PARTICLE TRAPPING AND NON-RESONANT INTERACTION IN A PROBLEM OF STOCHASTIC ACCELERATION

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Evolution of the velocity dispersion of particles undergoing the action of an external field of random waves is considered. For moderate Kubo numbers, the particle trapping effect is not negligible, so a discrepancy between the quasilinear diffusion and the results of simulations becomes evident. It is shown that the Fokker–Planck equation with the time dependent diffusion coefficient describes the particle trapping to some extent. Apart of this, such an equation accounts for the non-resonant wave-particle interaction.

1. Introduction

For a long time, the particle diffusion in turbulent fields is one of the fundamental problems of theoretical and practical interests. In non-equilibrium plasmas, the diffusion leads up to the saturation of instabilities and the establishment of a stationary state. In a beam-plasma system, a stationary level of turbulence depends on the rate of particle spread in a resonant region of the velocity space.

The quasilinear theory [1] gives a self-consistent description of the beam-field evolution for weak fields. Particle diffusion is governed by the Fokker–Planck equation with the velocity-dependent diffusion coefficient determined by a wave spectrum. A diffusion coefficient is calculated in the approximation of a free-particle propagator and under the assumption that the field correlation time is negligible on the time scale of the system evolution. The latter implies that the contribution to the particle spread on a scale of the order of the correlation time is relatively small. With a growth of the field intensity in a course of the system evolution, a discrepancy between the quasilinear theory and the results of numerical experiments becomes evident [2].

In a number of works, starting from work [3], it was proposed to extend the quasilinear description to stronger fields by renormalization of the diffusion coefficient. It was supposed to be calculated not on the free particle propagator, but on a such one that accounts for the particle diffusion described by the same diffusion coefficient. In such a way, a closed equation for the renormalized diffusion coefficient may be obtained. However, this scheme is inconsistent since the particle spread on the correlation time scale is assumed to be determined by an asymptotic diffusion coefficient. In contrast, the simulations of the diffusion of resonant particles in external fields [4] show a substantial growth of the velocity dispersion on the initial stage, in extent of a fraction of the correlation time, which is slowing down later on. Therefore, prior to develop a renormalization procedure, it is necessary to have correct description of the particle spread on a short-time scale.

In our previous works [5, 6], it was shown that, for moderate Kubo numbers ($Q = 1 \div 3$), the velocity dispersion evolution obtained from the Fokker–Planck equation with a time-dependent diffusion coefficient is in agreement with the results of simulation contrary to one calculated with the asymptotic diffusion coefficient. It proves to be important to account for a time variation of the diffusion coefficient on the scale of the correlation time. Along with fast initial growth, the oscillation of the dispersion was observed in simulations. Though the oscillations were partly recovered by solutions of the Fokker–Planck equation with the time-dependent diffusion coefficient, their origin nevertheless was not clear. In this paper, we examine the nature of the dispersion oscillation in more details.

For moderate Kubo numbers, the particle trapping effect should come into play, and this introduces difficulties in a description of the diffusion. It is commonly assumed that the effect of particle trapping by waves could not be described by equations obtained by the expansion in integer powers of a field. Here, we will show that the Fokker–Planck equation accounts for the particle trapping to some extent if the dependence of the diffusion coefficient on time is retained. Other significant feature of the time-dependent diffusion coefficient is its capability to account for the non-resonant interaction of particles with waves. Both these effects, the particle trapping and the non-resonant wave-particle interaction are not described by the diffusion equation with an asymptotic diffusion coefficient.

In Section 2, we briefly recall the model which was used in [5, 6] and discuss some details of the derivation of a diffusion equation for the transition probability. Oscillatory features of the velocity dispersion due to the bouncing of resonant particles and the velocity modulation of non-resonant particles are considered in Section 3.

2. Model for Simulation and Equation for Transition Probability

In simulations, we consider the motion of noninteracting particles in a 1D external electric field that is a superposition of waves with the same frequency ω, various wave numbers, and random phases. In more details, this model is described in works [5, 6]. The total intensity of waves φ_0^2 is distributed among the partial harmonics over a finite interval of wave numbers according to the Gaussian law. A spectrum is located around the central wave number k_0 and characterized by the width Δk. The overlap parameter for the central harmonic is taken large enough so, when the total intensity of waves is fixed, the results do not depend on the number of waves, and the approximation of the continuous spectrum is valid. Dimensionless parameters of the field are the dimensionless amplitude of the potential $\sigma = (k_0/\omega)^2(e/m)\varphi_0$, the dimensionless spectrum width $d = \Delta k/k_0$, and the Kubo number $Q = \sqrt{\sigma}/d$ that is the ratio of the field correlation time to the characteristic bounce period of particles.

In simulations, we found the temporal evolution of a distribution function in the velocity space which initially was taken as the delta function, along with evolution of its second moment, the velocity dispersion. The latter is compared with numerical and analytical solutions of the equation for transition probability.

We now discuss some details of the derivation of the diffusion equation for the transition probability averaged over random phases, W. The diffusion equation for W has been obtained by a number of authors in various approximations. Here, we start from a rather general equation which is exact provided the field statistical properties are completely characterized by the pair correlation function [8]

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) W(x,v,t;x',v',t') =$$

$$= \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v} \int\limits_{t'}^{t} d\tau \int dy du W(x,v,t;y,u,\tau) \times$$

$$\times \langle E(x,t)E(y,\tau)\rangle \frac{\partial}{\partial u} W(y,u,\tau;x',v',t'),$$

$$W(x,v,t';x',v',t') = \delta(x-x)\delta(v-v'). \tag{1}$$

This equation is non-linear and non-local. So it should be simplified in order to obtain an explicit solution.

For $t-\tau_{\rm cor} > t'$ ($\tau_{\rm cor} \equiv 2/\Delta k\, v$ is the correlation time of the Lagrangian correlation function, $\langle E^2\rangle_{x-y,\,t-\tau}$), the interval $[t-\tau_{\rm cor}, t]$ gives the main contribution to the integral over τ on the right-hand side of Eq. (1). So the following substitution is justified:

$$\frac{\partial}{\partial u} W(y,u,\tau;x',v',t') \Rightarrow \frac{\partial}{\partial v} W(x,v,t;x',v',t'). \tag{2}$$

Then Eq. (1) transforms to a local equation. Note that we will use such a substitution also for $t-\tau_{\rm cor} < t'$, though it is not so evident for such a case. To some extent, it is supported by a fast relaxation of the transition probability from the initial δ-like form on a fraction of the interval $[0, \tau_{\rm cor}]$.

The Fokker–Planck diffusion equation familiar from the quasilinear theory can be obtained in two further steps given by Eqs. (3) and (4). The first one is a substitution of the free particle propagator

$$W_0(x,v,t;y,u,\tau) \equiv \delta(x-y-v(t-\tau))\,\delta(v-u)$$

for the transition probability at the interval of correlation time

$$W(x,v,t;y,u,\tau) \Rightarrow W_0(x,v,t;y,u,\tau), \tag{3}$$

and the second step is a transition to the time independent diffusion coefficient (asymptotic value)

$$\int\limits_{t'}^{t} ...d\tau \Rightarrow \int\limits_{-\infty}^{t} ...d\tau. \tag{4}$$

As far as a justification for both approximations given by Eqs. (3) and (4) is a decay of the field correlation function $\langle E^2\rangle_{x,\,t}$ at $t > \tau_{\rm cor}$, it may look like one of them implies other one as well. Probably, for this reason, the

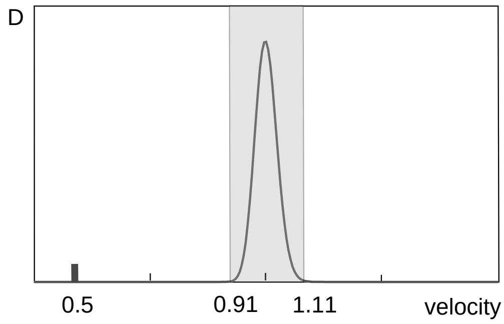

Fig. 1. Asymptotic value of the diffusion coefficient $D(v)$. Gray strip indicates the interval of the phase velocities of waves taken in simulations. Resonant particles, $v_0 = 1$, are in the middle of this interval; non-resonant particles, $v_0 = 0.5$ and 1.8, are far beyond of it

previous efforts to extend the quasilinear description for stronger fields were mostly directed to the renormalization of the asymptotic value of the diffusion coefficient. Our results [5, 6] show that, for moderate Kubo numbers, it is more important to account for the dependence of the diffusion coefficient on time than to renormalize a propagator:

$$\int_{t'}^{t} W...d\tau \Rightarrow \int_{t'}^{t} W_0...d\tau. \tag{5}$$

As a result, for the transition probability in the velocity space

$$w(v, v_0, t) = \int dxW(x - x', v, v_0, t),$$

we obtain the Fokker–Planck equation

$$\frac{\partial w(v, v_0, t)}{\partial t} = \frac{\partial}{\partial v} D(v, t) \frac{\partial}{\partial v} w(v, v_0, t) \tag{6}$$

with the time-dependent diffusion coefficient

$$D(v, t) = \left(\frac{e}{m}\right)^2 \int_0^t \langle E^2 \rangle_{v\tau,\tau} d\tau. \tag{7}$$

The well-known quasilinear diffusion coefficient $D(v)$ is the asymptotic value of $D(v, t)$.

Solutions of Eq. (6) are found numerically. In addition, the WKB approximation for this solution was proposed in [5]. Both numerical and WKB solutions of the Fokker–Planck equation are used to calculate the evolution of the velocity dispersion to compare it with the results of simulations.

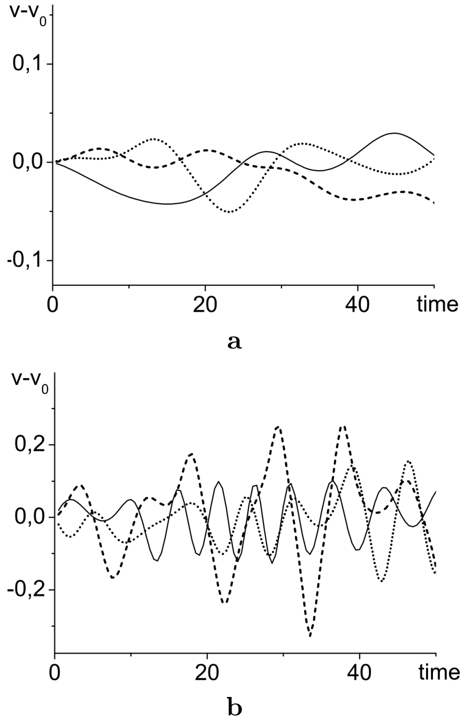

Fig. 2. $v(t)$ for three realizations of a field. Resonant particles, $v_0 = 1$; $\sigma = 0.0016$, $Q = 1$ (*a*) and $\sigma = 0.0256$, $Q = 4$ (*b*). For resonant particles, the period of oscillations increases with the field. Oscillations are attributed to the bouncing of trapped particles

3. Oscillatory Features of Velocity Dispersion at Small Times

Comparison of the results of simulations and the solutions of the Fokker–Planck equation (6) in a wide range of parameters for small Kubo numbers $Q \ll 1$ gives no noticeable difference between solutions with $D(v)$ and $D(v, t)$. For $Q > 1$, only solutions with $D(v, t)$ agree with the results of simulations either on the scale of the correlation time or on that by two orders larger [5, 6].

Here, we will discuss oscillatory features of the velocity dispersion observed on a small time scale. By resonant particles, we would mean those whose initial velocities are equal to the phase velocity of the central harmonic, i.e. $v_0 = 1$. Non-resonant particles have the initial velocities beyond the interval of phase velocities of the waves taken in the simulation (Fig. 1). Length and time in the plots are normalized to the wavelength of the central harmonic $2\pi/k_0$ and the period of waves $2\pi/\omega$. All plots except Fig. 5 are given for the spectrum width $d = 0.04$.

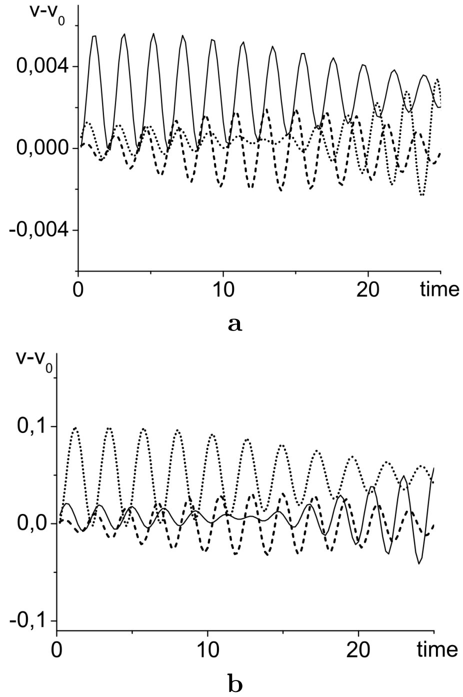

Fig. 3. $v(t)$ for three realizations of a field. Non-resonant particles, $v_0 = 0.5$. $\sigma = 0.0016$, $Q = 1$ (a) and $\sigma = 0.0256$, $Q = 4$ (b). For non-resonant particles, contrary to resonant ones, the period of oscillations is not affected by the field. Oscillations are attributed to the modulation of the velocity by a field

Before considering the oscillations of the velocity dispersion, let us compare the oscillations of velocities for arbitrary resonant and non-resonant particles. The scaling of the periods of oscillation with field amplitudes for two types of particles is different. A temporal variations of the velocity, $v(t)$, observed in simulations are given in Figs. 2 and 3 for Kubo numbers $Q = 1$ and $Q = 4$. For resonant particles, the period of oscillations depends on the field amplitude (and consequently on Q) and corresponds to the particle bouncing, (Fig. 2), whereas, for non-resonant particles, there is no such dependence (Fig. 3).

Figure 4 shows that solutions of the Fokker–Planck equation for resonant particles recover not only the fast initial growth of the dispersion observed in simulations for the moderate Kubo numbers $Q = 1, 2, 3, 4$ but, to a certain extent, its oscillations and afterwards the correct transition to a slow growth. Oscillations of the velocity dispersion of resonant particles appear for $Q > 1$, when the bounce period becomes less than the correlation time. We attribute such oscillations of the dispersion to the bouncing of resonant particles trapped by the waves. The reason for this assumption is given

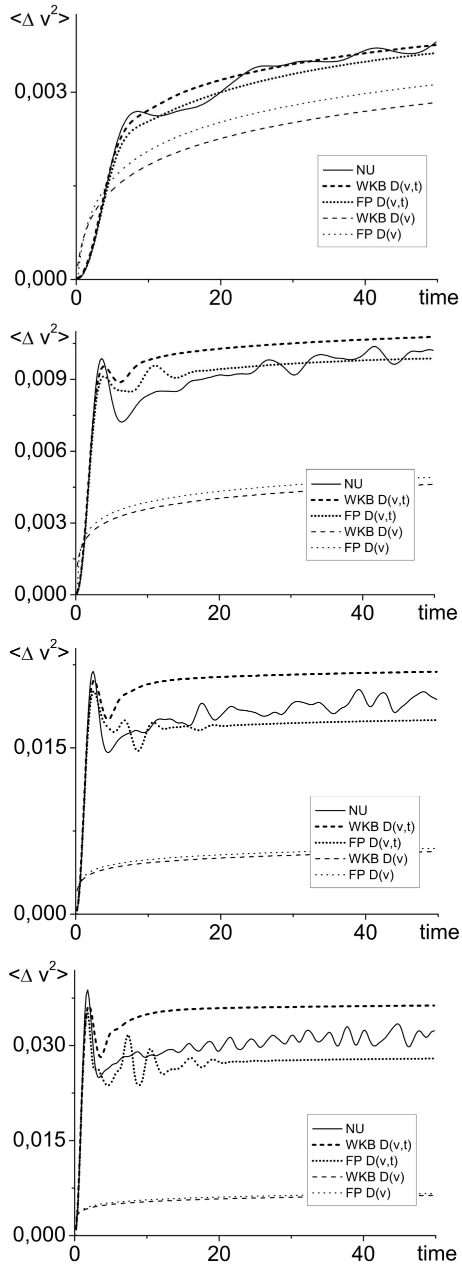

Fig. 4. Resonant particles, $v_0 = 1$. The results of simulations (NE) are compared with the numerical (FP) and analytical (WKB) solutions of the Fokker–Planck equation with the time dependent $D(v, t)$ and asymptotic $D(v, t)$ diffusion coefficients. Fast initial growth of the dispersion for the Kubo numbers $Q = 1, 2, 3, 4$; correspondingly, $\sigma = 0.0016, 0.0064, 0.0144, 0.0256$. The period of oscillations corresponds to the bounce period that is scaled with the field amplitude as $T_{\rm res} \sim \sigma^{-1/2}$. Oscillations of the dispersion due to the bouncing of trapped particles are partly recovered by solutions of the Fokker–Planck equation with the time dependent diffusion coefficient

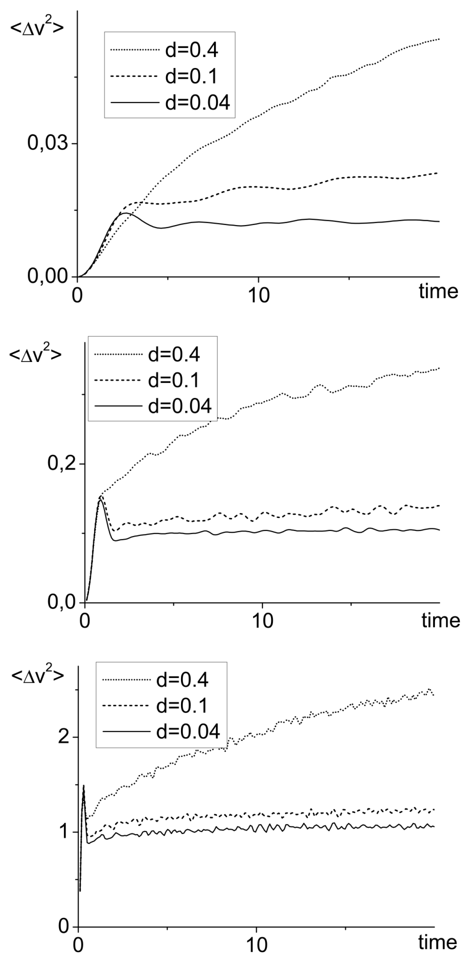

Fig. 5. Dispersion of resonant particles, $v_0 = 1$, obtained in simulations for various field amplitudes $\sigma = 0.01$, 0.1, and 1. On each of the plots, three curves are given for $d = 0.4, 0.1$, and 0.04. The period of oscillations corresponds to the bounce period and does not depend on the spectrum width d

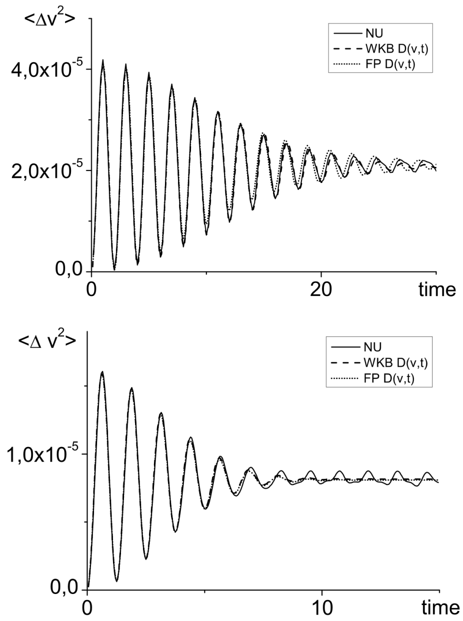

Fig. 6. Non-resonant particles, $v_0 = 0.5$ and 1.8. Kubo numbers $Q = 1$, $\sigma = 0.0016$. The results of simulations are compared with numerical and analytical solutions of the Fokker–Planck equation with the time dependent diffusion coefficient. The period of oscillations $T_{\rm nonres}$ does not depend on the field amplitude σ. Oscillations of the dispersion occur due to a modulation of particle velocities by waves

by the same scaling of its period with amplitude (Figs. 4 and 5) as of the bounce period

$$T_{\rm res} = \frac{\lambda}{v_{\rm bounce}} = \frac{1}{\sqrt{\sigma}}. \tag{8}$$

In addition, Fig. 5 shows that the oscillation period does not depend on the spectrum width d.

It is important that the effect of particle trapping is partly recovered by Eq. (6). Due to a common notion, trapping effects could not be described by equations obtained by the expansion in integer powers of the field, as far as the bounce period depends on a fractional power of the field. Note that the quasilinear diffusion equation does not reproduce neither the fast initial spread nor oscillations (Fig. 4).

The comparison of the non-resonant particle dispersions given in Fig. 6 shows that Eq. (6) recovers the non-local interaction in the velocity space between particles and waves as well. The initial particle velocities, $v_0 = 0.5$ and 1.8, are far beyond the interval of phase velocity localization (0.91, 1.11) taken in simulations. The quasilinear diffusivity is zero for such a case.

For non-resonant particles, oscillations of the dispersion is attributed to the particle motion modulation by waves. Consequently, the period of oscillation is estimated as

$$T_{\rm nonres} = \frac{1}{v_0 - 1}. \tag{9}$$

Contrary to resonant particles, it does not depend on the field intensity (Figs. 6 and 7).

In Fig. 8, the corresponding oscillations of the distribution functions of resonant and non-resonant particles are shown.

A difference in the scalings of a period of oscillations of the velocity dispersion between resonant and non-resonant particles can be also seen from the surface plot

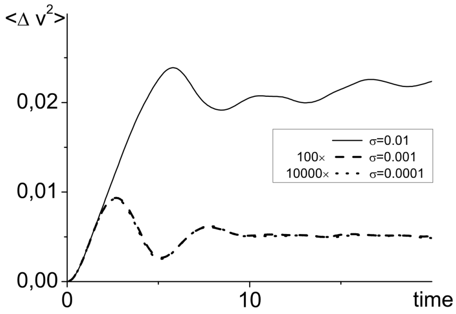

Fig. 7. Dispersion of non-resonant particles, $v_0 = 1.2$, obtained in simulations for various field amplitudes. The period of oscillations does not depend on the field amplitude

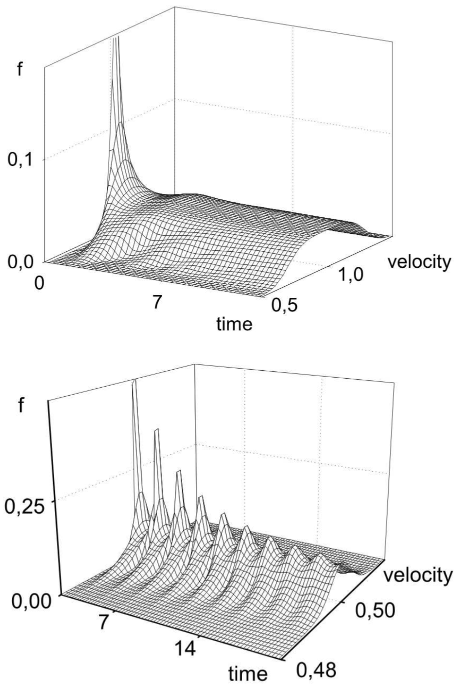

Fig. 8. Distribution functions of resonant particles, $v_0 = 1$, $\tau_{\rm cor} \simeq 8$, and non-resonant particles, $v_0 = 0.5$, $\tau_{\rm cor} \simeq 16$, obtained in the WKB approximation

of the diffusion coefficient (Fig. 9). In the region of resonant velocities near $v = 1$, the valleys in the plot of the diffusion coefficient are oriented along the time axis, and such orientation brings to the bouncing of resonant particles.

In the region of velocities far from $v = 1$, the diffusivity decays to the wings. Its valleys are bent to be oriented along the velocity axes. Small diffusivity causes a small displacement of non-resonant particles, and the particle diffusion is not restricted in velocity due to a changed orientation of valleys; so the oscillation of the

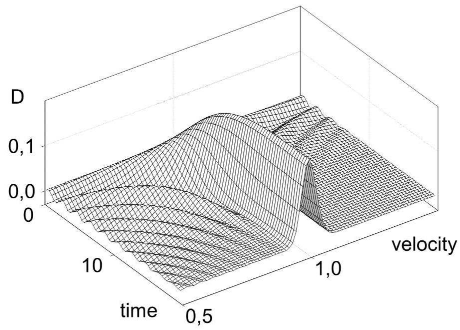

Fig. 9. Diffusion coefficient D(v,t). Valleys distant from $v_0 = 1$ are extended along the v-axis; near the resonant region, they make a bend and are oriented along the t-axis

dispersion is caused in this case by the temporal variation of the diffusion coefficient in the vicinity of a fixed velocity.

4. Conclusions

Simulations of the particle diffusion in an external field of random waves show a fast initial spread of velocities for a moderate Kubo numbers: the early stage gives a substantial contribution in the overall dispersion. It may be expected that a higher initial rate of velocity spread causes a lower saturation level of instability.

The solutions of the Fokker–Planck equation with the time dependent diffusion coefficient recover the oscillations observed in the simulation. Oscillations are attributed to the particle trapping and the non-resonant particle-wave interaction, the effects of which are not described by the diffusion equation known from the quasilinear theory.

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ЗАХОПЛЕННЯ ЧАСТИНОК
ТА НЕРЕЗОНАНСНА ВЗАЄМОДІЯ
В ЗАДАЧІ СТОХАСТИЧНОГО ПРИСКОРЕННЯ

В.І. Засенко, А.Г. Загородній, Я. Вейланд

Р е з ю м е

Розглянуто еволюцію дисперсії швидкості частинок, що рухаються в зовнішньому полі випадкових хвиль. Для помірних чисел Кубо стає помітним ефект захоплення частинок хвилями, що веде до розбіжності між результатами квазілінійного опису дифузії та моделювання. Показано, що рівняння Фоккера–Планка із залежним від часу коефіцієнтом дифузії деякою мірою описує захоплення частинок. Крім того, це рівняння враховує нерезонансну взаємодію частинок з хвилями.