CONFORMAL GRAVITY IN THE WEAK-FIELD LIMIT

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UDC 530.12;531.51

In the weak-field limit, we have found the solution of the field equations of conformal gravity in the case of an arbitrary spherically symmetric stationary distribution of matter in the gauge with the value of a scalar field invariable in space-time. The analysis of the solution obtained allows us to conclude that the consequences of conformal gravity do not agree with the data of observations in the scope of the Sun's system.

1. Introduction

The conformal theory of gravitation was proposed by Weyl [1] and was aimed at the unification of electromagnetic and gravitational forces into a single interaction which would have the purely geometric nature. To this end, in addition to the introduction of the principle of conformal invariance (i.e. the invariance of the action with respect to the below-presented transformation (1)), Weyl was forced to change the Riemann geometry, by generalizing the affine connectedness. Weyl considered a geometry, in which the scalar product of two vectors does not remain invariable on a parallel translation. In this case, an additional vector field which was identified by Weyl with an electromagnetic field appeared in the theory. But, as was noted else by Einstein in his comment to the work by Weyl, the variability of a scalar product leads to the contradiction with experimental data (e.g., the wavelength emitted by an atom would depend on its prehistory).

From the 1980s, the conformal gravity becomes again actual due to works by Mannheim and Kazanas [5]–[12]. As distinct from Weyl, these authors accept the principle of conformal invariance *ad hoc*, by retaining the Riemann geometry invariable. In such a version, the conformal theory does not lead to the contradictions indicated by Einstein and preserves its most attractive features.

The conformal theory is based on the Weyl principle of conformal invariance, i.e. the invariance of the action with respect to the transformation

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}, \qquad (1)$$

where $\Omega^2(x)$ is an arbitrary function of coordinates. This principle restricts strongly the possible form of the Lagrangian. For example, for the gravitational and electromagnetic fields, there exists only the single conformally invariant action (moreover, only in the four-dimensional space-time. As for this point, Weyl considered that it gives a possible answer to the question: Why is our Universe four-dimensional?).

The conformal theory has some advantages as compared with the standard theory. For example, the former has no problem of a cosmological singularity and contains the justification of the modern data of measurements of the deceleration parameter and the distribution of the energy densities of matter and vacuum in the Universe [11]. The conformal theory of gravitation simulates the effects of dark matter and dark energy [12], which levels the problem of clarification of their nature.

Despite these exciting successes of the conformal theory, we should not forget that the necessary condition for any alternative theory of gravitation to be suitable consists in the reproduction of results of the general relativity theory (GRT) at distances of the order of the size of the Sun's system, for which the Einstein theory is well verified.

It would be hoped for that the vacuum solutions of the conformal theory, as those of any other theory of gravitation constructed on the basis of the Ricci tensor $R_{\mu\nu}$, contain the Schwartzschild solution and, hence, explain the same experimental facts, as the Einstein theory does: the displacement of the Mercury perihelion, deviation of light rays in the gravitational field, gravitational shift of spectral lines, etc. Such a thought was supported by works, where the spherically symmetric and static solutions of the equations of the conformal theory in vacuum were obtained [4].

Recently, work [3], in which the qualitative calculations showed that the conformal theory contradicts the data of observations in the Sun's system, was published. Answering this work, Mannheim in [13] called its results in question and proposed a distribution of the special form for a point source (containing the singularities of a higher order than that of the ordinary δ-function), for which the Schwartzschild metric would be reconstructed.

The purpose of the present work is the determination of the exact solution of the field equations of conformal gravity for an arbitrary spherically symmetric stationary distribution of matter in the weak-field approximation[1]. The analysis of this solution answers the question about the consistency of the conformal theory with the known data of observations in the Sun's system.

2. Problem of the Existence of the Schwartzschild Solution in Conformal Gravity

In the conformal theory, the action for gravity and the fields of matter (the scalar S and fermionic ψ ones) looks as [8]

$$I_W = \int d^4x\sqrt{-g}\Big(-\alpha_g C_{\lambda\mu\nu\sigma}C^{\lambda\mu\nu\sigma} - \partial^\mu S\partial_\mu S/2-$$

$$-\lambda S^4 + S^2R/12 - i\bar{\psi}\gamma^\mu(x)\nabla_\mu\psi + \zeta S\bar{\psi}\psi\Big),$$

where $C_{\lambda\mu\nu\sigma}$ is the Weyl tensor, and α_g, λ, and ζ are dimensionless constants (in units of $c = \hbar = 1$). This action is invariant under a transformation of the metric and the fields of the form

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \quad \psi \to \Omega^{-3/2}\psi, \quad S \to S/\Omega.$$

The transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ is called conformal, because it does not change the angles between 4-vectors. The variation of the action with respect to $g_{\mu\nu}$, S, and ψ yields the equations of motion [11], [14]

$$4\alpha W_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

$$S^{\mu}_{;\mu} + SR/6 - 4\lambda S^3 + \zeta\bar{\psi}\psi = 0, \tag{3}$$

$$i\gamma^\mu\nabla_\mu\psi - \zeta S\psi = 0, \tag{4}$$

where

$$T_{\mu\nu} = 2S_\mu S_\nu/3 - g_{\mu\nu}S^\alpha S_\alpha/6 - SS_{\mu;\nu}/3+$$

$$+g_{\mu\nu}SS^{\alpha}_{;\alpha}/3 - S^2(R_{\mu\nu} - g_{\mu\nu}R/2)/6- \tag{5}$$

$$-\lambda S^4 g_{\mu\nu} + i\bar{\psi}\gamma_\mu\nabla_\nu\psi + g_{\mu\nu}(\zeta S\bar{\psi}\psi - i\bar{\psi}\gamma^\lambda\nabla_\lambda\psi),$$

and the tensor $W_{\mu\nu}$ was first constructed in [14] and is expressed through the Ricci tensor and its derivatives.

In the conformal theory, the material bodies move along a trajectory which is set by the equation [10]

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\frac{S_\mu}{S}\left(g^{\lambda\mu} + \frac{dx^\lambda}{ds}\frac{dx^\mu}{ds}\right), \tag{6}$$

rather than along a geodesic line (the equation for a geodesic line is not conformally invariant). Equation (6) and Eqs. (2)–(4) are conformally invariant. The conformal invariance of the equations guarantees the independence of physical results of the choice of a gauge. However, it is convenient to fix a gauge by the condition[2]

$$S = S_0 = \text{const}.$$

In this case, we have the standard theory of a massive fermionic field ψ and, what is especially important for our study, the ordinary equation for a geodesic line in the case of the motion of a free body. The equation for a geodesic line is well checked by standard gravitational tests for massive bodies and for rays of light, if we use the Schwartzschild metric as a the metric of space-time[3]. Thus, we deal with a good test for the verification of the conformal theory: in the gauge $S = S_0 = \text{const}$, the external metric for a massive source should be reduced

[1]In this case, as the energy density of a point source, we use the commonly accepted formula $\varepsilon(\vec{r}) = m\delta(\vec{r})$, because, in our opinion, there are no reasons to introduce the higher-order singularities.

[2]In this case, it is assumed that the field S has a constant sign and is not equal to zero everywhere. For specificity, we assume that $S > 0$ everywhere in space-time (in order that this condition be satisfied, it is sufficient that the inequality $S > 0$ be satisfied at least on one space-like surface). The inverse inequality $S < 0$ leads to the same action [3], if we pass from the field ψ to the field $\hat{\psi}$ which is connected with ψ by the unitary transformation $\hat{\psi} = \gamma_5\psi$.

[3]There are, however, the data of radar observations obtained with the Pioneer 10/11 satellites and others which indicate that the metric on the scale of the Sun's system differs somewhat from the Schwartzschild metric [15]. The satellites fixed a constant (independent of the distance from the Sun) additional radial acceleration directed to the Sun in the limits of $\sim 8.5 \times 10^{-10}$ m/s^2. In [2], the attempt was made to explain this additional acceleration in the frame of the conformal theory.

to the Schwartzschild metric on the scale of the Sun's system.

It is easy to verify that the Schwartzschild metric

$$-g_{00} = g_{11}^{-1} = 1 - \frac{2m}{r} \tag{7}$$

is, indeed, a solution of Eqs. (2) in vacuum ($\psi = 0$) on small scales, where we can neglect the cosmological term $\lambda S_0^4 g_{\mu\nu}$ in (5). As known, metric (7) is a solution of the equation $R_{\mu\nu} = 0$, but since the tensor $W_{\mu\nu}$ is constructed only from the Ricci tensor and its derivatives, it will become zero. For the tensor $T_{\mu\nu}$, we have also $T_{\mu\nu} = 0$, which follows from (5) in the gauge $S = S_0$ and for $\psi = 0$.

However, the above-presented consideration has a formally mathematical character and does not correspond to the physical statement of the problem. The proper metric outside a source must be consistent with the internal metric. The conditions of sewing can be violated for the external solution (7) or can lead to a negative value of the constant m. For example, the external spherically symmetric solution of Eqs. (2) in the gauge $S = S_0$ in the weak-field limit was obtained in [4]. For a metric in the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega_2, \tag{8}$$

$$B(r) = 1 - b(r), \quad A(r) = 1 + a(r),$$

the solution looks as [4]

$$a(r) = \frac{2m}{r} + N\left[\frac{\sin(kr+\phi)}{r} - k\cos(kr+\phi)\right],$$

$$b(r) = \frac{2m}{r} + 2N\,\frac{\sin(kr+\phi)}{r}. \tag{9}$$

If we assume that the constant $m > 0$ and set $\phi = 0$, then metric (9) passes to the Schwartzschild metric at small distances ($kr \ll 1$). However, as will be shown below, the conditions of sewing imply that both assumptions ($m > 0$ and $\phi = 0$) turn out to be wrong [see (31)].

3. Solution of the Equations for a Field in Matter

In the gauge $S = S_0$, the tensor of energy-momentum of a matter (5) in the hydrodynamic approximation takes the form

$$T_{\mu\nu} = \varepsilon u_\mu u_\nu - S_0^2(R_{\mu\nu} - g_{\mu\nu}R/2)/6 - \lambda S_0^4 g_{\mu\nu}. \tag{10}$$

We note that the trace of the tensor $W_{\mu\nu}$ is equal to zero identically, which is a result of the conformal invariance of the theory. By equating the trace $T_{\mu\nu}$ from (10) to zero (by virtue of Eq. (2) and the above-mentioned property of the tensor $W_{\mu\nu}$), we get

$$R = 24\lambda S_0^2 + \frac{6}{S_0^2}\varepsilon. \tag{11}$$

For a spherically symmetric and static distribution of matter, the metric has the standard form (8). With the help of a transformation of the radial variable, metric (8) can be represented in a form more convenient for calculations,

$$ds^2 = C^2(\rho)[-D(\rho)dt^2 + d\rho^2/D(\rho) + \rho^2 d\Omega_2], \tag{12}$$

where $r(\rho) = \rho C(\rho)$, and the functions $C(\rho) = 1 + c(\rho)$ and $D(\rho) = 1 - d(\rho)$ are connected with the metric coefficients $A(r) = 1 + a(r)$ and $B(r) = 1 - b(r)$ by the relations (in the linear approximation)

$$a(r) = d(r) - 2rc'(r), \quad b(r) = d(r) - 2c(r). \tag{13}$$

In this case, all the quantities $a(r)$, $b(r)$, $c(\rho)$, and $d(\rho)$ should be considered as small values of the same order $\sim \epsilon \ll 1$ (the condition of applicability of the linear approximation). In the case under consideration, only two equations from ten ones (2) are independent. It is convenient to choose them in the form

$$4\alpha(W_0^0 - W_1^1) = \mathcal{T}_0^0 - \mathcal{T}_1^1, \quad 4\alpha W_{11} = \mathcal{T}_{11}, \tag{14}$$

where

$$\mathcal{T}_{\mu\nu} = \varepsilon u_\mu u_\nu + \varepsilon g_{\mu\nu}/4 - S_0^2(R_{\mu\nu} - g_{\mu\nu}R/4)/6$$

is the traceless part of the tensor $T_{\mu\nu}$, and the index "1" designates the radial coordinate ρ. The formulas for the left-hand sides of equalities (14) in metric (12) were obtained in [5] and look as

$$W_0^0 - W_1^1 = \frac{D(\rho D)''''}{3\rho C^4},$$

$$W_{11} = \frac{1}{3C^2 D}\left(\frac{D'D'''}{2} - \frac{D''^2}{4} - \frac{DD''' - D'D''}{\rho} - \right.$$

$$\left. - \frac{DD'' + D'^2}{\rho^2} + \frac{2DD'}{\rho^3} - \frac{D^2}{\rho^4} + \frac{1}{\rho^4}\right).$$

For the right-hand sides of Eqs. (14) and the scalar curvature R, we have

$$\mathcal{T}_0^0 - \mathcal{T}_1^1 = -\varepsilon + \frac{S_0^2 D}{3C}\left(\frac{C'}{C^2}\right)',$$

$$\mathcal{T}_{11} = -\frac{S_0^2}{12}\left(\frac{D''}{2D} + \frac{1-D}{\rho^2 D} + 3F'' + \right.$$

$$\left. + \frac{F'D'}{D} - \frac{2F'}{\rho} - 3F'^2\right) + \frac{C^2}{4D}\varepsilon,$$

$$R = \frac{6(\rho^2 DC')' - C(\rho^2(1-D))''}{\rho^2 C^3},$$

where $F = \ln C$. By linearizing Eq. (14) and relation (11) with respect to $c(\rho)$ and $d(\rho)$, we obtain the equations

$$-(\rho d)'''' = 6p\rho c'' - \frac{3\rho\varepsilon}{4\alpha}, \tag{15}$$

$$\frac{1}{3\rho}\left(d''' + \frac{d''}{\rho} - \frac{2d'}{\rho^2} + \frac{2d}{\rho^3}\right) =$$

$$= \frac{p}{2}\left(\frac{d''}{2} - \frac{d}{\rho^2} - 3c'' + \frac{2c'}{\rho}\right) + \frac{\varepsilon}{16\alpha}, \tag{16}$$

$$6(\rho^2 c')' = (\rho^2 d)'' + 24q\rho^2 + \frac{6}{S_0^2}\varepsilon\rho^2, \tag{17}$$

where we use the notations

$$p = \frac{S_0^2}{24\alpha_g}, \quad q = \lambda S_0^4.$$

Before proceeding to the solution of Eqs. (15)–(17), we make a reservation. As known, the dynamics of the field of matter and the metric field are closely connected with each other in the GRT: the energy-momentum tensor of matter determines a metric, whereas the metric itself sets the dynamics of matter. As a result, we cannot, generally saying, arbitrarily set a distribution of matter $\varepsilon(r)$, $p(r)$, and its dynamics u_μ. This property of the mutual influence of the metric on the dynamics of matter and conversely is caused by the nonlinearity of the field equations (and, in particular, by the absence of the principle of superposition). But, in the linear approximation, the metric which enters the tensor $T_{\mu\nu}$ is replaced by the Minkowski metric. As a result, the dynamics of matter is considered against the background of a plane space-time and can be set arbitrarily but so that the laws of conservation are satisfied. For example, it is easy to verify that the Einstein equations in the linear approximation admit the stationary solutions for an arbitrary spherically symmetric distribution of matter at the zero pressure. This is also true for the conformal theory, which follows from the self-consistency of Eqs. (2) in the linear approximation at $p = 0$ and arbitrary $\varepsilon(r)$. In particular, the system of three equations (15) – (17) for two unknown variables $c(\rho)$ and $d(\rho)$ is consistent for any $\varepsilon(\rho)$.

According to [4], we set

$$d(\rho) = -2q\rho^2 + v(\rho) \tag{18}$$

(this allows us to exclude the cosmological term $q\rho^2$ from Eqs. (15) – (17)) and pass to new independent functions $\tilde{a}(\rho)$ and $\tilde{b}(\rho)$ which are connected with $v(\rho)$ and $c(\rho)$ by the relations [see Eqs. (13) and (18)]

$$\tilde{a}(\rho) = v(\rho) - 2\rho c'(\rho), \quad \tilde{b}(\rho) = v(\rho) - 2c(\rho).$$

As a result, Eqs. (15)–(17) take the form

$$-\left(\frac{(\rho y)''}{\rho}\right)' = -\frac{18\rho p}{S_0^2}\varepsilon + 3p(\tilde{b} - \tilde{a})', \tag{19}$$

$$\left(\frac{(\rho y)'}{\rho^2}\right)' = \frac{3p}{2}\left(\frac{3}{2}\rho\tilde{b}'' - y' - \frac{\tilde{a}}{\rho}\right) + \frac{9p}{2S_0^2}\rho\varepsilon, \tag{20}$$

$$(\rho(\rho\tilde{b}' + 2\tilde{a}))' = -\frac{6\rho^2\varepsilon}{S_0^2}, \tag{21}$$

where $y = y(\rho) = \rho\tilde{b}'(\rho) - \tilde{a}(\rho)$. By integrating Eqs. (19) and (21) in the limits from 0 to ρ, we get

$$-(\rho y)'' = 3p\rho(\tilde{b} - \tilde{a}) + 6p\rho g(\rho) + C_1\rho,$$

$$\rho\tilde{b}' + 2\tilde{a} = \varphi(\rho) + C_2/\rho, \tag{22}$$

where

$$g(\rho) = -\frac{3}{S_0^2}\int\limits_0^\rho r\varepsilon(r)\,dr, \qquad \varphi = \varphi(\rho) = -\frac{3M(\rho)}{2\pi S_0^2\rho},$$

$$M(\rho) = 4\pi\int\limits_0^\rho \varepsilon(r)r^2 dr.$$

The integration constant C_2 in (22) should be set to zero, because the metric must not be singular for a nonsingular distribution $\varepsilon(\rho)$. Thus, Eq. (22) takes the form

$$\rho\tilde{b}' + 2\tilde{a} = \varphi(\rho). \tag{23}$$

In order to exclude $\tilde{a}(\rho)$ from this equation, we divide (23) by ρ, integrate in the limits from ρ to infinity, and then obtain the integral with respect to the quantity $\tilde{a}(\rho)/\rho$ from Eq. (20). As a result, we get

$$\tilde{b}'' + \frac{2}{\rho}\tilde{b}' + p\tilde{b} = -\frac{2\varepsilon}{S_0^2} - \frac{3p}{2\pi S_0^2}\int\limits_\rho^\infty \frac{M(x)}{x^2}\,dx. \tag{24}$$

While integrating, we assume that the function $\tilde{b}(\rho)$ tends to zero at infinity.

Equation (24) can be integrated in a simple way. To this end, we find firstly the solution of (24) in the case of a point source with the density $\varepsilon(\vec{\rho}) = m\delta(\vec{\rho})$. The corresponding metric function is denoted as $\beta(\rho)$. The equation for $\beta(\rho)$ has the form

$$\Delta\beta(\rho) + p\beta(\rho) = -\frac{2m}{S_0^2}\delta(\vec{\rho}) - \frac{3pm}{2\pi S_0^2\rho}.$$

To solve this equation, we separate a part of $\beta(\rho)$ that is singular at zero. This part is due to the source

$$\beta(\rho) = \frac{\alpha}{\rho} + \beta_1(\rho),$$

where the function $\beta_1(\rho)$ is regular at zero. We now use the known relation

$$\Delta\frac{1}{r} = -4\pi\delta(\vec{r}),$$

which implies that the constant α should be chosen to be equal to $m/2\pi S_0^2$. The equation for β_1,

$$\beta_1'' + \frac{2}{\rho}\beta_1' + p\beta_1 = -\frac{2pm}{\pi S_0^2\rho},$$

is easily integrated after the substitution $\beta_1(\rho) = u(\rho)/\rho$ under the boundary condition $u(0) = 0$. Finally, we obtain

$$\beta(\rho) = \frac{m}{2\pi S_0^2\rho} + \frac{n}{4\pi S_0^2}\frac{\sin(k\rho)}{\rho} - \frac{2m}{\pi S_0^2}\frac{1-\cos(k\rho)}{\rho},$$

where $n = \text{const}$, and $k = \sqrt{p}$ (for now, we consider the case where $\alpha_g > 0$). For the metric of a point source called below as a nucleon, we obtain

$$g_{00} = 1 - \frac{m}{2\pi S_0^2 r} - \frac{n}{4\pi S_0^2}\frac{\sin(kr)}{r} + \frac{2m}{\pi S_0^2}\frac{1-\cos(kr)}{r} + 2qr^2,$$

$$g_{11} = 1 - \frac{m}{2\pi S_0^2 r} + \frac{n}{8\pi S_0^2}\left(\frac{\sin(kr)}{r} - k\cos(kr)\right) + \frac{m}{\pi S_0^2}\left(k\sin(kr) - \frac{1-\cos(kr)}{r}\right) - 2qr^2. \qquad (25)$$

In what follows, we will omit the cosmological term qr^2, which is obviously small on the scale of the Sun's system, in the metric. The solution of (25) depends not only on the nucleon mass m, but also on the constant n, whose physical content will not be discussed in the present work.

Let us now turn to the question about the integration of Eq. (24). The solution obtained for a point source $\beta(\rho)$ plays the role of the Green's function of Eq. (24) in the sense that

$$\tilde{b}(\vec{r}) = \int \beta(\vec{r} - \vec{r}\,')n(\vec{r}\,')\,dV', \qquad (26)$$

where $n(\vec{r}) = \varepsilon(\vec{r})/m$ is the concentration of nucleons. Since the distribution of matter $n(r)$ is assumed to be spherically symmetric, we can integrate (26) with respect to angular variables, which yields the following formula for the metric of a distributed source:

$$b(r) = -\frac{3}{2\pi S_0^2}\int\limits_r^\infty \frac{M(r')dr'}{r'^2} + \frac{8}{S_0^2}\int\limits_r^\infty \varepsilon(r')\frac{\sin[k(r-r')]}{kr}r'dr' + \left[\frac{\eta}{S_0^2}\frac{\sin kr}{kr} + \frac{8}{S_0^2}\frac{\cos kr}{kr}\right]\int\limits_0^\infty \varepsilon(r')\sin(kr')r'dr', \qquad (27)$$

$$a(r) = -\frac{3}{2\pi S_0^2}\frac{M(r)}{r} + \frac{4}{S_0^2}\int\limits_r^\infty \varepsilon(r')\left(\frac{\sin[k(r-r')]}{kr} - \cos[k(r-r')]\right) - \frac{1}{2}\left[\frac{\eta}{S_0^2}\left(\cos kr - \frac{\sin kr}{kr}\right) - \frac{8}{S_0^2}\left(\sin kr + \frac{\cos kr}{kr}\right)\right]\int\limits_0^\infty \varepsilon(r')\sin(kr')r'dr', \qquad (28)$$

where $\eta = n/m$ is the dimensionless constant characterizing a nucleon. In particular, for a spherically symmetric object (we will call it as a star) with mass M and radius R (i.e., $\varepsilon(r > R) = 0$), relations (27) and (28) take the form

$$b(r > R) = -\frac{3M}{2\pi S_0^2 r} + \frac{\eta\sin kr + 8\cos kr}{krS_0^2}C,$$

$$a(r > R) = -\frac{3M}{2\pi S_0^2 r} - \frac{C}{2S_0^2}\left[\eta\left(\cos kr - \frac{\sin kr}{kr}\right) - 8\left(\sin kr + \frac{\cos kr}{kr}\right)\right], \qquad (29)$$

where

$$C = \int_0^R \varepsilon(r)\sin(kr) r dr.$$

Solution (29) corrects the external solution for a spherically symmetric source. The latter,

$$a(r > R) = \frac{2m}{r} + N\left[\frac{\sin(kr+\phi)}{r} - k\cos(kr+\phi)\right],$$

$$b(r > R) = \frac{2m}{r} + 2N\frac{\sin(kr+\phi)}{r}, \quad (30)$$

was obtained earlier in [4] and includes three integration constants m, N, and ϕ, whose relation to the source mass was not clarified in [4]. By comparing solution (29) with (30), we obtain the following relations:

$$m = -\frac{3M}{4\pi S_0^2}, \quad \phi = \mathrm{arctg}\,(8/\eta),$$

$$N = \frac{C}{kS_0^2}\left(1 + \frac{\eta^2}{64}\right)^{1/2}. \quad (31)$$

4. Analysis of the Solution and the Discussion of Results

Let us analyze solution (29) obtained for the cases where $kR \ll 1$, $kR \gg 1$, and $kR \approx 1$.

4.1. Case 1: $kR \ll 1$

As known, the function $b(\vec{r})$ in the case of a weak gravitational field is related to the Newton potential $\Phi(\vec{r})$ by the formula $\Phi(\vec{r}) = -b(\vec{r})/2$. Substituting it in (24) and taking into account that the terms $p\tilde{b}$ and $\frac{3p}{2\pi S_0^2}\int_\rho^\infty \frac{M(x)}{x^2}dx$ are small on the scale $kr \ll 1$, we can conclude that Eq. (24) agrees[4] with the Newton law for gravity $\Delta\Phi(\vec{r}) = 4\pi\varepsilon(\vec{r})$, if we set

$$G = \frac{1}{4\pi S_0^2}. \quad (32)$$

Then, in the case under consideration, we can replace $\sin kr$ in the integrand for the constant C by kr, which gives $C = kM/4\pi$. As a result, solution (29) takes the form

$$b(r > R) = \frac{2GM}{r} + MG\eta\frac{\sin(kr)}{r} - 8MG\frac{1-\cos(kr)}{r},$$

$$a(r > R) = -\frac{2GM}{r} + \frac{MG\eta}{2}\left(\frac{\sin(kr)}{r} - k\cos(kr)\right) +$$

$$+4MG\left(k\sin(kr) - \frac{1-\cos(kr)}{r}\right) \quad (33)$$

and coincides with solution (25) obtained earlier for a nucleon with the replacements $m \to Nm$ and $n \to Nn$, where $N = M/m$ is the number of nucleons. Near the source, $kr \ll 1$, and the main contribution to the metric is given only by the first terms on the right-hand side of (33). Therefore, we can write

$$b(r > R) = \frac{2GM}{r}, \quad a(r > R) = -\frac{2GM}{r}, \quad (kr \ll 1).$$

The formula for $b(r)$ coincides with the Schwartzschild solution for the source with mass M and, hence, correctly describes the motion of bodies with nonrelativistic velocities and yields the proper value for the gravitational red shift of spectral lines. However, the formula for $a(r)$ has erroneous sign. Thus, the metric is conformally planar at small distances from a source, which leads to a wrong law for the deviation of rays of light in the field of a star. Namely, for the deviation angle $\triangle\varphi$, metric (33) yields the expression

$$\triangle\varphi = \rho\int_\rho^\infty \frac{a(r) - rb'(r)}{r(r^2-\rho^2)^{1/2}}dr =$$

$$= 3\pi r_g k\left[k\rho\left(\frac{\eta}{16} - \frac{1}{\pi}(\ln\frac{k\rho}{2} + \gamma + 1/2)\right) + O(k^2\rho^2)\right],$$

where ρ is the impact parameter, γ is the Euler constant, and $r_g = 2MG$. This formula is essentially different from the well-known experimentally verified law $\triangle\varphi = 2r_g/\rho$. Their ratio $\sim k^2\rho^2\ln(1/k\rho) \ll 1$.

4.2. Case 2: $kR \gg 1$

Integrating the formula for the constant C by parts and taking into account that $\varepsilon(R)$ and its derivatives are zero on the star surface, we get

$$C \simeq -\frac{2\varepsilon'(0)}{k^3} \sim -kM\left(\frac{1}{kR}\right)^4.$$

Thus, everywhere outside the star, the ratio of the last term in formula (29) for $b(r)$ to the first term has the order of $(1/kR)^4$ and can be considered equal to zero

[4] We recall that the function $b(r)$ differs from $\tilde{b}(r)$ only by the cosmological term $2qr^2$.

in the limiting case under consideration. As a result, we get

$$b(r > R) = -\frac{3M}{2\pi S_0^2 r}, \quad (kR \gg 1),$$

which corresponds to the gravitational repulsion instead of the attraction.

4.3. Case 3: $kR \approx 1$

In this case, all terms in (29) for $b(r)$ are of the same order of smallness, and the functions $\sin kr$ and $\cos kr$ vary significantly on the scale of the Sun's system. As a result, we would not obtain the ordinary Kepler orbits for planets, and the strongly elongated orbits of comets would be unclosed and would have a significant displacement of the perihelion per one rotation.

Thus, we are forced to conclude that, for all the possible values of the parameter kR, the consequences of the conformal gravity do not agree with the data of observations on the scale of the Sun's system. This result is completely consistent with the conclusions made in [3].

Previously, we have considered the case where $\alpha_g > 0$. But if $\alpha_g < 0$, then the hyperbolic functions $\mathrm{sh}\, kr$ and $\mathrm{ch}\, kr$ appear instead of $\sin kr$ and $\cos kr$ in the solution. This implies that the metric will exponentially grow at infinity, which is physically inadmissible. Though the linear approximation stops to be true at great distances, the presence of increasing exponential functions would mean that any arbitrarily small mass would create a very strong gravitational field at distances of the order of $1/\sqrt{-p}$. We also note that conformal gravity includes guests ($\alpha_g < 0$) or tachions ($\alpha_g > 0$). The presence of tachions in the theory means the instability of a wide class of classical solutions, including the solutions for the flat space-time.

5. Conclusions

We have found the solution of the field equations of conformal gravity for an arbitrary spherically symmetric stationary distribution of matter in the weak-field limit. The analysis of the solution obtained implies that the conformal gravity contradicts the data of observations in the scope of the Sun's system for all the values of parameters of the theory.

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Received 18.10.07.
Translated from Ukrainian by V.V. Kukhtin

КОНФОРМНА ТЕОРІЯ ГРАВІТАЦІЇ В НАБЛИЖЕННІ СЛАБКОГО ПОЛЯ

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Р е з ю м е

У наближенні слабкого поля знайдено розв'язок польових рівнянь конформної теорії гравітації для випадку довільного сферично-симетричного стаціонарного розподілу речовини у калібровці з постійним в просторі-часі значенням скалярного поля. Аналіз отриманого розв'язку приводить до висновку, що наслідки конформної теорії гравітації не узгоджуються з даними спостережень у межах Сонячної системи.