ON ISOSPECTRAL PARTNERS FOR THE $\mathcal{PT}$-SYMMETRIC COMPLEX POTENTIALS

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We have discussed the Darboux method to derive isospectral partners of a complex $\mathcal{PT}$-invariant potential $V(x) = -V_1 \frac{q_c}{q_0} \mathrm{sech}^2_{q_c} \lambda x + V_2 \frac{q_c}{q_0} \mathrm{cosech}^2_{q_c} \lambda x$. We have constructed the Pöschl–Teller potential which gives rise to the real energy spectrum. The supersymmetric partner potential has the same energy including the zero-energy state with the emphasis on a particular type.

1. Introduction

The introduction of $\mathcal{PT}$-symmetric quantum mechanics [1] generated renewed interests in the analysis of quantum mechanical potentials including their physically relevant solutions, their energy spectrum, and their physical and mathematical interpretations. $\mathcal{PT}$-symmetric Hamiltonians are required to be invariant under the simultaneous action of space($\mathcal{P}$) and time($\mathcal{T}$) reversals. One-dimensional potentials $V(x)$ incorporating $\mathcal{PT}$-symmetry have to satisfy the relation $V(x) = [V(-x)]^*$ or $[V(x)] = [V(\xi - x)]^*$. Based on the results of various numerical studies, Bender and his co-workers [1,2] have found certain examples of one-dimensional $\mathcal{PT}$-invariant non-Hermitian Hamiltonians that possess real spectra, and their spectral properties are linked with $\mathcal{PT}$-symmetry. The complex $\mathcal{PT}$-symmetric Hamiltonians have been found to have a real discrete spectrum provided the energy eigenstates are also eigenstates of $\mathcal{PT}$; if not, then the $\mathcal{PT}$-symmetry is called spontaneously broken, and there are complex conjugate pairs of energy eigenvalues [2–10].

The existence of several non-$\mathcal{PT}$-symmetric complex potential models [11,12] with real energy spectra confirms that this symmetry is not a sufficient nor necessary condition for the reality of the energy spectrum. Recently, Mostafazadeh [13] has introduced, in his very noteworthy work, the concept of pseudo-Hermiticity, in which he has pointed out that all the $\mathcal{PT}$-symmetric Hamiltonians regarded so far are actually $\mathcal{P}$-pseudo Hermitian, namely $\mathcal{P}H\mathcal{P}^{-1} = H^\dagger$. By highlighting the concept of pseudo-Hermiticity, he has addressed that it is a generalization of Hermiticity. He has claimed that the necessary and sufficient condition for the reality of the energy spectrum of any Hamiltonian is that the Hamiltonian admits a complete set of biorthonormal eigenvectors.

In the present work, we construct new complex potentials by using the Darboux method [14], where the new potentials are not necessarily $\mathcal{PT}$-symmetric but also give rise to a real and discrete spectrum, provided the original potential admits real energies only.

2. Pöschl–Teller Potential

In the literature [15], the general pseudo-Hermitian Pöschl–Teller potential is usually given in the form

$$V(x) = -V_1 \frac{q_c}{q_0} \mathrm{sech}^2_{q_c} \lambda x + V_2 \frac{q_c}{q_0} \mathrm{cosech}^2_{q_c} \lambda x, \tag{1}$$

where $\sqrt{q} = iq_c,\ q_c = q_0 e^{2i\lambda\epsilon},\ q_0 > 0,\ \alpha = 2\lambda$, and the deformed hyperbolic function is defined as

$$\sinh_q x = \frac{e^x - qe^{-x}}{2}, \cosh_q x = \frac{e^x + qe^{-x}}{2},$$

$$\tanh_q x = \frac{\sinh_q x}{\cosh_q x}.$$

If the parity operator ($\mathcal{P}$) and the time reversal one ($\mathcal{T}$) are defined, respectively, as $\mathcal{P} : x \longrightarrow \frac{\ln q_0}{\lambda} - x$ and $\mathcal{T} : i \longrightarrow -i$, then one can obtain the relations $\mathcal{P}V(x)\mathcal{P}^{-1} = V(x)^*$ i.e. $V(x)$ is $\mathcal{P}$ pseudo-Hermitian and $\mathcal{PT}V(x)(\mathcal{PT})^{-1} = V(x)$, $V(x)$ is the $\mathcal{PT}$-symmetric.

3. Darboux Method

We now give a brief review of the Darboux method. This method relates the spectral properties of a pair of standard Schrödinger Hamiltonians

$$H_\pm = -\frac{\hbar^2}{2m}\frac{d^2}{dx} + V_\pm(x). \tag{2}$$

Let us assume that one of the spectral properties of one of these Hamiltonians, say H_+, is exactly known. Thus,

$$H_+\phi_n(x) = E_n\phi_n(x). \quad (3)$$

For simplicity, we will assume that H_+ has a purely discrete positive spectrum enumerated by $n = 0, 1, 2, ...$ so that $0 < E_0 < E_1 < E_2 < ......$ Let there exist a linear operator A satisfying an intertwining relation

$$AH_+ = H_-A. \quad (4)$$

Then the functions $\psi_n = A\phi_n \neq 0$ are obviously eigenfunctions of the other Hamiltonian H_-, and

$$H_-\psi_n(x) = E_n\psi_n(x) \quad (5)$$

with the same energy $E_n > 0$. A general form for an intertwining operator satisfying (4) is given by

$$A = \sum_{k=0}^{N} f_k(x)\frac{d^k}{dx^k}, \quad (6)$$

where the $f_k : \mathbb{R} \longrightarrow \mathbb{C}; k = 0, 1, 2, ...N-1$, are (at least twice) differentiable functions to be determined via (4), and f_N is an arbitrary constant.

We take the simplest nontrivial choice for A ($N = 1$)

$$A = -\frac{d}{dx} + f(x);\ f : \mathbb{R} \longrightarrow \mathbb{C} \quad (7)$$

Using (4) and (7), we have

$$[V_- - V_+ + 2f']\frac{d}{dx} - [(V_- - V_+)f + V_+' - f'']1 = 0, \quad (8)$$

where prime denotes the differentiation with respect to x and '1' is the unit operator. We have

$$V_- = V_+ - 2f', \quad (9)$$

$$[V_- - V_+]f + V_+' - 2f'' = 0. \quad (10)$$

Substituting (10) into (9), we obtain

$$f^2 + f' - V_+ = -\varepsilon = \text{const}. \quad (11)$$

Setting $f = \frac{g'}{g}$ in (11), one can obtain the Schrödinger-like equation

$$\left(-\frac{d^2}{dx^2} + V_+\right) g(x) = \varepsilon g(x). \quad (12)$$

It is worth noting here that g(x) need not be square-integrable. So we are not restricted to normalizable solutions of (12). However, for A to be well defined, g must not have any zero on the real line. For this, it requires the condition $\varepsilon < E_0$ to hold.

In terms of $f(x)$, the two potentials are expressed as

$$V_\pm(x) = f^2(x) \pm f'(x) + \varepsilon. \quad (13)$$

It is noted that the potentials $V_\pm$ are the supersymmetric partner potentials, when $\varepsilon = 0$, and the Hamiltonians $H_\pm$ are the supersymmetric partner Hamiltonians. In terms of linear operator A, we can express

$$H_+ = A^+A + \varepsilon,$$

$$H_- = AA^+ + \varepsilon,$$

and the two partner potentials are connected by the relation

$$V_-(x) = 2\left(\frac{g'(x)}{g(x)}\right)^2 - V_+(x) + 2\varepsilon. \quad (14)$$

though $\mathcal{PT}$-symmetry may be either restored or broken. The potential $V_-(x)$ has the same eigenvalues as $V_+(x)$ (with the possible exception of the ground state). The corresponding eigenfunctions are given by

$$\psi_n(x) = c_n\left(-\frac{d\phi_n(x)}{dx} + \frac{g'(x)}{g(x)}\phi_n(x)\right), \quad (15)$$

where the normalization constant c_n is given as

$$|c|_n^{-2} = \langle A\phi_n|A\phi_n\rangle = E_n - \epsilon. \quad (16)$$

We illustrate the above results concerning $\mathcal{PT}$-symmetric quantum mechanics using the Pöschl–Teller potential given in (1). The discrete eigenvalues of this potential are shown to be only complex conjugate pairs, when $V_2 < -\frac{q_0\lambda^2}{4}$, and real otherwise. After some algebraic manipulation and setting $k = \sqrt{\varepsilon}$, we write the Schrödinger-like equation for the complex potential (1) in the form

$$\frac{d^2g(x)}{dx^2} + \{k^2 - \frac{2q_c^2}{q_0}(V_1+V_2)\text{sech}_q^2\alpha x - \frac{2q_c}{q_0}(V_2-V_1)\times$$

$$\times\text{sech}_q\alpha x\tanh_q\alpha x\}g(x) = 0. \quad (17)$$

With the help of two substitutions

$$z = \frac{1 - iq^{-1/2}\sinh_q\alpha x}{2} \quad (18)$$

and

$$g(x) = z^{-p}(1-z)^{-r}G(z), \tag{19}$$

Eq. (17) becomes

$$z(1-z)\frac{d^2G(z)}{dz^2} + \left(-2p + \frac{1}{2} - (-2p-2r+1)\right)\times$$

$$\times\frac{dG(z)}{dz} - \left\{(p+r)^2 - (\frac{ik}{\alpha})^2 + \frac{-p^2 - \frac{p}{2} + \frac{V_2}{\alpha^2 q_0}}{z(1-z)} + \right.$$

$$\left. + \frac{-r^2 - \frac{r}{2} + p^2 + \frac{p}{2} + \frac{V_1 - V_2}{\alpha^2 q_0}}{1-z}\right\}G(z) = 0. \tag{20}$$

If the conditions hold

$$-p^2 - \frac{p}{2} + \frac{V_2}{\alpha^2 q_0} = 0, \tag{21}$$

$$-r^2 - \frac{r}{2} + p^2 + \frac{p}{2} + \frac{V_1 - V_2}{\alpha^2 q_0} = 0 \tag{22}$$

hold, Eq. (20) will have a solution of the hypergeometric type [16]

$$G(z) = \mu F(a,b,c,z) + \nu z^{1-c}(1-z)^{c-a-b}\times$$

$$\times F(1-a, 1-b, 2-c, z), \tag{23}$$

where

$$a = -p - r + \frac{ik}{\alpha}, \tag{24}$$

$$b = -p - r - \frac{ik}{\alpha}, \tag{25}$$

$$c = -2p + \frac{1}{2}. \tag{26}$$

Hence,

$$g(x) = \mu z^{-p}(1-z)^{-r}F(a,b,c,z) + \nu z^{p+\frac{1}{2}}(1-z)^{r+\frac{1}{2}}\times$$

$$\times F(1-a, 1-b, 2-c, z). \tag{27}$$

Since g(x) must not have any real zero, μ must be non-zero. Solving Eqs. (21) and (22), we have

$$p = \frac{\tau}{2\lambda}\sqrt{\frac{\lambda^2}{4} + \frac{V_2}{q_0}} - \frac{1}{4}, \tag{28}$$

$$r = \frac{\sigma}{2\lambda}\sqrt{\frac{\lambda^2}{4} + \frac{V_1}{q_0}} - \frac{1}{4}, \tag{29}$$

where $\sigma = \pm 1$, $\tau = \pm 1$. The real energy eigenvalues and the unnormalized eigenfunctions for the potential given in (1) are given by [15]

$$E_n^{(\sigma,\tau)} = -\left(2n\lambda + \lambda - \sigma\sqrt{\frac{\lambda^2}{4} + \frac{V_1}{q_0}} - \tau\sqrt{\frac{\lambda^2}{4} + \frac{V_2}{q_0}}\right)^2, \tag{30}$$

$$\phi_n(x) = \left(\frac{1 - iq^{-1/2}\sinh_q \alpha x}{2}\right)^{-p}\times$$

$$\times\left(\frac{1 + iq^{-1/2}\sinh_q \alpha x}{2}\right)^{-r}\times$$

$$P_n^{(-2p-\frac{1}{2}, -2r-\frac{1}{2})}\left(iq^{-1/2}\sinh_q \alpha x\right), \tag{31}$$

$$n = 0, 1, 2, \ldots$$

$$< \mathrm{Re}\left(\frac{\sigma}{2\lambda}\sqrt{\frac{\lambda^2}{4} + \frac{V_1}{q_0}} + \frac{\tau}{2\lambda}\sqrt{\frac{\lambda^2}{4} + \frac{V_2}{q_0}}\right) - \frac{1}{2}, \tag{32}$$

where $P_n^{(a,b)}(z)$ are the Jacobi polynomials [16].

4. Two Partner Potentials

We now discuss the following two cases:
(i) Case I:- $\mu = 1$, $\nu = 0$, and $b = c$;
(ii) Case II:- $\mu = 1$, $\nu = 0$, $\alpha = 1$, and $a = c = b + 1$.
Now for Case I, we have

$$\varepsilon = -\alpha^2\left(p - r - \frac{1}{2}\right), \tag{33}$$

$$F(a,b,b,z) = (1-z)^{-a}. \tag{34}$$

Hence, $g(x)$ takes the simple form

$$g(x) = \left(\frac{1 - iq^{-1/2}\sinh_q \alpha x}{2}\right)^{-p}\times$$

$$\times\left(\frac{1 + iq^{-1/2}\sinh_q \alpha x}{2}\right)^{r+\frac{1}{2}}, \tag{35}$$

and $f(x) = \frac{g'(x)}{g(x)}$ becomes

$$f(x) = i\alpha q^{1/2}\left(p + r + \frac{1}{2}\right)\mathrm{sech}_q\alpha x -$$

$$-\alpha\left(p - r - \frac{1}{2}\right)\tanh_q\alpha x. \qquad (36)$$

From relation (14), we have the isospectral partner potential $V_-(x)$ of (1):

$$V_-(x) = -S_1\frac{q_c}{q_0}\mathrm{sech}^2_{q_c}\lambda\, x + S_2\frac{q_c}{q_0}\mathrm{cosech}^2_{q_c}\lambda x. \qquad (37)$$

Here,

$$S_1 = 2\alpha^2 q_0\left(r + \frac{1}{2}\right)^2 - V_1, \qquad (38)$$

$$S_2 = 2\alpha^2 q_0 p^2 - V_2. \qquad (39)$$

Using relation (15), the corresponding eigenfunction is

$$\psi_n(x) = \left(\frac{1 - iq^{-1/2}\sinh_q\alpha x}{2}\right)^{-p}\times$$

$$\times\left(\frac{1 + iq^{-1/2}\sinh_q\alpha x}{2}\right)^{-r}\times$$

$$\times\left\{c_n\left(2r + \frac{1}{2}\right)\left(i\alpha q^{1/2}\mathrm{sech}_q\alpha x + \alpha\tanh_q\alpha x\right)\times\right.$$

$$\times P_n^{(-2p-\frac{1}{2},-2r-\frac{1}{2})}\left(iq^{-1/2}\sinh_q\alpha x\right) +$$

$$+ic_n q^{-1/2}\frac{(p+r)^2 + \frac{E_n}{4\lambda^2}}{2n}\cosh_q\alpha x\times$$

$$\left.\times P_{n-1}^{(-2p+\frac{1}{2},-2r+\frac{1}{2})}\left(iq^{-1/2}\sinh_q\alpha x\right)\right\}. \qquad (40)$$

Therefore, $V_-(x)$ and $V_+(x)$ have exactly the same form with different coefficients. Now for $V_1 = 80$, $V_2 = 120$, $\sqrt{q} = iq_c$, $q_0 = 1$, $\lambda = 2$, $\varepsilon = \frac{\pi}{16}$, $V_+(x)$ becomes

$$V_+(x) = -80e^{\frac{\pi i}{4}}\mathrm{sech}^2_{q_c}2x + 120e^{\frac{\pi i}{4}}\mathrm{cosech}^2_{q_c}2x \qquad (41)$$

with energies

$$E_n^{(+,+)} = -(4n - 18)^2, n = 0, 1, 2, 3, 4, \qquad (42)$$

and there is no real energy of the type $E_n^{(+,-)}$. By using Eqs. (28) and (29), we have

$$p = \frac{5}{2}, -3; r = 2, -\frac{5}{2}.$$

Since the wave function vanishes asymptotically, the sign of p is positive. For

$$p = \frac{5}{2}, r = -\frac{5}{2},$$

$$V_-(x) = -48e^{\frac{\pi i}{4}}\mathrm{sech}^2_{q_c}2x + 80e^{\frac{\pi i}{4}}\mathrm{cosech}^2_{q_c}2x \qquad (43)$$

with energies

$$E_n^{(+,+)} = -(4n - 14)^2,\ n = 0, 1, 2, 3. \qquad (44)$$

Hence, $V_+(x)$ and $V_-(x)$ share the same energy with the exception of the ground state as shown in the Table.

Again, for $p = \frac{5}{2}, r = 2$ implies $p = r + \frac{1}{2}$, $\varepsilon = 0$, and $f(x) = i\alpha q^{1/2}\left(p + r + \frac{1}{2}\right)\mathrm{sech}_q 2x$. The isospectral partner reduces to

$$V_-(x) = -120e^{\frac{\pi i}{4}}\mathrm{sech}^2_{q_c}2x + 80e^{\frac{\pi i}{4}}\mathrm{cosech}^2_{q_c}2x. \qquad (45)$$

Hence,

$$V_\pm(x) = f^2(x) \pm f'(x), \qquad (46)$$

and they are invariant under the interchange of $V_1 \longleftrightarrow V_2$.

The potentials in (46) are totally degenerate having identical energies, including the zero energy ground state. We note that SUSY (supersymmetry) [17] is broken differently from the conventional SUSY breaking. In the conventional supersymmetric quantum mechanics [18], supersymmetry is broken, when the zero energy state does not exist, and both $V_\pm(x)$ have the same energy. The unnormalized ground-state wave functions $\phi_0(x)$ and $\psi_0(x)$ are given by

$$\phi_0(x) = 16q^{5/2}\mathrm{sech}^4_q 2x\times$$

$$\times\left[\mathrm{sech}_q 2x\sqrt{1 + q^{-1/2}\cosh_q 2x} + \frac{iq^{-1/2}\tanh_q 2x}{\sqrt{1 + q^{-1/2}\cosh_q 2x}}\right], \qquad (47)$$

n	$E_n^{(+,+)}[V_+(x)]$	$E_n^{(+,+)}[V_-(x)]$
0	–324	–196
1	–196	–100
2	–100	–36
3	–36	–4
4	–4	

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$$\psi_0(x) = 144q^{5/2}\text{sech}_q^5 2x\times$$
$$\times\left[\frac{\sinh_q 2x}{\sqrt{1+q^{-1/2}\cosh_q 2x}} + i\sqrt{1+q^{-1/2}\cosh_q 2x}\right]. \quad (48)$$

Again for Case II, we have $\lambda = \frac{1}{2}$, $\varepsilon = -\frac{1}{4}$, $p = r$, $V_1 = V_2 = V_0$ and

$$F(a, a-1, a, z) = (1-z)^{-(a-1)}. \quad (49)$$

Then

$$g(x) = \left(\frac{1 - iq^{-1/2}\sinh_q x}{2}\right)^{-r}\left(\frac{1 + iq^{-1/2}\sinh_q x}{2}\right)^{r+\frac{1}{2}}, \quad (50)$$

$$f(x) = -2r\tanh_q x + \left(2r + \frac{1}{2}\right)[iq^{1/2}\text{sech}_q x + \tanh_q x], \quad (51)$$

and the two partner potentials becomes

$$V_+(x) = -V_0\frac{q_c}{q_0}\text{sech}^2_{q_c} x + V_2\frac{q_c}{q_0}\text{cosech}^2_{q_c} x, \quad (52)$$

$$V_-(x) = -\left(2q_0(r+1/2)^2 - V_0\right)\frac{q_c}{q_0}\times$$
$$\times\text{sech}^2_{q_c} x + \left(2q_0 r^2 - V_0\right)\frac{q_c}{q_0}\text{cosech}^2_{q_c} x. \quad (53)$$

The corresponding eigenfunctions are

$$\psi_n(x) = \left(2q^{1/2}\cosh_q x\right)^{2r}\times$$
$$\times[c_n\left(2r + \frac{1}{2}\right)\left(iq^{1/2}\text{sech}_q x + \tanh_q x\right)\times$$
$$\times P_n^{(-2r-\frac{1}{2}, -2r-\frac{1}{2})}\left(iq^{-1/2}\sinh_q x\right) + ic_n q^{-1/2}\times$$
$$\frac{(4r)^2 + E_n}{2n}\cosh_q x P_{n-1}^{(-2r+\frac{1}{2}, -2r+\frac{1}{2})}\left(iq^{-1/2}\sinh_q x\right)]. \quad (54)$$

5. Conclusion

In this study, using the Darboux method, we have constructed the isospectral partner potentials of the general $\mathcal{PT}$-invariant Pöschl–Teller potential. The constructed potentials are obtained by choosing ν and ε. They share the same energy spectrum as the original potential with the exception of the ground state. If the original potential has only real energies, we have a series of nontrivial complex potentials generating the real-valued spectrum, and the new wave functions can also be obtained by this method.

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ПРО ІЗОСПЕКТРАЛЬНІ ВАРІАНТИ ДЛЯ $\mathcal{PT}$-СИМЕТРИЧНИХ КОМПЛЕКСНИХ ПОТЕНЦІАЛІВ

Санджи Меюр, С. Дебнас

Р е з ю м е

Обговорюється метод Дарбу для отримання ізоспектральних варіантів для комплексного $\mathcal{PT}$-інваріантного потенціалу $V(x) = -V_1\frac{q_c}{q_0}\text{sech}^2_{q_c}\lambda x + V_2\frac{q_c}{q_0}\text{cosech}^2_{q_c}\lambda x$. Побудовано потенціал Пешла – Теллера, з якого отримано спектр з дійсними значеннями енергії. Суперсиметричний варіант потенціалу дає ті ж самі енергії, в тому числі і нульову; виокремлено особливий випадок.