ON FINITE QUANTUM OSCILLATORS

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We construct and study new models of a finite quantum oscillator satisfying the quantum mechanics relations $[H,Q] = -\mathrm{i}P$ and $[H,P] = \mathrm{i}Q$. These models are related to finite dimensional representations of the quantum algebra $\mathrm{su}_q(2)$. The position and momentum operators and the Hamiltonian are elements of the algebra $\mathrm{su}_q(2)$. The models depend on the index j of an irreducible representation of $\mathrm{su}_q(2)$. The spectra of the position and momentum operators consist of a finite set of points (for this reason, these models are called finite). The time evolution for each of the models is evaluated explicitly. The limit $j \to 1$ leads to a Macfarlane–Biedenharn q-oscillator.

1. Introduction

The aim of this paper is to construct and to study finite models of a quantum oscillator which satisfy the fundamental quantum mechanics relations $[H,Q] = -\mathrm{i}P$ and $[H,P] = \mathrm{i}Q$.

Discrete oscillator models which are counterparts to the well-known continuous systems are of a fundamental interest in theoretical physics. Moreover, finite discrete models (that is, models with a finite number of states) are of interest for the parallel processing of signals, where the input and output are registered by a finite sensor array (see [1]).

There exist many algebraic constructions which can be used for building up different models of quantum oscillators. They are constructed on a base of different associative algebras or their deformations. For most of them, it is difficult to construct a complete theory of such an oscillator: the spectra of observables, explicit form of eigenfunctions of observables, description of time evolution, etc. Only a few of such models admits a development of the corresponding theory.

Models based on infinite dimensional representations of the Lie algebra $\mathrm{su}(1,1)$ are constructed in [2]. In [3], models based on discrete series representations of the quantum algebra $\mathrm{su}_q(1,1)$ are studied. All these models are continuous, that is, the spectra of their position and momentum operators are continuous.

Finite models of quantum oscillators are constructed in [4] and [5]. Models of work [4] are based on representations of the classical group SU(2), whereas models of [5] are constructed by means of representations of the quantum algebra $\mathrm{su}_q(2)$.

Our models are based also on finite dimensional representations of the quantum algebra $\mathrm{su}_q(2)$. However, our position and momentum operators differ from those in work [5]. Our models are better than those in [5], since the former give good limit to the Macfarlane–Biedenharn q-oscillator constructed in [6] and [7].

Since the quantum group $\mathrm{SU}_q(2)$ is a group of motions in a non-commutative geometry (see, e.g., [8]), our oscillators can be useful for applications to quantum systems in the non-commutative space-time and to quantum systems with the quantum algebra $\mathrm{su}_q(2)$ describing their dynamical symmetry. These oscillators can be considered (along with the well-known q-oscillator) as new non-trivial deformations of the standard quantum harmonic oscillator. Moreover, our models have the Macfarlane–Biedenharn q-oscillator as a limit case.

For deriving properties of our oscillators, we essentially use the theory of special functions and q-orthogonal polynomials. Namely, using properties of the dual q-Krawchouk polynomials, which are orthogonal on finite sets, we find the spectra of the position and momentum operators and derive an explicit form of their eigenfunctions. These spectra consist of the points $x_m = \frac{1}{2}[2m]_q$, $m = j, j-1, \ldots, -j$, where $[a]_q$ are the so-called q-numbers. These points are not equally spaced. The spectrum of energies is equally spaced and consists of points $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2j+1}{2}$.

We derive an explicit form of the evolution operator in the coordinate space. It is an operator which is given by an explicit matrix.

In this paper, we use the standard notations of the theory of basic hypergeometric functions which can be found in [9]. For any complex number a, q-numbers $[a]_q$

are defined as

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}.$$

The same notation $[A]_q$ is used for operators.

We assume that the basic number q is a fixed real number such that $0 < q < 1$. Each fixed value of q, as well as a finite dimensional representation of $\mathrm{su}_q(2)$, determines a model of the quantum oscillator.

2. The Algebra $\mathrm{su}_q(2)$ and Its Representations

In order to describe our models, we need some information on the quantum algebra $\mathrm{su}_q(2)$ and its finite dimensional irreducible representations (more detailed information can be found in [8, Ch. 3] and [10 Ch. 5])

The quantum algebra $\mathrm{su}_q(2)$ is an associative algebra generated by the elements J_+, J_-, and J_3 satisfying the relations

$$[J_+, J_-] = [2J_3]_q\,, \quad [J_3, J_\pm] = \pm J_\pm.$$

Equivalently, writing $J_\pm = J_1 \pm \mathrm{i} J_2$, we characterize the algebra $\mathrm{su}_q(2)$ by

$$[J_2, J_3] = \mathrm{i}\, J_1, \quad [J_3, J_1] = \mathrm{i}\, J_2, \quad [J_1, J_2] = \tfrac{\mathrm{i}}{2}[2J_3]_q. \quad (1)$$

The element

$$C_q = J_+J_- + [J_3 - \tfrac{1}{2}]_q^2 - \tfrac{1}{4}$$

in the algebra $\mathrm{su}_q(2)$ commutes with all other elements and is called its *Casimir* operator.

Nontrivial finite dimensional irreducible representations of the algebra $U_q(\mathrm{su}_2)$ are given by positive integers or half-integers j (see [8 Ch. 3]). We denote such a representation acting on a $2j + 1$-dimensional linear space by T_j.

The linear space of the irreducible representation T_j can be realized as the space $\mathcal{H}_j$ of all polynomials in y of degrees less or equal to $2j$. The operators J_3 and $J_\pm$ are realized on this space as

$$J_3 = y\frac{d}{dy} - j, \quad J_+ = y\left[2j - y\frac{d}{dy}\right]_q, \quad J_- = \frac{1}{y}\left[y\frac{d}{dy}\right]_q$$

(see [5]). The canonical basis of the space $\mathcal{H}_j$ consists of monomials

$$e^j_m(y) = c^j_m\, y^{j+m}, \quad m = -j, -j+1, \cdots, j, \qquad (2)$$

$$c^j_m = q^{(m^2-j^2)/4}\left(\frac{(q,q)_m}{(q,q)_n (q,q)_{m-n}}\right)^{1/2},$$

where $(a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$.

We introduce a scalar product $\langle\cdot,\cdot\rangle$ into $\mathcal{H}_j$, by assuming that $\langle e^j_m, e^j_n\rangle = \delta_{mn}$. This turns $\mathcal{H}_j$ into a finite dimensional Hilbert space. The operators J_3 and $J_\pm$ act upon the canonical basis (2) as

$$J_+ e^j_m = \sqrt{[j+m+1]_q [j-m]_q}\, e^j_{m+1},$$

$$J_- e^j_m = \sqrt{[j-m+1]_q [j+m]_q}\, e^j_{m-1}$$

$$J_3 e^j_m = m\, e^j_m(x) \equiv (n-j)\, e^j_m,$$

where $n = j + m$. Obviously, the operator J_3 is diagonal in the canonical basis. For the operators J_+ and J_-, we have $J_+^* = J_-$.

3. Models of a Quantum Oscillator

Our models of a quantum oscillator are based on the irreducible representations T_j of the algebra $\mathrm{su}_q(2)$. We define the Hamiltonian H and the position and momentum operators Q and P in terms of the generators J_1, J_2, J_3 of this representation as

$$Q = q^{-J_3/4} J_1 q^{-J_3/4}, \quad P = -q^{-J_3/4} J_2 q^{-J_3/4},$$

$$H = J_3 + j + 1/2.$$

Then, due to (1), for Q, P, and H we have the commutation relations

$$[H, Q] = -\mathrm{i}\, P\,, \quad [H, P] = \mathrm{i}\, Q\,,$$

$$[Q,P] = \tfrac{\mathrm{i}}{2} q^{-\frac{1}{2}J_3}(q^{\frac{1}{2}} J_+J_- - q^{-\frac{1}{2}} J_-J_+) q^{-\frac{1}{2}J_3} = \mathrm{i}F(C_q, J_3) =$$

$$= \mathrm{i}\{e^{hJ_3}[(C_q + \tfrac{1}{4})\sinh\tfrac{h}{2} + \tfrac{1}{2}\sinh^{-1}\tfrac{h}{2}] - \tfrac{1}{2}e^{-2hJ_3}\coth\tfrac{h}{2}\},$$

where $q := \exp h$ (the expression for $[Q,P]$ is calculated in the same way as in [5]). The first of these relations determines the moment operator and the second describes the oscillator dynamics.

The operator $F(C_q, J_3)$ commutes with J_3 and therefore is also diagonal in basis (2); in the irreducible representation T_j, we have

$$F(C_q, J_3)\, e^j_m = \frac{e^{mh}\cosh(j+\frac{1}{2})h - e^{2mh}\cosh\frac{h}{2}}{2\sinh\frac{h}{2}}\, e^j_m.$$

The basis vectors e^j_{n-j}, $n = 0, 1, 2, \cdots, 2j$, of the Hilbert space $\mathcal{H}_j$ consist of eigenfunctions of the Hamiltonian operator H:

$$H\, e^j_{n-j} = (n+1/2)\, e^l_{n-j}, \quad n \equiv j+m = 0, 1, 2, \cdots, 2j,$$

that is, the spectrum of H coincides with a part of the spectrum of the Hamiltonian of a standard quantum harmonic oscillator. Our models are similar to but do not coincide with models in [5]. Moreover, our models give a good limit to the Macfarlane–Biedenharn q-oscillator.

The time evolution of our system is the harmonic motion with

$$e^{\mathrm{i}\tau H}\begin{pmatrix}Q\\P\end{pmatrix}e^{-\mathrm{i}\tau H}=\begin{pmatrix}Q(\tau)\\P(\tau)\end{pmatrix}=\begin{pmatrix}\cos\tau & \sin\tau\\-\sin\tau & \cos\tau\end{pmatrix}\begin{pmatrix}Q\\P\end{pmatrix}.$$

This is a group $U(1)$ of inner automorphisms of the algebra $\mathrm{su}_q(2)$ and of rotations of the phase-space surface. We have

$$\exp(\mathrm{i}\tau H)=e^{\mathrm{i}(j+1/2)\tau}\exp(\mathrm{i}\tau J_3)\,. \tag{3}$$

The explicit form of the time evolution in the coordinate space will be derived below.

4. Spectrum of the Position Operator

Since $Q=q^{-J_3/4}J_1q^{-J_3/4}$, the direct calculation shows that the position operator Q in the basis of the Hamiltonian eigenfunctions e^j_{n-j}, $n=0,1,2,\cdots,2j$, has the form

$$Q\,e^j_m=\tfrac{1}{2}q^{-m/2}\left(q^{-1/4}\sqrt{[j+m+1]_q[j-m]_q}e^j_{m+1}+\right.$$

$$\left.+q^{1/4}\sqrt{[j+m]_q[j-m+1]_q}e^j_{m-1}\right).$$

Let us find the spectrum and eigenfunctions of this operator. For this, we use the method given in [11].

If $\psi_x(y)$ is an eigenfunction of Q corresponding to the eigenvalue x, $Q\,\psi_x(y)=x\,\psi_x(y)$, then

$$\psi_x(y)=\sum_{n=0}^{2j}h_n(x)\,e^j_{n-j}(y), \tag{4}$$

where $h_n(x)$ are coefficients depending on x. Since our space of states is $2j+1$-dimensional, the operator Q has $2j+1$ independent eigenfunctions.

In order to find the explicit form of eigenfunctions $\psi_x(y)$, we substitute expression (4) for $\psi_x(y)$ and then the expression for Qe^j_m into the equation $Q\,\psi_x(y)=x\,\psi_x(y)$. After equating the coefficients at a fixed basis element e^j_m, we obtain a recurrence relation for the coefficients $h_n(x)$:

$$q^{-\frac{n}{2}}\left[q^{-1/4}\sqrt{[2j-n]_q[n+1]_q}h_{n+1}(x)+\right.$$

$$\left.+q^{1/4}\sqrt{[2j-n+1]_q[n]_q}h_{n-1}\right]=2xq^{-j/2}h_n(x). \tag{5}$$

We have to find the coefficients $h_n(x)$, taking into account that $h_{-1}(x)=0$. Since eigenfunctions (4) can be found up to a constant, we put $h_0(x)=1$. Then the coefficients $h_n(x)$ are uniquely determined by relation (5). In order to find them, we make the substitution

$$h_n(x)=\left[(-1)^nq^n(q^{-2j};q)_n(q;q)_n\right)^{1/2}h'_n(x) \tag{6}$$

in (5), where $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$. Then relation (5) reduces to

$$q^{-n}\left[(1-q^{n-2j})h'_{n+1}(x)-(1-q^n)h'_{n-1}(x)\right]=$$

$$=2xq^{-j}\lambda h'_n(x),$$

where $\lambda=q^{1/2}-q^{-1/2}$. Comparing this relation with the recurrence relation for dual q-Krawtchouk polynomials (see, e.g., [12], section 3.17) we find that

$$h'_n(x)=K_n(q^j\lambda x;-1,2j|q^{-1}),$$

where K_n are dual q-Krawtchouk polynomials determined as

$$K_n(q^{-z}+cq^{z-N};c,N|q)={}_3\phi_2\left(\begin{matrix}q^{-n},\,q^{-z},\,cq^{z-N}\\q^{-N},\,0\end{matrix}\;\middle|\;q\,;\,q\right).$$

Therefore, eigenfunctions of the position operator Q are given by the formula

$$h_n(x)=\left(\frac{q^{n(1-2j)/2}[2j]_q!}{[n]_q![2j-n]_q!}\right)^{1/2}K_n(q^j\lambda x;-1,2j|q^{-1}),$$

where $[m]_q!=[1]_q[2]_q\cdots[m]_q$.

In order to find eigenvalues of Q, we have to take into account the orthogonality relation for dual q-Krawtchouk polynomials which is given by formula (3.17.2) in [12]. It follows from this orthogonality relation that the q-Krawtchouk polynomials $K_n(q^j\lambda x;-1,2j|q^{-1})$ are orthogonal on the finite set of points

$$x\equiv x_s=\frac{1}{2}[2s]_q=\frac{1}{2}\frac{q^s-q^{-s}}{q^{1/2}-q^{-1/2}}=-x_{-s}, \tag{7}$$

$$s=-j,-j+1,\cdots,\quad j.$$

The orthogonality relation shows that the $(2j+1)\times(2j+1)$-matrix $\left(\tilde{h}_n(x_s)\right)$ with

$$\tilde{h}_n(x_s)=\left(\frac{q^{s^2-j/2}[4s]_q[2j]_q!}{2[2s]_q[2j-2s]_q!![2j+2s]_q!!}\right)^{1/2}h_n(x_s),$$

where $[2m]_q!! = [2m]_q[2m-2]_q \cdots [2]_q$, has orthogonal rows, that is,

$$\sum_{s=-j}^{j} \tilde{h}_n(x_s)\tilde{h}_{n'}(x_s) = \delta_{nn'}. \quad (8)$$

It is well-known that if a matrix is orthogonal in rows then it is orthogonal in columns, that is,

$$\sum_{n=0}^{2j} \tilde{h}_n(x_s)\tilde{h}_n(x_{s'}) = \delta_{ss'}.$$

Considering functions (4) for values of x from (7) and taking the orthogonality of the basis $\{e^j_m\}$ into account, we have

$$\langle \psi_{x_s}(y), \psi_{x_{s'}}(y)\rangle = \sum_n h_n(x_s)\overline{h_n(x_{s'})} = M_s^{-2}\delta_{ss'},$$

in the Hilbert space $\mathcal{H}_j$. Here,

$$M_s = \left(\frac{q^{s^2-j/2}[4s]_q[2j]_q!}{2[2s]_q[2j-2s]_q!![2j+2s]_q!!} \right)^{1/2}.$$

This means that, for x_s taken from (7), the functions

$$\psi_{x_s}(y), \quad s = -j, -j+1, \cdots, j, \quad (9)$$

constitute a full set of eigenfunctions of the position operator Q, that is, our system can take only the coordinate values (7). The set of coordinates (7) has the following properties:

(a) $x_s = -x_{-s}$, $s = -j, -j+1, \cdots, j$, i.e. set (7) is symmetric with respect to the point 0;

(b) the set of points x_s is not equidistant;

(c) a distance between neighboring points x_s increases with increase in a distance of the points from the point 0.

We can find a more explicit form for eigenfunctions (9). Substituting the explicit expressions for $h_n(x_s)$ and $e^j_n(y)$ into (4) and then taking formula (3.17.11) in [12] into account, we obtain, after some calculation,

$$\psi_{x_s}(y)=(-1)^{j-s}y^{2j}q^{-s^2+j/2}(cy^{-1};q)_{j-s}(-cy^{-1};q)_{j+s},$$

where $c = q^{(1-2j)/4}$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. The functions

$$\tilde{\psi}_{x_s}(y) = M_s\psi_{x_s}(y), \quad s = -j, -j+1, \cdots, j,$$

constitute the orthonormal basis of the Hilbert space $\mathcal{H}_j$, consisting of eigenfunctions of the position operator Q.

5. Coordinate Realization of the Models

In Section 3, we have constructed a realization of the oscillator on the space of functions of the supplementary variable y. It is natural to look for its realization on the space of functions of the coordinate x_s.

Let us realize the space of states in a form of functions of x_s. In order to obtain this realization, we associate the function

$$f(x_s) \equiv \langle g(y), \tilde{\psi}_{x_s}(y)\rangle$$

with a function $g(y) \in \mathcal{H}_j$, where the scalar product is taken in the Hilbert space $\mathcal{H}_j$. We have to introduce a scalar product, which is conserved under the transformation $g(y) \to f(x_s)$, in the space of functions $f(x_s)$. To this end, we note that the orthonormal elements e^j_{n-j}, $n = 0, 1, 2, \cdots, 2j$, of $\mathcal{H}_j$ correspond to the oscillator wave functions

$$\langle e^j_{n-j}, \tilde{\psi}_{x_s}(y)\rangle = \tilde{h}_n(x_s),$$

where we have taken formula (4) into account. Due to (8), the functions $\tilde{h}_n(x_s)$, $n = 0, 1, \cdots, 2j$, are orthonormal with respect to the scalar product

$$\langle f_1(x_s), f_2(x_s)\rangle = \sum_{s=-j}^{j} f_1(x_s)\overline{f_2(x_s)}. \quad (10)$$

Then relation (10) gives a scalar product in the space of functions $f(x_s)$. The linear space of functions $f(x_s)$ with the scalar product (10) is a Hilbert space which will be denoted as $L^2(X_j)$, where X_j is the set of points (7). This is the space of states realized as a space of functions on the coordinate set.

The position operator Q in $\mathcal{H}_j$ corresponds to the position operator Q in $L^2(X_j)$. Since $Qg(y)$ corresponds to the function

$$Qf(x_s) = \langle Qg(y), \tilde{\psi}_{x_s}(y)\rangle = \langle g(y), Q\tilde{\psi}_{x_s}(y)\rangle =$$
$$= x_s\langle g(y), \tilde{\psi}_{x_s}(y)\rangle = x_s f(x_s),$$

Q is the multiplication operator in $L^2(X_j)$, $Qf(x_s) = x_s f(x_s)$.

Since $H\, e^j_{n-j}(y) = (n+\frac{1}{2})e^j_{n-j}$, we have

$$\langle H\, e^j_{n-j}(y), \tilde{\psi}_{x_s}(y)\rangle = (n+\tfrac{1}{2})\langle e^j_{n-j}(y), \tilde{\psi}_{x_s}(y)\rangle =$$
$$= (n+\tfrac{1}{2})\tilde{h}_n(x_s).$$

This means that $H\, \tilde{h}_n(x_s) = (n+\frac{1}{2})\tilde{h}_n(x_s)$. That is, the basis $\{\tilde{h}_n(x_s)\}$ consists of eigenfunctions of Hamiltonian H. It can be shown that

$$\tilde{h}_n(-x_s) = \tilde{h}_n(x_{-s}) = (-1)^n\tilde{h}_n(x_s).$$

6. Spectrum of the Momentum Operator

The momentum operator P in the basis e^j_{n-j}, $n = 0, 1, 2, \cdots$, has the form

$$P\, e^j_m = \tfrac{\mathrm{i}}{2} q^{-m/2} \left(q^{-1/4} \sqrt{[j+m+1]_q [j-m]_q} e^j_{m+1} - \right.$$

$$\left. - q^{\frac14} \sqrt{[j+m]_q [j-m+1]_q} e^j_{m-1} \right).$$

By changing the basis $\{e^j_{n-j}\}$ to the basis $\{\tilde{e}^j_{n-j}\}$, where $\tilde{e}^j_{n-j} = \mathrm{i}^n e^j_{n-j}$, we see that the momentum operator P is given in the latter basis by the same formula as the position operator is given in the former basis $\{e^j_{n-j}\}$ (see Section 4). This means that the spectrum of the operator P coincides with that of Q. That is, this spectrum consists of the points

$$p_s = \frac{1}{2}[2s]_q, \qquad s = -j, -j+1, \cdots, j.$$

Eigenfunctions of the momentum operator can be found (by using the basis $\{\tilde{e}^j_{n-j}\}$) in the same way as in the case of the position operator. For this reason, we exhibit here only the result.

If $\varphi_{p_s}(y)$ is the eigenfunction of P corresponding to the eigenvalue p_s, $P\varphi_{p_s}(y) = p_s \varphi_{p_s}(y)$, then

$$\varphi_{p_s}(y) = \sum_{n=0}^{2j} \hat{h}_n(p_s)\, e^j_{n-j}(y), \tag{11}$$

where, as before, $e^j_m(y)$ are given by (2) and $\hat{h}_n(p_s)$ are coefficients depending on the eigenvalues p_s.

Repeating the reasoning of the previous section, we get a three-term recurrence relation for the coefficients $\hat{h}_n(p_s)$ and conclude that

$$\hat{h}_n(p_s) = \mathrm{i}^n h_n(p_s) =$$

$$= \mathrm{i}^n \left(\frac{q^{n(1-2j)/2}[2j]_q!}{[n]_q![2j-n]_q!} \right)^{1/2} K_n(q^j \lambda p_s; -1, 2j | q^{-1}),$$

where, as before, $\lambda = q^{1/2} - q^{-1/2}$.

Substituting expressions for $\hat{h}_n(p_s)$ and e^j_m into (11), we can sum up the expression for $\varphi_{p_s}(y)$ by the same method as in the case of functions (4). The eigenfunctions of the momentum operator P are of the form

$$\varphi_{p_s}(y) = (-1)^s y^{2j} q^{-s^2+j/2} (\mathrm{i}cy^{-1}; q)_{j-s} (-\mathrm{i}cy^{-1}; q)_{j+s},$$

where $c = q^{(1-2j)/4}$. The functions $\tilde{\varphi}_{p_s}(y) = M_s \varphi_{p_s}(y)$, $s = -j, -j+1, \cdots, j$, constitute the orthonormal basis of the Hilbert space $\mathcal{H}_j$.

7. Momentum Realization of the Models

We can realize our oscillator on the space of functions of the momentum p_s. For this, we realize the space of states $\mathcal{H}_j$ in the form of functions of p_s. In order to obtain this realization, we associate a function $g(y) \in \mathcal{H}_j$ with the function

$$F(p_s) \equiv \langle g(y), \tilde{\varphi}_{p_s}(y) \rangle,$$

where the scalar product is taken in the Hilbert space $\mathcal{H}_j$. Further, we introduce a scalar product in the space of functions $F(p_s)$ which is conserved under the transformation $g(y) \to F(p_s)$. Then, we take into account that the orthonormal elements e^j_{n-j}, $n = 0, 1, 2, \cdots, 2j$, correspond to the functions

$$\langle e^j_{n-j}, \tilde{\varphi}_{p_s}(y) \rangle = M_s \hat{h}_n(p_s),$$

where we have taken formula (11) into account. The functions $M_s \hat{h}_n(x_s)$ are orthonormal with respect to the scalar product

$$\langle F_1(p_s), F_2(p_s) \rangle = \sum_{s=-j}^{j} F_1(p_s) \overline{F_2(p_s)},$$

and this scalar product induces a scalar product in the space of functions $F(p_s)$. We denote the linear space of functions $F(p_s)$ with this scalar product by $\hat{L}^2(Y_j)$, where Y_j is the set of points $p_s = \frac12 [2s]_q$, $s = -j, -j+1, \cdots, j$.

The momentum operator P in $\mathcal{H}_j$ corresponds to the momentum operator P in $\hat{L}^2(Y_j)$, which is given as the multiplication operator, $P\,F(p_s) = p_s F(p_s)$.

For the Hamiltonian H, we have

$$H\, \hat{h}_n(p_s) = (n + \tfrac12) \hat{h}_n(p_s),$$

i.e. the basis $\{\hat{h}_n(x_s)\}$ of $\hat{L}^2(Y_j)$ consists of eigenfunctions of the Hamiltonian H.

8. Evolution Operator in the Coordinate Space

According to (3), the time evolution operator $\exp(\mathrm{i}\tau H)$ acts on the basis elements e^j_{n-j}, $n = 0, 1, 2, \cdots, 2j$, of the Hilbert space $\mathcal{H}_j$ in the following manner:

$$e^{\mathrm{i}\tau H}\, e^j_{n-j} = e^{\mathrm{i}(n+1/2)\tau}\, e^j_{n-j}.$$

We wish to find how this operator acts in the coordinate space, i.e. in the Hilbert space $L^2(X_j)$. If, under a transition from $\mathcal{H}_j$ to $L^2(X_j)$, a function $g(y) \in \mathcal{H}_j$

maps onto a function $f(x_s) \in L^2(X_j)$, then the function $\exp(\mathrm{i}\tau H)\, g(y) \in \mathcal{H}_j$ corresponds to the function

$$e^{\mathrm{i}\tau H} f(x_s) = \langle e^{\mathrm{i}\tau H} g(y), \tilde{\psi}_{x_s}(y)\rangle = \langle g(y), e^{-\mathrm{i}\tau H}\tilde{\psi}_{x_s}(y)\rangle =$$

$$= \sum_{n=0}^{2j} \langle g(y), e^j_{n-j}\rangle \langle e^j_{n-j}, e^{-\mathrm{i}\tau H}\tilde{\psi}_{x_s}(y)\rangle =$$

$$= \sum_{n=0}^{2j} \langle e(y), e^j_{n-j}\rangle \langle e^{\mathrm{i}\tau H} e^j_{n-j}, \tilde{\psi}_{x_s}(y)\rangle =$$

$$= \sum_{n=0}^{2j} \sum_{s'=-j}^{j} \langle g(y), \tilde{\psi}_{x_{s'}}(y)\rangle \langle \tilde{\psi}_{x_{s'}}(y), e^j_{n-j}\rangle \times$$

$$\times g^{\mathrm{i}\tau(n+1/2)} \langle e^j_{n-j}, \tilde{\psi}_{x_s}(y)\rangle = \sum_{s'=-j}^{j} f(x_{s'}) K^\tau(x_s, x_{s'}),$$

where scalar products are taken in the space $\mathcal{H}_j$, and the kernel $K^\tau(x_s, x_{s'})$ is given by

$$K^\tau(x_s, x_{s'}) =$$

$$= \sum_{n=0}^{2j} \langle \tilde{\psi}_{x_{s'}}(y), e^j_{n-j}\rangle \langle e^j_{n-j}, \tilde{\psi}_{x_s}(y)\rangle e^{\mathrm{i}\tau(n+1/2)} =$$

$$= \sum_{n=0}^{2j} M_s M_{s'} h_n(x_s) h_n(x_{s'}) e^{\mathrm{i}\tau(n+1/2)},$$

where the coefficients M_s and the functions $h_n(x_s)$ are such as in Section 4. In order to sum up the expression for $K^\tau(x_s, x_{s'})$, we use formula (8.15) in [13]. After some calculations and transformations, we obtain the explicit expression for the kernel $K^\tau(x_s, x_{s'})$ in the form

$$K^\tau(x_s, x_{s'}) = e^{\mathrm{i}\tau/2} a^j_{ss'}(e^{\mathrm{i}\tau}) W(e^{\mathrm{i}\tau}), \qquad (12)$$

where

$$a^j_{ss'}(e^{\mathrm{i}\tau}) = (q^{-s}t;q)_{j-s'} (q^{-s'}t;q)_{j-s} \times$$

$$\times \frac{(q^s t;q)_{j-s'} (q^{s'} t;q)_{j-s} (q^{j-s-s'} t;q)_{2(s+s')}}{(-1)^{s+s'} q^{s^2+s'^2+j(2j-1)} t^{2j} (q^{-j}t;q)_{2j}}$$

and

$$W(t) = \frac{(q^{-j}t;q)_{j-s} (-q^{s-j}t;q)_{j-s}}{(-q^s t;q)_{j-s} (q^{-2j}t;q)_{j-s}} \times$$

$$\times {}_4\phi_3 \left(\begin{matrix} q^{s-j},\ -q^{-j-s},\ q^j t,\ -q^j t \\ q^{-s'}t,\ -q^{-s'}t,\ q \end{matrix} \middle| q\,;\,q \right)$$

with $t = -q^{-j} e^{-\mathrm{i}\tau}$. Thus, the evolution operator $\exp(\mathrm{i}\tau H)$ is a kernel operator given by the formula

$$\exp(\mathrm{i}\tau H)\ f(x_s) = \sum_{s'=-j}^{j} f(x_{s'}) K^\tau(x_s, x_{s'}),$$

where the kernel $K^\tau(x, x')$ is given by (12).

Since $e^{\mathrm{i}\tau H} e^{\mathrm{i}\tau' H} = e^{\mathrm{i}(\tau+\tau')H}$, this kernel satisfies the relation

$$\sum_{s'=j}^{j} K^\tau(x_s, x_{s'}) \overline{K^\tau(x_{s'}, x_{s''})} = K^{\tau+\tau'}(x_s, x_{s''}).$$

We note that this relation leads to the corresponding integral relation for the basic hypergeometric function ${}_4\phi_3$ from the expression for $K^\tau(x_s, x_{s'})$.

9. The Limit Transition to a Macfarlane–Biedenharn *q*-oscillator

We denote the models of a quantum oscillator characterized by the number j constructed above by osc_j. We now show that

$$\lim_{j\to\infty} \sqrt{2(1-q)}\sqrt{q^{-1}-1}\ \mathrm{osc}_j = \mathrm{osc}_q,$$

where osc_q denotes a Macfarlane–Biedenharn q-oscillator. This limit means that

$$\lim_{j\to\infty} \sqrt{2(1-q)}\ Q_j = \mathcal{Q},$$

$$\lim_{j\to\infty} \sqrt{2(1-q)}\ P_j = \mathcal{P}, \qquad (13)$$

where $Q \equiv Q_j$, $P \equiv P_j$ are the position and momentum operators for osc_j, and $\mathcal{Q}$ and $\mathcal{P}$ are the position and momentum operators for osc_q. At limit (13), we have

$$\lim_{j\to\infty} q^{nj/2} e^j_{j-n}(y) = \tilde{e}_n(y) = \sqrt{q^{n^2/2}(q;q)_n^{-1}}\ y^n.$$

Relations (13) follow from the fact that, in the limit $j \to \infty$, the operators Q_j and P_j turn into

$$\mathcal{Q}\,\tilde{e}_n = \frac{1}{\sqrt{2}} \left[\left(\frac{1-q^{-n-1}}{1-q^{-1}} \right)^{\frac{1}{2}} \tilde{e}_{n+1} + \left(\frac{1-q^{-n}}{1-q^{-1}} \right)^{\frac{1}{2}} \tilde{e}_{n-1} \right],$$

$$\mathcal{P}\,\tilde{e}_n = \frac{\mathrm{i}}{\sqrt{2}} \left[\left(\frac{1-q^{-n-1}}{1-q^{-1}} \right)^{\frac{1}{2}} \tilde{e}_{n+1} - \left(\frac{1-q^{-n}}{1-q^{-1}} \right)^{\frac{1}{2}} \tilde{e}_{n-1} \right].$$

Considering that

$$\mathcal{Q} = \frac{1}{\sqrt{2}}(a^+ + a), \quad \mathcal{P} = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a),$$

we get

$$a^{+}\tilde{e}_n = \left(\frac{1-q^{-n-1}}{1-q^{-1}}\right)^{\frac{1}{2}} \tilde{e}_{n+1},$$

$$a\tilde{e}_n=\left(\frac{1-q^{-n}}{1-q^{-1}}\right)^{\frac{1}{2}} \tilde{e}_{n-1}.$$

These operators together with the operator q^N given as $q^N\tilde{e}_n = q^n\tilde{e}_n$ satisfy the relations

$$aa^{+}-q^{-1}a^{+}a=1, \quad q^N a^{+}=q^{-1}a^{+}q^N, \quad q^N a=qaq^N,$$

i.e. they generate the Macfarlane–Biedenharn q-oscillator algebra.

10. Concluding Remarks

We have constructed models of a finite quantum oscillator which are related to dual q-Krawtchouk polynomials (i.e., the models can be realized on the basis of the coordinate and momentum Hilbert spaces expressed in terms of these q-orthogonal polynomials) and to irreducible representations of the quantum algebra $\mathrm{su}_q(2)$. The characteristic properties of our models consist in that the spectra of the position and momentum operators cover a finite set of points of the real line.

We consider that our models of the oscillator can be useful for the description of models of quantum mechanics and quantum field theory on lattices.

Our models can be also useful for the description of quantum systems in a non-commutative space-time (for which a "motion group" is the quantum group $SU_q(2)$) and quantum systems with the quantum algebra $\mathrm{su}_q(2)$ describing their dynamical symmetry. The principles of these applications are the same as those in the case of the Biedenharn–Macfarlane q-oscillator.

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ПРО СКІНЧЕННІ КВАНТОВІ ОСЦИЛЯТОРИ

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Р е з ю м е

Побудовано та вивчено нові моделі скінченного квантового осцилятора, що задовольняє квантово-механічні співвідношення $[H,Q] = -\mathrm{i}P$ і $[H,P] = \mathrm{i}Q$. Ці моделі зв'язані зі скінченновимірними представленнями квантової алгебри $\mathrm{su}_q(2)$. Оператори положення і момента та гамільтоніан є елементами алгебри $\mathrm{su}_q(2)$. Спектри операторів положення і момента є скінченними множинами точок (з цієї причини ці моделі називаються скінченними). Часова еволюція для кожної із цих моделей знайдена в явному вигляді. Границя $j \to \infty$ приводить до q-осцилятора Біденгарна–Макфарлейна.