# OCCURRENCE OF PAIRWISE ENERGY LEVEL DEGENERACIES IN $q, p$-OSCILLATOR MODEL

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It is demonstrated that a two-parameter deformed oscillator with the deformation parameters $q, p$ such that $0 < q, p \leq 1$ exhibits the property of “accidental” two-fold (pairwise) energy level degeneracy of the classes $E_m = E_{m+1}$ and $E_0 = E_m$. The most general case of degeneracy of $q, p$-oscillators of the form $E_{m+k} = E_m$ (with $k \geq 1$ for $m \geq 1$ or $k \geq 2$ for $m = 0$) is briefly discussed.

## 1. Introduction

The so-called $q$-deformed oscillators ($q$-oscillators) remain, from their appearance till now, a very popular subject of investigations including their diverse applications (see, e.g., [1, 2] and references therein). Much attention has been paid to the two most distinguished versions of $q$-oscillators: the one proposed by Biedenharn and Macfarlane [3, 4] (BM $q$-oscillator) and the other introduced by Arik and Coon [5] (AC $q$-oscillator).

It is well known that, unlike the AC $q$-oscillator, the BM version admits not only real but also phase-like complex values of the deformation parameter $q$. Such a distinction leads to essentially differing aspects of their particular applications. Let us note that, among others, there is the well-known property of the BM $q$-oscillator consisting in a possibility of certain degeneracies and periodicity appearing in case of $q$ being a root of unity, the most popular values for the BM-type $q$-oscillator. Say, for $q = \exp(\frac{i\pi}{2n+2})$, the following two neighboring energy levels coincide: $E_{n+1} = E_n$. This equality along with other coincidences leads to a kind of periodicity and naturally makes the corresponding phase space both discrete and finite [6].

One can then wonder whether some kind of 'accidental' degeneracy (occurring without any obvious underlying symmetry) can be a peculiar feature of the AC $q$-oscillator, and the answer is negative: the only possible case requires the value $q = 0$, but usually this value is excluded from the treatment.

The latter conclusion is however not the ultimate statement concerning $q$-deformed oscillators and, as recently shown [7], yet another version of $q$-oscillator which has been termed the “Tamm-Dancoff cutoff” deformed oscillator in [8,9] and does possess the property of “accidental” degeneracy of the kinds $E_m = E_{m+1}$, $E_0 = E_m$, and some others.

The goal of the present paper is to analyze the analogous question about possible 'accidental' degeneracies if one deals with more general two-parameter (or $q, p$-)deformed oscillators. The $q, p$-deformed oscillators introduced in [10] more than 15 years ago provide the valuable and perspective tools for obtaining nonstandard $q$-oscillators and for elaborating diverse applications. It suffices to mention only a few following ones.

First, the $q, p$-deformed oscillators turn out to be rather effective [11] in the phenomenological description of the rotational spectra of (super)deformed nuclei.

Second, the concept of $q, p$-deformation, unlike the standard harmonic oscillator, allows to account for more involved reasons/aspects of the extension of the standard oscillator: for the situation where the included interaction is highly nonlinear (non-polynomial, with inclusion of all anharmonisms) and/or employs the momentum operator; it may also involve the non-constant position-dependent mass of the quantum-mechanical particle [12].

Third, the application elaborated in [13] incorporates the appropriate set of $q, p$-deformed oscillators ($q, p$-bosons) for developing the corresponding $q, p$-Bose gas model. Such a model is based on the analytical expressions for the intercepts (strengths) of the general $n$-particle momentum correlation functions obtained for the first time in explicit form in [14] (note that these results generalize the previously known formulas for two-particle correlations in the AC and BM versions of the $q$-Bose gas model). As such, the mentioned results were analyzed [13] in the context of their direct relevance to experimental data on the 2- and 3-pion correlations collected during the RHIC/STAR and CERN/SPS runs of relativistic heavy ion collisions.

The paper is organized as follows. In Section 2, we recall the main facts about the phenomenon of 'accidental' double degeneracy of energy levels within two particular (BM or TD) versions of $q$-oscillators, respectively for $q$ being a root of unity or $q$ being real. The peculiarities of an analogous sort of degeneracies manifested by the two-parameter (or the $q, p$-) deformed oscillators are disclosed and explained in Section 3, where we formulate, prove, and illustrate our basic statements. Section 4 is devoted to concluding remarks.

## 2. "Accidental" Degeneracies of $q$-Oscillators

To explain the idea of 'accidental' degeneracies of energy levels, consider first the famous Biedenharn–Macfarlane (or BM) $q$-oscillator [3, 4], whose defining relations are

$$aa^\dagger - qa^\dagger a = q^{-N}, \qquad aa^\dagger - q^{-1}a^\dagger a = q^N, \tag{1}$$

$$[N, a] = -a, \qquad [N, a^\dagger] = a^\dagger. \tag{2}$$

Then, $a^\dagger a = [N]_q$ and $aa^\dagger = [N+1]_q$ where the $q$-bracket reads

$$[X]_q \equiv \frac{q^X - q^{-X}}{q - q^{-1}}, \qquad [X]_q \xrightarrow{q\to 1} X. \tag{3}$$

The Hamiltonian of the BM $q$-oscillator is taken to be

$$H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a).$$

For convenience, we put $\hbar\omega = 1$ in what follows. Using the $q$-Fock space and its vacuum state $|0\rangle$ such that

$$a|0\rangle = 0\ , \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle, \qquad N|n\rangle = n\ |n\rangle,$$

where $[n]_q! = [n]_q[n-1]_q...[2]_q[1]_q$, $[1]_q = 1$, $[0]_q = 1$, the creation/annihilation operators act by the formulas

$$a\ |n\rangle = \sqrt{[n]_q}\ |n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]_q}\ |n+1\rangle.$$

The spectrum $H|n\rangle = E_n|n\rangle$ of the Hamiltonian reads

$$E_n = \frac{1}{2}\Big([n+1]_q + [n]_q\Big). \tag{4}$$

If $q \to 1$, $E_n = n + \frac{1}{2}$; also $E_0 = \frac{1}{2}$ for any value of $q$.

For real $q \neq 1$, the *spectrum is not equidistant.* The most interesting situation arises for phase-like $q$, $q = \exp(i\theta)$.

### *2.1. Level degeneracy of a q-oscillator with* $q = e^{i\theta}$

In the next two statements, $n$ is any positive integer.

**Proposition 1**.

(i) Fix the angle $\theta$ to be

$$\theta = \frac{\pi(2k+1)}{2n+2} \quad \text{with} \quad k = 0, \pm1, \pm2, ...$$

Then Eq. (4) yields $E_{n+1} - E_n = \cos\frac{(2n+2)\theta}{2}$, and, with the indicated $\theta$, the degeneracy $E_{n+1} = E_n$ follows,

(ii) Fix the angle $\theta$ to be

$$\theta = \frac{\pi(2k+1)}{2n+3} \quad \text{where} \quad k = 0, \pm1, \pm2, \ldots$$

Then Eq. (4) yields $E_{n+2} - E_n = 2\cos\frac{(2n+3)\theta}{2}\cos\frac{\theta}{2}$, and, with this $\theta$, the degeneracy $E_{n+2} = E_n$ follows.

This statement can be generalized as follows.

**Proposition 2**. For $r \geq 1$, let us fix the angle $\theta$ as

$$\theta = \frac{\pi(2k+1)}{2n+r+1} \quad \text{with} \quad k = 0, \pm1, \pm2, \ldots \tag{5}$$

Then Eq. (4) yields the equality

$$E_{n+r} - E_n = 2\frac{\sin(r\theta/2)}{\sin(\theta)}\cos\frac{(2n+1+r)\theta}{2}\cos(\theta/2), \tag{6}$$

from which the degeneracy

$$E_{n+r} = E_n, \qquad r \geq 1 \tag{7}$$

follows for the values of $\theta$ given in (5).

The indicated degeneracies, for $q = \exp(\mathrm{i}\theta)$ being the corresponding root of unity with $\theta$ a rational fraction of $\pi$, lead to such consequences as periodicity, discreteness, and finiteness [6] of the phase space of a BM-type $q$-oscillator.

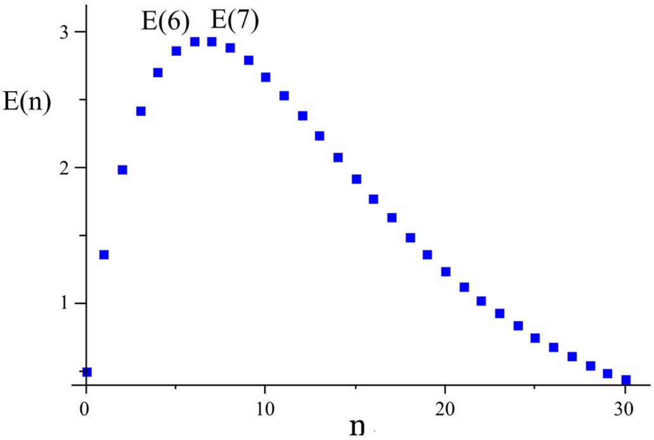


Fig. 1. Spectrum of a $q$-oscillator (8) at fixed $q = \sqrt{6/8}$. Observe the degeneracy $E_6 = E_7$

### *2.2. q-Oscillator with level degeneracy at real q*

Although the AC $q$-oscillator does not allow any accidental degeneracy, it was demonstrated in [7] that the 'accidental' degeneracy at real values of the $q$-parameter can occur in the case of a $q$-oscillator called [8, 9] the Tamm–Dancoff deformed oscillator. Its creation, destruction, and number operators obey the following defining relation:

$$bb^\dagger - qb^\dagger b = q^N, \tag{8}$$

$$[N, b] = -b, \qquad [N, b^\dagger] = b^\dagger. \tag{9}$$

Taking the Hamiltonian of this $q$-oscillator as

$$H = \frac{\hbar\omega}{2}(bb^\dagger + b^\dagger b) \tag{10}$$

and putting $\hbar\omega = 1$, we consider its eigenvalues in the states of the corresponding $q$-Fock space. With the vacuum state $|0\rangle$, the relevant relations are

$$b|0\rangle = 0, \quad |n\rangle = \frac{(b^\dagger)^n}{\sqrt{\{n\}_q!}}|0\rangle, \qquad N|n\rangle = n\;|n\rangle, \tag{11}$$

where $\{n\}_q! = \{n\}_q\{n-1\}_q...\{2\}_q\{1\}_q$, $\{0\}! = 1$, $\{1\}! = 1$, and the $q$-bracket in this case being

$$\{X\}_q \equiv Xq^{X-1}, \qquad \{X\}_q \xrightarrow{q\to 1} X \tag{12}$$

[compare it with (3)],

$$b^\dagger b = \{N\}_q, \quad bb^\dagger = \{N+1\}_q, \tag{13}$$

and the operators $b$, $b^\dagger$ act by the formulas

$$b|n\rangle = \sqrt{\{n\}_q}|n{-}1\rangle, \quad b^\dagger|n\rangle = \sqrt{\{n{+}1\}_q}\;|n{+}1\rangle. \tag{14}$$

Note that, for any real $q \geq 0$, the operators $b$ and $b^\dagger$ are adjoint to each other.

From (12)–(14), the spectrum $H|n\rangle = E_n|n\rangle$ of the Hamiltonian reads

$$E_n = \frac{1}{2}\Big((n+1)q^n + nq^{n-1}\Big) = \frac{1}{2}q^{n-1}\Big(q + n(1+q)\Big). \tag{15}$$

At $q \to 1$, we recover $E_n = n + \frac{1}{2}$; note also that $E_0 = \frac{1}{2}$ for any value of $q$.

If $q \neq 1$, the *spectrum is not uniformly spaced* (not equidistant). Moreover, if $q > 1$, the spacing $E_{n+1} - E_n$ gradually increases with growing $n$, so that $E_n \to \infty$ as $n \to \infty$. However, more interesting possibilities arise when $q$ belongs to the interval $0 < q < 1$.

The energy spectrum given by expression (15) manifests some sorts of degeneracies [7], with the strong dependence on the particular fixed value of $q$. Let us consider relevant cases.

*Degeneracies $E_m = E_{m+1}$ and $E_m = E_{m+2}$*

**Proposition 3.** If the parameter $q$ is fixed as $q = \sqrt{\frac{m}{m+2}}$, where $m \geq 1$ , then the following degeneracy of the energy levels does occur:

$$E_m = E_{m+1}. \tag{16}$$

Note that $m{=}0$ is excluded from (16) as the degeneracy $E_0{=}E_1$ would require the (excluded) value $q{=}0$.

**Proposition 4.** Let the parameter $q$ be fixed as

$$q = \frac{1 + \sqrt{4m^2 + 12m + 1}}{2(m+3)} \qquad \text{with} \qquad m \geq 0\;. \tag{17}$$

Then the following degeneracy of $E_m$ does occur:

$$E_m = E_{m+2}. \tag{18}$$

For illustration, we show the particular cases $E_6 = E_7$ and $E_4 = E_6$ of (16) and (18) in Figs. 1 and 2.

*Degeneracy of the type $E_0 = E_m$*

One more type of degeneracy, $E_0 = E_m$, was also pointed out in [7], see the next proposition.

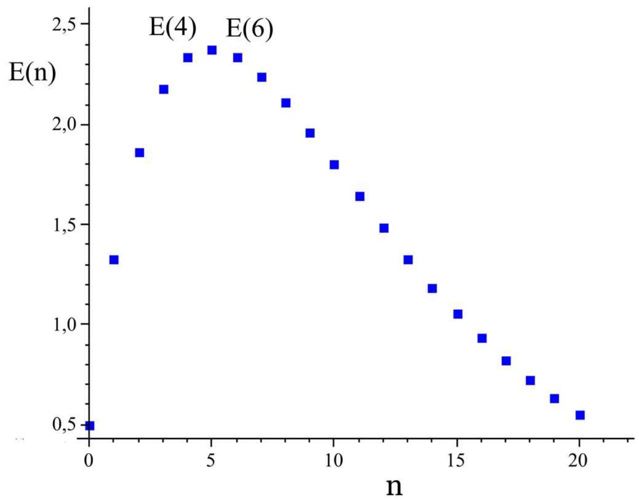


Fig. 2. Spectrum of a $q$-oscillator (8) at fixed $q = \frac{1+\sqrt{113}}{14}$. Observe the degeneracy $E_4 = E_6$

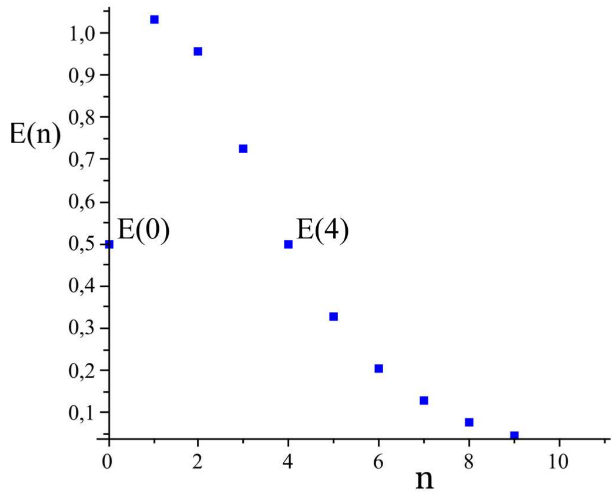


Fig. 3. Spectrum of the $q$-oscillator (8) at fixed $q \simeq 0.5315645$. Observe the degeneracy $E_0 = E_4$

**Proposition 5.** For any integer $m = 2, 3, 4, \ldots$, there exists an appropriate $q_m = q(m)$ such that

$$E_0 = E_m. \tag{19}$$

The proof of this statement uses a graphical treatment as demonstrated in [7].

Some values of the $q$-parameter which provide degeneracy (19) are listed in the Table. The first three values $q_2, q_3$, and $q_4$ in the Table can be given in radicals, while, for $m \geq 5$ the values $q_m$ are found approximately. Clearly, all the $q_m$ obey the relation $0 < q_m < 1$.

In Fig. 3, the particular case $m = 4$ of Eq. (19) is presented. The latter degeneracy occurs, as seen from the Table, at $q \simeq 0.5315645$.

*"Accidental" degeneracy of the general type $E_m = E_{m+k}$*

As was mentioned in [7], the diversity of possible cases of two-fold degeneracy are given by the relation $E_m = E_{m+k}$. The equation for the values $q = q(m, k)$ of $q$-parameter responsible for such degeneracies looks as

$$(m + k + 1)q^{m+k} + (m + k)q^{m+k-1} -$$

$$-(m + 1)q^m - mq^{m-1} = 0,$$

or

$$(m + k + 1)\ q^{k+1} + (m + k)\ q^k - (m + 1)\ q - m = 0. \tag{20}$$

For each pair $(m,\ m + k)$, it can be proved that there exists such real solution $q = q(m, k)$ of (20) that $0 < q < 1$. Let us comment on few low $k$ values. Obviously, $k = 1$ resp. $k = 2$ correspond to the particular series of degeneracies already considered, see (16) and (18) above. For the next two cases, the equations to be solved are

$$(k = 3) \quad q^4 + \frac{m + 3}{m + 4}\, q^3 - \frac{m + 1}{m + 4}\, q - \frac{m}{m + 4} = 0, \tag{21}$$

$$(k=4) \quad q^4 - \frac{1}{m + 5}\, q^3 + \frac{1}{m + 5}\, q^2 - \frac{1}{m + 5}\, q - \frac{m}{m + 5} = 0, \tag{22}$$

where, for $k = 4$, we have taken into account that the fifth-degree equation divides exactly by $q + 1$. Note also that the root $q = -1$ exists in all the cases of higher even $k$ in (20). Equations (21) and (22) can be solved in radicals, which yields awkward expressions. Analogously to the above Table, a set of values $q = q(m, k)$ can be found numerically and tabulated. Let us finally remark that the case $E_0 = E_m$ (see Proposition 5 above) is obviously covered by the most general situation, $E_m = E_{m+k}$.

**Some values $q_m$ that yield degenerace $E_0 = E_m$**

| | |
|---|---|
| $m = 2$ | $q_2 = \frac{1}{3} \simeq 0.333333$ |
| $m = 3$ | $q_3 \simeq 0.45541$ |
| $m = 4$ | $q_4 \simeq 0.5315645$ |
| $m = 5$ | $q_5 \simeq 0.585442$ |
| $m = 6$ | $q_6 \simeq 0.626225$ |
| $m = 10$ | $q_{10} \simeq 0.725405$ |
| $m = 25$ | $q_{25} \simeq 0.851675$ |
| $m = 100$ | $q_{100} \simeq 0.948094$ |
| $m = 400$ | $q_{400} \simeq 0.983404$ |

## 3. "Accidental" degeneracies of $q,p$-oscillators

The following main part of our paper deals with the issue of degeneracies for the two-parameter extended or $q,p$-deformed oscillators defined [10] by the relations

$$AA^\dagger - q\,A^\dagger A = p^{\mathcal{N}}, \qquad AA^\dagger - p\,A^\dagger A = q^{\mathcal{N}}, \tag{23}$$

along with two relations involving $A^\dagger$, $A$ and $\mathcal{N}$ completely analogous to (2) and (9).

The pair of relations (23) is symmetric under $q \leftrightarrow p$ and leads to the formulas

$$A^\dagger A = [\![\mathcal{N}]\!]_{q,p}\ , \quad AA^\dagger = [\![\mathcal{N}+1]\!]_{q,p}\ , \tag{24}$$

where the $q,p$-bracket is

$$[\![X]\!]_{q,p} \equiv \frac{q^X - p^X}{q-p}\ . \tag{25}$$

Obviously, with $p = q^{-1}$, we are back to the BM case [3, 4] of $q$-oscillators and to the AC case [5] at $p = 1$. The other special case [8, 9] of TD deformed oscillators corresponds to $p = q$.

Similarly to BM and TD $q$-oscillators, we take the Hamiltonian in the form

$$H = \frac{1}{2}(AA^\dagger + A^\dagger A). \tag{26}$$

In the $q,p$-deformed Fock space for which $A|0\rangle = 0$,

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{[\![n]\!]_{q,p}!}}|0\rangle\ , \qquad N|n\rangle = n\,|n\rangle\ , \tag{27}$$

the creation/annihilation operators act by the formulas

$$A\,|n\rangle = \sqrt{[\![n]\!]_{q,p}}\ |n-1\rangle\ , \quad A^\dagger\,|n\rangle = \sqrt{[\![n+1]\!]_{q,p}}\ |n+1\rangle\ . \tag{28}$$

The spectrum $H|n\rangle = E_n|n\rangle$ of the Hamiltonian reads

$$E_n = \frac{1}{2}\Big([\![n+1]\!]_{q,p} + [\![n]\!]_{q,p}\Big). \tag{29}$$

As $q,p \to 1$, $E_n = n + \frac{1}{2}$. In addition, $E_0 = \frac{1}{2}$ for any $q,p$.

To study the degeneracy properties of $q,p$-oscillators, we consider $q,p$ as real parameters valued in the intervals

$$0 \le q \le 1, \qquad\qquad 0 \le p \le 1, \tag{30}$$

where the point (0,0) is excluded.

Now we go over to 'accidental' degeneracies and demonstrate the validity of relevant statements.

*Degeneracy of the type $E_m = E_0$*

**Proposition 6.** There exists a continuum of pairs of the values $(q,p)$ or the equivalent continuum of points of the curve $F_{m,0}(p,q) = 0$, for which the degeneracy

$$E_m - E_0 = 0\ , \qquad\qquad m = 2,3,4,\ldots \tag{31}$$

does hold. The curve is given by the equation

$$F_{m,0}(q,p) \equiv \sum_{r=0}^{m} p^{m-r}q^r + \sum_{s=0}^{m-1} p^{m-1-s}q^s - 1 = 0. \tag{32}$$

To prove the statement, take account of Eqs. (29), (25) in Eq. (31). Then, Eq. (32) obviously follows. This formula implies nothing but a certain implicit function $p = f_{m,0}(q)$ which is continuous and monotonically decreases on the $q$-interval in (30). To confirm this assertion, let us consider the derivative

$$\frac{dp}{dq} = f'_{m,0}(q) = -\frac{\partial F_{m,0}}{\partial q}\left(\frac{\partial F_{m,0}}{\partial p}\right)^{-1} =$$

$$= -\frac{\sum\limits_{r=1}^{m} rp^{m-r}q^{r-1} + \sum\limits_{s=1}^{m-1} sp^{m-1-s}q^{s-1}}{\sum\limits_{r=0}^{m-1} q^r(m-r)p^{m-1-r} + \sum\limits_{s=0}^{m-2} q^s(m-1-s)p^{m-2-s}}\ . \tag{33}$$

One can prove the following two facts: 1) for none point $(p,q)$ obeying (30), the derivative $\frac{\partial F_{m,0}}{\partial p}$ in the denominator of (33) turns into zero. 2) For the intervals in (30), both $\frac{\partial F_{m,0}}{\partial q}$ and $\frac{\partial F_{m,0}}{\partial p}$ are positive. Then, the derivative $\frac{dp}{dq}$ in Eq.(33) is always negative, and, thus, $f_{m,0}(q)$ is a continuously decreasing implicit function represented by a flat curve in the quadrant given by (30).

**Remark 1.** In fact, the values of $p$ and $q$ from the admissible pairs $(p,q)$, i.e., those solving Eq. (32), belong to the intervals $0 < q < q_m$ and $0 < p < p_m$ which, since $p_m, q_m < 1$, are smaller than the intervals in (30) (the value $p_m = q_m$ that solve (32) at either $q = 0$ or $p = 0$ being put, clearly depends on the fixed $m$). Moreover, denoting $q_\infty \equiv 1$ (since $q_m \stackrel{m\to\infty}{\longrightarrow} 1$), we have

$$q_2 < q_3 < q_4 < \ldots < q_{m-1} < q_m < \ldots < q_\infty = 1. \tag{34}$$

Now consider, for all $m \ge 2$, the above derivative $f'_{m,0}(q)$ at the end points $(q,p) = (0,p_m)$ and $(q,p) =$

$(q_m, 0)$, where $q_m = p_m$, as well as the derivative of each $f_{m,0}(q)$ at the midpoint of the curve fixed by $p = q$. It is easy to see that $f'_{m,0}(q)|_{q=p} = -1$ for any $m$, whereas at the both endpoints the derivatives are negative and such that

$$f'_{m,0}(q)|_{q=q_m,p=0} < -1 < f'_{m,0}(q)|_{q=0,p=p_m} \quad < 0.$$

As a result, with $q$ growing from zero to $q_m$, the derivative $f'_{m,0}(q)$ is always negative and continuously decreases from $f'_{m,0}(q)|_{q=0,p=p_m}$ through $-1$ to $f'_{m,0}(q)|_{q=q_m,p=0}$ .

**Example 1.** Let $m = 2$. In this case, the relation

$$F_{2,0}(q, p) = p^2 + pq + q^2 + p + q - 1 = 0$$

yields the function (explicit for this case only)

$$p = f_{2,0}(q) = \frac{-1 - q + \sqrt{(1 + q)(1 - 3q) + 4}}{2} \qquad (35)$$

which monotonically decreases for $0 \le q \le q_2$, where

$$q_2 = (\sqrt{5} - 1)/2, \qquad p_2 = q_2. \qquad (36)$$

Then, with account of (33) and (36), we have

$$f'_{2,0}(q) = -\frac{p + 2q + 1}{2p + q + 1} = \begin{cases} -\frac{p_2+1}{2p_2+1} \simeq -0.7236, & q = 0; \\ -1, & p = q; \\ -\frac{2q_2+1}{q_2+1} \simeq -1.382, & p = 0. \end{cases}$$

Figure 4 illustrates this case (and also the cases $m = 4$ and 7).

**Remark 2.** The equations, from which the values $q_m$ are deduced (see Remark 1), can be presented in the form $q + 1 = \frac{1}{q}$ for $m = 2$, $q + 1 = \frac{1}{q^2}$ for $m = 3$, ... , $q+1 = \frac{1}{q^{m-1}}$ for any $m$. Such a form is convenient for applying the graphical treatment. From these equalities, the above inequalities (34) become more obvious.

<u>*Degeneracy of the type $E_{m+1} = E_m$*</u>

**Proposition 7.** There exists a continuous curve $F_{m+1,m}(q, p) = 0$ given by the continuum of pairs $(q, p)$, for which the degeneracy

$$E_{m+1} - E_m = 0 \,, \qquad m \ge 1 \,, \qquad (37)$$

does hold. The equation for this curve is

$$F_{m+1,m}(q, p) \equiv \sum_{r=0}^{m+1} p^{m+1-r} q^r - \sum_{s=0}^{m-1} p^{m-1-s} q^s = 0. \qquad (38)$$

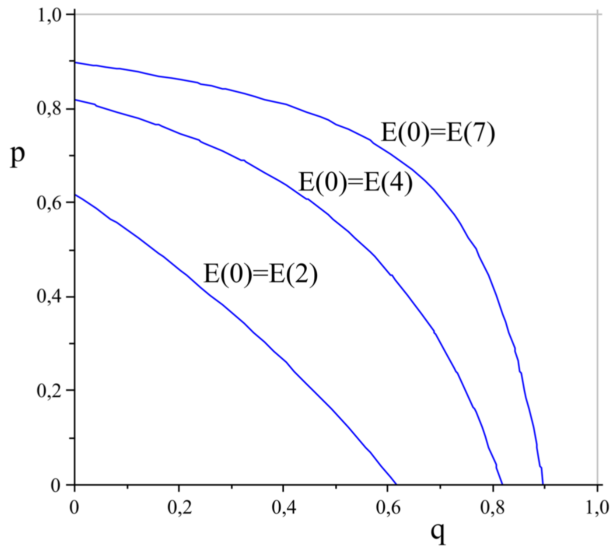


Fig. 4. Three cases of pairwise degeneracies: $E_0 = E_2$, $E_0 = E_4$ and $E_0 = E_7$ in the energy spectrum (29) of a $q, p$-oscillator. The corresponding curves are given by (31)–(32)

In order to prove the statement, we substitute formula (29) in Eq. (37), and Eq. (38) readily follows. Clearly, this equation implies the continuous implicit function $p = f_{m+1,m}(q)$. To prove that this implicit function monotonically decreases on the $q$-interval in (30), we examine the derivative

$$\frac{dp}{dq} = f'_{m+1,m}(q) = -\frac{\frac{\partial}{\partial q} F_{m+1,m}(q, p)}{\frac{\partial}{\partial p} F_{m+1,m}(q, p)}, \qquad (39)$$

where

$$\frac{\partial}{\partial q} F_{m+1,m}(q, p) =$$

$$= \sum_{r=1}^{m+1} r p^{m+1-r} q^{r-1} - \sum_{s=1}^{m-1} s p^{m-1-s} q^{s-1}, \qquad (40)$$

$$\frac{\partial}{\partial p} F_{m+1,m}(q, p) =$$

$$= \sum_{r=0}^{m} (m+1-r) q^r p^{m-r} - \sum_{s=0}^{m-2} (m-1-s) p^{m-2-s} q^s. \qquad (41)$$

Obviously, the both partial derivatives are continuous (polynomial) functions of two variables. The derivative in (41) should be nonzero for each point of the flat curve

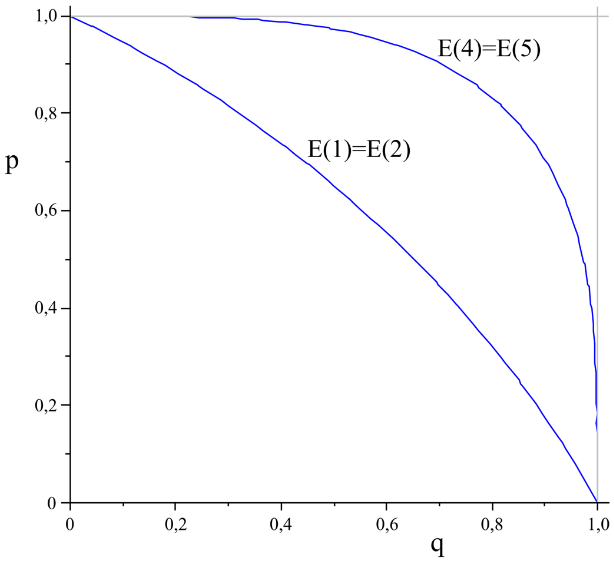


Fig. 5. Two cases of pairwise degeneracies, $E_1 = E_2$ and $E_4 = E_5$, in the energy spectrum (29) of a $q,p$-oscillator

given by (38). To check this, we consider the set of zeros of (41), that is, the set of pairs $(q_0, p_0)$ which solve

$$\frac{\partial}{\partial p} F_{m+1,m}(q,p) = 0. \tag{42}$$

One can show, for generic $m$, that such solutions $(q_0, p_0)$ form a set of points none of which belongs to curve (38). Let us see this in the particular cases of $m = 1, 2, 3$.

For $m = 1$ from (38)–(41), we have

$$F_{2,1} \equiv p^2 + pq + q^2 - 1 = 0, \quad \frac{dp}{dq} = -\frac{2q+p}{2p+q} < 0.$$

Inserting $p + q = \frac{1}{p}(1 - q^2)$ drawn from the equation of the curve $F_{2,1} = 0$ (or $E_2 - E_1 = 0$) in the denominator of $\frac{dp}{dq}$, we find that *for the points of the curve* the denominator is $p + \frac{1}{p}(1-q^2) = \frac{p^2+1-q^2}{p}$ which is strictly positive. Note also that $2p + q > 0$ for all $q, p$ from (30).

For $m = 2$, relations (38)–(41) yield

$$F_{3,2} \equiv p^3 + p^2 q + p q^2 + q^3 - p - q = 0,$$

$$\frac{dp}{dq} = -\frac{p^2 + 2qp + 3q^2 - 1}{q^2 + 2pq + 3p^2 - 1} < 0. \tag{43}$$

Inserting $p^2 + pq + q^2 - 1 = \frac{q}{p}(1 - q^2)$ drawn from the equation of the curve $F_{3,2} = 0$ (or $E_3 - E_2 = 0$) in the denominator of $\frac{dp}{dq}$ in (43), we find that *for the points of the curve* the denominator is $\frac{q}{p}(1 - q^2) + pq + 2p^2$. That is, it is always strictly positive (never turns into zero). Since the same conclusion about strict positivity can be deduced for the numerator in (43), the overall negative sign of the derivative $\frac{dp}{dq}$ in (43) then follows.

For $m = 3$, relations (38)–(41) yield

$$F_{4,3} \equiv p^4 + p^3 q + p^2 q^2 + p q^3 + q^4 - p^2 - pq - q^2 = 0,$$

$$\frac{dp}{dq} = -\frac{p^3 + 2qp^2 + 3q^2 p + 4q^3 - p - 2q}{q^3 + 2pq^2 + 3p^2 q + 4p^3 - q - 2p} < 0. \tag{44}$$

From the above equation $F_{4,3} = 0$ for the curve of degeneracy $E_4 = E_3$, we draw $p^3 + qp^2 + q^2 p + q^3 - p - q = \frac{q^2}{p}(1 - q^2)$ and insert it in the denominator of $\frac{dp}{dq}$ to get: *for the points of the curve*, the denominator is $\frac{q}{p}(1-q^2) + p(q^2 + 2pq + 3p^2 - 1)$. The latter is always positive[1] never turning into zero except for the single point $(1, 0)$. Since the same conclusion about strict positivity can be deduced for the numerator in (44), the overall negative sign of the derivative $\frac{dp}{dq}$ is confirmed.

So, for the cases of $m = 1, 2, 3$, we have demonstrated that the above implicit function $p = f_{m+1,m}(q)$ is a *continuous monotonically decreasing* one. The proof can be extended to higher values of $m$ and also to arbitrary $m$. In Fig. 5, the two particular (different) degeneracy cases $E_2 - E_1 = 0$ and $E_5 - E_4 = 0$ are shown.

**Remark 3.** The case $m = 1$ of (38) (i.e. $E_1 = E_2$) differs from all other cases $m \geq 2$ since, at the end points $(q, p) = (0, 1)$ and $(q, p) = (1, 0)$, the above derivative $f'_{m+1,m}(q)$ has, for the $m = 1$ case, the values differing from the rest $m \geq 2$ cases. Namely, $f'_{2,1}(q)|_{q=0} = -\frac{1}{2}$ and $f'_{2,1}(q)|_{q=1} = -2$. This implies that, as $q$ runs from zero to one, the derivative $f'_{2,1}(q)$ continuously changes from $-\frac{1}{2}$ to $-2$. On the other hand, for all $m \geq 2$, we have $f'_{m+1,m}(q)|_{q=0} = 0$ and $f'_{m+1,m}(q) \xrightarrow{q \to 1} -\infty$, i.e., $f'_{m+1,m}(q)$ continuously decreases from 0 to $-\infty$ as $q$ grows from zero to one. The distinction of $m = 1$ (i.e. $E_2 = E_1$) case from all other $m \geq 2$ cases (e.g., $E_5 = E_4$) is clearly seen in Fig. 5.

Let us emphasize that, contrary to the distinction just discussed in Remark 3, all the degeneracy curves (38) of $E_{m+1} - E_m = 0$, with $m = 1, 2, \ldots$, share the same value of the derivative at their midpoints given by $p = q$: $f'_{m+1,m}(q)|_{q=p} = -1$ (note also its coincidence with the value $f'_{m,0}(q)|_{q=p} = -1$ mentioned in the paragraph immediately after Eq.(34)). Clearly, this is rooted in the $q \leftrightarrow p$ symmetry of the energy function [see (29) and (25)] inherited by curves (38) and (32).

[1] Note that the polynomial in the second parenthesis is identical to the denominator in (43) and, as argued there, is strictly positive.

**Remark 4.** Recall that the one-parameter deformed Tamm–Dancoff oscillator, which stems from the $q,p$-oscillator if $p = q$, possesses double degeneracy [7] of energy levels $E_{m_1} = E_{m_2}$ at a certain value of the parameter $q$. In the present paper, the two-parameter $q,p$-oscillator was shown to possess the same type of degeneracy, $E_{m_1} = E_{m_2}$, for the appropriate (continuum of) pairs $(q,p)$, where $q,p \in (0,1]$. This gives a hint of how is it possible to obtain, besides the TD, numerous other $q$-deformed oscillators with a similar property of double (pairwise) degeneracy of energy levels [15]. For such a degeneracy to occur in the chosen pair $E_{m_1} = E_{m_2}$, it is necessary that the curve (in $q,p$-plane) of the relation $p = f(q)$ generating the particular $q$-oscillator intersects the curve of degeneracy $E_{m_1} - E_{m_2} = 0$ at least once. This is displayed in Fig. 6 for a sample relation $p = q^5$ which crosses the indicated degeneracy curves $E_3 - E_0 = 0$ and $E_5 - E_4 = 0$.

It is clearly seen from Fig. 6 that the non-standard $q$-oscillator inferred by substituting in (23)–(25) and (29) the relation $p = q^5$ does possess the degeneracy $E_3 - E_0 = 0$ at a definite value of $q$ and the degeneracy $E_5 - E_4 = 0$ at a distinct value of $q$. Details of this approach with many particular cases are given in [15].

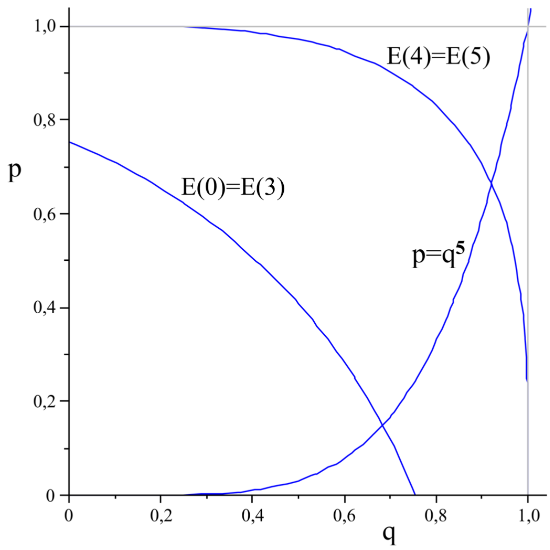


Fig. 6. Curve $p = q^5$ yielding a respective $q$-oscillator crosses the degeneracy curves $E_0 = E_3$ and $E_4 = E_5$ at different values of $q$

## 4. Conclusions and Outlook

The study of deformed oscillators demonstrates that, due to modified commutation relations, such oscillators possess nontrivial properties very different from those of the standard quantum oscillator. In our papers [7,15,16], we studied the unusual property of accidental two-fold or double two-fold energy level degeneracies of definite one-parameter deformed oscillators. The present paper deals with the degeneracy of energy levels of two-parameter deformed $q,p$-oscillators.

After recalling the special degeneracies occurring for the Biedenharn–Macfarlane $q$-oscillator at $q$ being some roots of unity, we placed a sketch of the 'accidental' double degeneracy properties [7] of energy levels of the Tamm–Dancoff deformed oscillator. The peculiarity of the latter consists in the fact that, for each pair $E_{m+k} = E_m$ of energy levels, there exists a special **real** value of the $q$-parameter which provides their degeneracy.

In the main part of the paper, we have examined the ability of the two-parameter $q,p$-oscillators to have pairwise energy level degeneracies. As is shown, the $q,p$-oscillator possesses the two-fold (pairwise) degeneracy of a definite type, i.e., within some pair $E_{m_1} = E_{m_2}$, at the corresponding values $(q,p)$ from a continual set identical to the curve of $E_{m_2} - E_{m_1} = 0$ in the $q,p$-plane.

What is important, the pairwise degeneracy of the energy levels of $q,p$-oscillators observed at certain values of $q$ and $p$ is "accidental" (as it occurs without any underlying symmetry) and involves a single fixed pair of levels.

Let us also remark that the degeneracy in $q,p$-oscillators shown in this paper is not in conflict, as it was already commented in [16], with the well-known "no-go" theorem [17, 18] about the absence, in one dimension, of degenerate discrete states in any standard quantum-mechanical system. Indeed, the $q,p$-oscillators analyzed in our paper go beyond the scope of customary systems of traditional quantum mechanics, due to such more general nontrivial features (see, e.g., [12]) as the non-constant position-dependent mass given by an inertia function, the complicated interaction depending on both the position and the momentum, etc.

On the base of the considered (seemingly, unnoticed earlier) important peculiarity of the $q,p$-oscillators, we can infer a plenty of new nonlinear one-parameter deformed oscillators which exhibit nontrivial and unusual degeneracy properties (diverse patterns of levels degeneracies, including rather complicated ones). As some step already made in this direction, let us quote the paper [16], where a number of $p$-oscillators is presented exhibiting a rather nontrivial pattern of two-fold double degeneracies (two pairwise degeneracies within each of two fixed pairs of energy levels, e.g., $E_1 = E_2$ and $E_3 = E_4$).

It is worth to mention again the fact of the applicability [13] of $q, p$-oscillators ($q, p$-bosons) in the context of the efficiency of a description of the observed non-Bose properties of the two- and multi-pion (-kaon) correlations in the experiments on relativistic heavy-ion collisions like that of the other types, e.g., the one-parameter BM-type $q$-oscillator and the $q$-Bose gas model [19, 20]. In that context, it would be interesting to find some peculiarity (if any) connected with the feature of 'accidental' double degeneracy of $q, p$-oscillators considered in the present paper. The same can be said about the usage (see [13]) of TD $q$-oscillators and the "TD $q$-Bose gas" model which is just the one-parameter $p = q$ limit of the $q, p$-Bose gas model.

One may also hope that the unusual novel $q$-bosons (related with non-standard $q$-oscillators treated in [7, 15, 16]) and possibly some others will be useful in the study of explicitly solvable problems, say, along the lines similar to those described in [21], and also for diverse physical applications.

Of course, it is desirable to give explicit and exhaustive proofs of the pairwise degeneracy of the energy levels of $q, p$-oscillators for more involved cases like $E_{m+2} = E_m$ and, also, for the most general case of degeneracy: $E_{m+k_1} = E_{m+k_2}$, $k_1 \neq k_2$. This will be done in a separate paper.

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ДВОКРАТНЕ ВИПАДКОВЕ ВИРОДЖЕННЯ ЕНЕРГЕТИЧНИХ РІВНІВ У МОДЕЛІ $q$, $p$-ОСЦИЛЯТОРА

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Р е з ю м е

Показано, що двопараметрично деформовані осцилятори з параметрами деформації $q, p$, де $0 < q,\ p \leq 1$, мають властивість "випадкового" двократного виродження енергетичних рівнів типу $E_m = E_{m+1}$ та типу $E_0 = E_m$ при відповідних значеннях $q$ і $p$. Коротко обговорено також найбільш загальний випадок виродження $E_{m+k} = E_m$, де $k \geq 1$ для $m \geq 1$ або $k \geq 2$ для $m = 0$.