisuperspace [27] is used, the normalization factor

$$N_{n,k} = \Bigg\{ 2^{n-1} n! \sqrt{\pi}\, [\mathrm{erf}\, M_k + 1] - \\ - \mathrm{e}^{-M_k^2} \sum_{l=0}^{n-1} \frac{2^l n!}{(n-l)!} H_{n-l}(M_k)\, H_{n-l-1}(M_k) \Bigg\}^{-1/2}, \tag{36}$$

where

$$\mathrm{erf}\, M = \frac{2}{\sqrt{\pi}} \int_0^M dt\, \mathrm{e}^{-t^2}$$

is the probability integral. It follows from the properties of the Hermitian polynomials $H_n(M)$ that the exponential addition in Eq. (36) for the states of the quantum Universe with large enough masses M_k of the condensate is of the same order of magnitude as $O((2M_k)^{2n-1}\, \mathrm{e}^{-M_k^2})$ and can be neglected. In this case, $N_{n,k}$ does not depend on k, and its numerical value coincides with that of the normalization factor of the wave function of an ordinary harmonic oscillator in the state n.

According to (33) and (34), the quantum states of the Universe are characterized by two quantum numbers n and k. The mean value of the scale factor a in state (33),

$$\bar{a} = \langle f_{n,k} | a | f_{n,k} \rangle, \tag{37}$$

equals

$$\bar{a} = M_k + \bar{\xi}, \tag{38}$$

where

$$\bar{\xi} = N_{n,k}^2\, 2^{n-1} n!\, \mathrm{e}^{-M_k^2} \times \\ \times \left\{ 1 + \sum_{l=0}^{n-1} \frac{2^{l-n}}{(n-l)!} H_{n-l}(M_k)\, H_{n-l-1}(M_k) \right\}. \tag{39}$$

According to (38), the Universe with the Planck mass of the condensate, $M_k = 1$, in the ground (vacuum) state, $n = 0$, is characterized by the mean value $\bar{a}_{n=0} = 1.11$ which coincides with the Planck length by an order of magnitude. For the states with $M_k > 1$, the mean value (38) does not depend on n to within a small summand

[2] As shown in [26], this quantum model in the semiclassical limit leads to the equations of general relativity. In classical cosmology the Universe near the initial cosmological singularity point $a = 0$ is characterized by nontrivial values of energy density. Since $|f_k(0)|^2$ is the particle number density at $a = 0$, the choice of such a boundary condition is justified from the cosmological point of view (cf. [23]).

$\sim O((2M_k)^{2n-1}\,\mathrm{e}^{-M_k^2})$ and is determined by the mass M_k only,

$$\bar{a} = M_k \qquad \text{at} \qquad M_k \gg 1. \tag{40}$$

The mass M_k determines also the absolute minimum of the effective potential $U(a)$ (35) of Eq. (32), $U(M_k) = -\,M_k^2$.

§4. According to quantum field theory, a particle decay rate is determined by the expression

$$\Gamma_\psi = \overline{v\,\sigma_r}\,|\psi(0)|^2, \tag{41}$$

where v is the relative velocity of decay products, σ_r is the reaction cross-section, a bar means the averaging over non-recorded parameters (e.g., over initial spin states), and $\psi(0)$ is the wave function of a particle before the decay at the origin (at zero distance). In accordance with that, the quantity

$$\Gamma_{n,k} = \overline{v\,\sigma_r}\,|f_{n,k}(0)|^2 \tag{42}$$

can be interpreted as the rate of nucleation of the Universe in the n,k-state from the initial cosmological singularity point $a = 0$. In accordance with the classical viewpoint, the nucleation of the Universe is the process in time with the expansion initiation at some instant τ which we choose as $\tau = 0$ for convenience. The velocity v can be naturally identified with the expansion rate $v = \frac{da}{d\tau}$ at $\tau = 0$, where τ is some time parameter which ensures the boundary condition $a(\tau = 0) = 0$. The cross-section σ_r will be set equal to $\sigma_r = \pi\,a^2$. The dependence $a(\tau)$ can be found from the condition of finiteness of $\overline{v\,\sigma_r}$ at the point $a = 0$. We put

$$\overline{v\,\sigma_r} \equiv \lim_{a\to 0}\left(\frac{da}{d\tau}\,\pi\,a^2\right) = \text{const}. \tag{43}$$

It is easy to make sure that, at const $\neq 0$, the single possible case is $a(\tau) \sim \tau^{1/3}$ which is realized when the primordial matter is described by the extremely rigid equation of state at the point $a(\tau = 0) = 0$,

$$p_{in} = \rho_{in},$$

where p_{in} and ρ_{in} are the pressure and the energy density of a primordial scalar field at the moment of the nucleation of the Universe. From the viewpoint of the semiclassical approximation, such an equation of state is realized in the primordial scalar field ϕ taken in the state of its true vacuum $\phi = \sigma$, where $V(\sigma) = 0$,

$$\rho_\sigma = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2_\sigma = p_\sigma, \tag{44}$$

when the whole energy of the field ϕ is concentrated in its kinetic part. Such a state is unstable and should turn into the state with the condensate of ϕ quanta. This transition looks like a nucleation of the Universe from the point $a = 0$ with the extremely rigid equation of state (44) and the wave function $f_{n,k}(0)$, the square of which determines the rate of such a process in accordance with (42).

Using the explicit form of the function $f_{n,k}(a)$ (33) and keeping the main term only, we find

$$\Gamma_{n,k} \simeq \text{const}\,\frac{2^n}{\sqrt{\pi}}\,P(n), \tag{45}$$

where

$$P(n) = \frac{\langle n\rangle^n}{n!}\,\mathrm{e}^{-\langle n\rangle}$$

is the Poisson distribution with the mean value $\langle n\rangle = M_k^2$ of the quantum number n. The total nucleation rate of the Universe $\Gamma = \sum_n \Gamma_{n,k}$ is given by the mean value $\langle 2^n\rangle$ over the Poisson distribution. It appears to be exponentially high:

$$\Gamma \simeq \frac{\text{const}}{\sqrt{\pi}}\,\exp\{M_k^2\}. \tag{46}$$

According to quantum theory, the square of the wave function modulus at the origin determines the particle number density at this point. In the case of the quantum Universe,

$$|f_{n,k}(0)|^2 \sim L_{n,k}^{-3}, \tag{47}$$

where $L_{n,k}$ is the linear dimension of the region, from which the Universe nucleates. Hence, we find

$$L_{n,k} \sim \left(\frac{\sqrt{\pi}n!}{2^n M_k^2}\right)^{1/3} \exp\left\{\frac{M_k^2}{3}\right\}. \tag{48}$$

For $n = 0$ and $M_k = 1$, we have $L_{0,k} \sim 1.69$. This value is consistent with the mean value $\bar{a}_{n=0} = 1.11$ calculated above for the Universe with the Planck mass of the condensate.

The given estimations show that the Universe nucleates with a finite rate into the state with the Planck parameters (mass and spatial dimensions). The larger masses of the condensate of the nucleating Universe correspond to the larger primordial dimensions. The total nucleation rate of the Universe obeys an exponential law and corresponds to the "explosion" from the point $a = 0$. It can be identified with the zero time moment of the Big Bang.

As we have shown in [26], the condensate of ϕ quanta with the total mass $M_k \neq 0$ has an antigravitating property and is described by the vacuum equation of state $p_k = -\rho_k$, where $\rho_k = \frac{2M_k}{\bar{a}^3}$. The Universe from an unstable state with the extremely rigid equation of state at the point $a = 0$ passes into the ground state with the Planck mass of the condensate and the Planck scale factor. As soon as the condensate mass reaches nonzero values, the equation of state of a primordial scalar field changes from the extremely rigid equation of state to the vacuum one, and the condensate acquires an antigravitating property. The growth of M_k leads to the growth of antigravitation and, as a consequence, triggers a subsequent growth of $\bar{a}$ of the quantum Universe which undergoes, in this case, an accelerating expansion.

§5. According to (31) and (33), the function $f_{n,k}(a)$ can be interpreted as a state vector which describes the Universe in the n-th state with the condensate mass M_k. If the states $f_{n,k}$ and $f_{n',k'}$ correspond to different masses, $M_k \neq M_{k'}$, then they are eigenfunctions of different operators and therefore are nonorthogonal between themselves in the general case. The corresponding overlap integral is $\langle f_{n,k}|f_{n',k'}\rangle \neq 0$. The evolution of the Universe takes place in such a way that it passes from, say, the n,k-state in one potential well $U(a)$ (35) into the state with the quantum numbers n',k' in another well, where the number of the state n' may differ from n or be equal to it, but the index k' that is equal to the number of ϕ quanta differs from k.

Equation (32) describes the stationary states. Therefore, when calculating the transition probability $w(n,k \to n',k')$, we use the model of the instantaneous change of a state of the quantum system [21]. Then

$$w(n,k \to n',k') = |\langle f_{n',k'}|f_{n,k}\rangle|^2. \tag{49}$$

It is easy to show that the normalization condition

$$\sum_{n'} w(n,k \to n',k') = 1 \tag{50}$$

is satisfied.

Using the explicit form of the function $f_{n,k}(a)$ (33), we reduce the overlap integral of wave functions to the form

$$\langle f_{n',k'}|f_{n,k}\rangle = (-1)^{n'} N_{n',k'} N_{n,k} \times$$
$$\times \exp\left\{-\frac{\xi_0^2}{2}\right\} \int\limits_{-M_k}^{\infty} d\xi\, H_n(\xi)\, \mathrm{e}^{-\xi_0\xi} \frac{d^{n'}}{d\xi^{n'}}\, \mathrm{e}^{-\xi^2+2\xi_0\xi}, \tag{51}$$

where we denote $\xi_0 = M_{k'} - M_k$. Integrating by parts and neglecting the terms $\sim \exp\{-M_k^2\}$ for $M_k \gg 1$, we find

$$\langle f_{n',k'}|f_{n,k}\rangle = 2\sqrt{\pi}\, N_{n',k'} N_{n,k}\, \xi_0^{n-n'}\, \mathrm{e}^{-\xi_0^2} \times$$
$$\times \sum_{i=0}^{n'} (-1)^i \frac{2^{n'-i}\, n'!\, n!\, \xi_0^{2i}}{i!(n'-i)!(n-n'+i)!} \tag{52}$$

for $n \neq 0$ and

$$\langle f_{n',k'}|f_{0,k}\rangle = 2\sqrt{\pi}\, N_{n',k'} N_{0,k}\, \xi_0^{n'}\, \mathrm{e}^{-\frac{1}{4}\xi_0^2}. \tag{53}$$

The transition probability (49) at $n' > n \geq 1$ is equal to

$$w(n,k \to n',k') \simeq \frac{1}{2^{n'-n-2}} \frac{n'!}{n![(n'-n)!]^2}\, \xi_0^{2(n'-n)}\, \mathrm{e}^{-2\xi_0^2}, \tag{54}$$

where we have used expression (52) and keep only the main term with $i = n'-n$. The probability of a transition between the states with $n' < n$ follows from (54) after the substitution $n \leftrightarrow n'$ and can be obtained from (52) keeping the main term with $i = 0$. According to (54), when the difference $\xi_0 = m_\sigma(k'-k)$ grows, the transition probability decreases almost exponentially.

A nonzero transition probability indicates the possibility, in principle, for the Universe to evolve as a result of transitions between quantum states. An increase (decrease) in the condensate mass means an increase (decrease) in the mean value of the scale factor of the Universe. From the viewpoint of the semiclassical theory, the Universe will expand (contract).

Using Eq. (53), one can calculate the probability of a transition of the Universe from the ground (vacuum) state, $n = 0$, to any other state. It obeys the Poisson distribution

$$w(0,k \to n',k') = \frac{\langle n'\rangle^{n'}}{n'!}\, \mathrm{e}^{-\langle n'\rangle} \tag{55}$$

with the mean value $\langle n'\rangle = \xi_0^2/2$ of the quantum number n'.

The transition vacuum $\to$ vacuum from different potential wells $U(a)$ is given by an exponent. This means that the transitions from vacuum to non-vacuum states occur with the overwhelming probability. The ratio of the total transition probability (50) at $n = 0$ to the probabilities of transitions between the vacuum states is equal to

$$\text{Ratio} = \frac{\sum_{n'} w(0,k \to n',k')}{w(0,k \to 0,k')} = \mathrm{e}^{\langle n'\rangle}, \tag{56}$$

i.e., against the background of vacuum-vacuum transitions, the probabilities of transitions into non-vacuum states look like exponentially high.

§6. In this paper, we have calculated the spectrum of stationary states (34) and the wave functions (31) and (33) of the homogeneous isotropic Universe in the epoch of matter-energy production from the primordial uniform scalar field on the basis of the exact solution of Eq. (17) of a quantum model. The produced matter-energy represents itself the condensate of the excitation quanta of oscillations of a scalar field above its true vacuum state. The condensate mass (30) and the mean value of the scale factor (37) in a given state of the Universe are connected between themselves by the linear expression (38). We note that condition (40) is a mathematical formulation of the Mach's principle proposed by Sciama [28] (see also [29]). The Universe in an arbitrary state (31) is described by a superposition of the states with all possible masses of the condensate. The nucleation rate of the Universe from the initial cosmological singularity point (45) appears to be non-zero, while the total nucleation rate obeys the exponential (explosive) law (46). The nucleation of the Universe takes place as a result of its transition from the initial cosmological singularity point with the extremely rigid equation of state of a primordial scalar field into the state with the non-zero mass of the condensate with the vacuum equation of state. The Universe being nucleated in the ground (vacuum) state has the Planck parameters. The evolution of the Universe is described as transitions with the non-zero probability (54) between the states of the Universe with different masses of the condensate. An increase (decrease) in this mass leads to an expansion (contraction) of the Universe.

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КВАНТОВА ЕВОЛЮЦІЯ ПЕРВІСНОГО ВСЕСВІТУ

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Р е з ю м е

Запропоновано послідовну квантову космологію, що описує всесвіт поблизу початкового моменту Великого Вибуху. Розраховано спектр стаціонарних станів та хвильові функції квантового всесвіту Фрідмана, який від початку заповнений однорідним скалярним полем та ідеальною рідиною, яка задає систему відліку. Показано, що матерія-енергія у всесвіті має компоненту у формі конденсату масивних квантів збудження з нульовими імпульсами, які відповідають коливанням первісного скалярного поля. Процес народження всесвіту з точки початкової космологічної сингулярності може мати експоненціальну (вибухову) природу. Еволюція всесвіту описується як переходи з ненульовою імовірністю між станами всесвіту з різними масами конденсату.