ON LOW-FREQUENCY ALFVÉN INSTABILITIES IN STELLARATORS

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A theory of low frequency Alfvén eigenmodes [the global Alfvén eigenmodes (GAE) and non-conventional global Alfvén eigenmodes (NGAE)] in stellarators is developed taking into account the plasma compressibility. It is found that the conditions of existence of GAE/NGAE modes in stellarators differ from those in tokamaks and that the geodesic acoustic mode (GAM) may not exist in stellarators. Specific calculations modeling low-frequency Alfvén instabilities in a Wendelstein 7-AS are carried out.

1. Introduction

Theoretical studies of Alfvén instabilities (AI) driven by energetic ions were started well before any experimental research on this topic. In 1968, Belikov, Kolesnichenko, and Oraevskij showed that shear Alfvén waves can be destabilized in the presence of a rather small population of energetic ions, in particular, fusion-produced alpha particles [1]. This work was based on the local approach, which enabled to evaluate the growth rate of instabilities. However, the local approach is not sufficient, in particular, because it cannot give an answer to the question whether Alfvén modes exist in real inhomogeneous plasmas. The study of AIs with employing the eigenmode analysis was started by Rosenbluth and Rutherford in 1975 [2]. First experimental observations of AIs were reported at the beginning of the 1990s: it was found in tokamaks TFTR and DIII-D with the neutral beam injection (NBI) that AIs can be very dangerous, leading to the loss of more than 50% of the injected power [3, 4]. This stimulated very extensive experimental and theoretical investigations of AIs in all types of toroidal devices (tokamaks, stellarators, and spherical tori), see, e.g., overviews [5, 6]. As a result, it was concluded that AIs are not necessarily dangerous, and they can even be used for diagnostics in some cases. AIs in stellarators have a number of peculiarities, see overview [7].

In this paper, we concentrate on the instabilities associated with the destabilization of eigenmodes in the low-frequency part of the Alfvén spectrum in stellarators. A well-known mode of this kind is the global Alfvén eigenmode (GAE) [8]. The GAE is a mode with the frequency below a minimum of the continuum branch with given mode numbers (m and n). Another mode is the NGAE which is a mode with the frequency above a maximum of the continuum branch with given mode numbers. The NGAE was predicted for currentless stellarators with a slightly non-monotonic rotational transform; furthermore, it was argued that the NGAE can generate a kinetic Alfvén Wave (KAW) leading to anomalous electron heat transfer and sheared plasma rotation [9, 10]. This enabled one to explain strong drops of the energy content observed in a Wendelstein 7-AS stellarator [9, 10].

2. Global Alfvén Eigenmodes and Non-Conventional Global Alfvén Eigenmodes in Compessible Plasmas

Although Alfvén waves are studied in many cases in the approximation of incompressible plasma, features of these waves and even conditions of their existence may depend on the plasma compressibility, $\widetilde{\zeta} \equiv \operatorname{div} \boldsymbol{\xi}$, with $\boldsymbol{\xi}$ the plasma displacement. One can expect that the compressibility will most strongly affect low-frequency Alfvén eigenmodes. Therefore, it is of importance to take finite $\widetilde{\zeta}$ into account. We proceed from the equations derived for stellarators in [11]. Eliminating $\widetilde{\zeta}$, we obtain the equation

$$\frac{1}{r}\frac{d}{dr}r\delta_0\left(\frac{\omega^2-\omega_{G1}^2}{v_A^2}-k_{mn}^2\right)\frac{d\Phi_{mn}}{dr}-$$

$$-\left[\frac{m^2\delta_0}{r^2}\left(\frac{\omega^2-\omega_{G2}^2}{v_A^2}-k_{mn}^2\right)+\right.$$

$$+ \frac{k_{mn}}{r}(r\delta_0 k'_{mn})'\Big] \Phi_{mn} + C^{(j)}_{mn} + C^{(c)}_{mn} = 0, \quad (1)$$

where Φ_{mn} is a Fourier harmonic of the scalar potential of the electromagnetic field, $\Phi = \sum_{mn} \Phi_{mn}(r)\exp(im\vartheta - in\varphi - i\omega t)$; r, ϑ, φ are the radial, poloidal, and toroidal coordinates, respectively; $k_{mn} \equiv k_\parallel(m,n) = (m\iota - n)/R$ is the longitudinal wave number, ι is the rotational transform, R is the major radius of the torus; $C^{(j)}_{mn}$ describes a contribution of the plasma current, $C^{(j)}_{mn} = 0$ in currentless stellarators, whereas, in tokamaks,

$$C^{(j)}_{mn} = \frac{m}{r^2}k_{mn}(3\iota' + r\iota''), \quad (2)$$

$C^{(c)}_{mn}$ describes the coupling between the harmonics of Φ because of the inhomogeneity of the magnetic field, the plasma shaping and compressibility; δ_0 is associated with the plasma elongation, $\delta_0 = \delta_0(r) \gtrsim 1$;

$$\omega^2_{G1} = \frac{c_s^2}{\delta_0 r^2}\sum_{\mu\nu}\alpha^{(\mu\nu)}_{mn}\mu^2(\epsilon^{(\mu\nu)}_B)^2,$$

$$\omega^2_{G2} = \frac{c_s^2}{\delta_0}\sum_{\mu\nu}\alpha^{(\mu\nu)}_{mn}\left(\frac{d\epsilon^{(\mu\nu)}_B}{dr}\right)^2,$$

$$\alpha^{(\mu\nu)}_{mn} = \omega^2/(\omega^2 - k^2_{m+\mu,n+\nu N}), \quad (3)$$

$c_s = (\gamma p/\rho)^{1/2}$ is the sound velocity, p is the plasma pressure, $\rho = \sum_i M_i n_i$, M_i and n_i are the ion mass and density, respectively; $\epsilon^{(\mu\nu)}_B$ is a Fourier component of the equilibrium magnetic field, B_0,

$$B_0 = \bar{B}\left(1 + \frac{1}{2}\sum_{\mu\nu}\epsilon^{(\mu\nu)}_B e^{i\mu\vartheta - i\nu N\varphi}\right), \quad (4)$$

N is the number of the field periods.

It follows from Eq. (1) that the local Alfvén resonance is approximately described by the equation

$$\omega^2 = k_\parallel^2 v_A^2 + \omega^2_{G1}. \quad (5)$$

Let us analyze this equation. We consider first the simplest case of $k_\parallel = 0$, assuming that the equilibrium magnetic field contains only the toroidicity-induced Fourier harmonic, $\epsilon_t \equiv -\epsilon^{(10)}_B$ ($\epsilon_t > 0$). Then Eq. (5) yields

$$\omega^2 = \omega^2_G \equiv \frac{2c_s^2}{R^2}\frac{\epsilon_t^2}{\delta_0\epsilon^2}\left(1 + \frac{\iota^2}{2}\frac{\delta_0\epsilon^2}{\epsilon_t^2}\right), \quad (6)$$

where $\epsilon = r/R$; typically, $\epsilon_t < \epsilon$ in stellarators. Equation (6) is reduced to $\omega_G = \sqrt{2}(c_s/R)(1 + \iota^2/2)]$ for tokamaks with circular cross section [12]. The frequency ω_G is associated with the geodesic curvature, $\mathcal{K}_G \equiv 2B^{-1}[\boldsymbol{\mathcal{K}} \times \mathbf{B}] \cdot \nabla r$, as $\omega_G^2 \approx c_s^2\langle\mathcal{K}_G^2\rangle$, where $\langle\dots\rangle$ means averaging over the angles. Therefore, the mode with this frequency is called the geodesic acoustic mode (GAM) [12]. It plays an important role in the physics of zonal flows and Alfvén cascades in tokamaks [13, 14]. However, we should note that introducing ω_G makes sense only provided that the first term in Eq. (6) dominates, i.e., when

$$\mathfrak{S} \equiv \frac{\iota^2}{2}\frac{\delta_0\epsilon^2}{\epsilon_t^2} \ll 1. \quad (7)$$

In another limit case, $\mathfrak{S} \gg 1$, Eq. (6) describes sound waves, $\omega^2 = k_\parallel^2 c_s^2$, with $k_\parallel = \iota/R$. We will refer to $\mathfrak{S}$ as a "sound parameter" because, depending on its magnitude, the (m,n) Alfvén continuum branch either avoids the sound resonance (when $\mathfrak{S} \ll 1$) or reaches its vicinity (when $\mathfrak{S} \gg 1$) at the rational surface $\iota = m/n$. When $\mathfrak{S} \ll 1$, $\omega_s^2/\omega_G^2 \approx \mathfrak{S}$, with $\omega_s = \iota c_s/R$. Equation (7) is typically satisfied in tokamaks, at least, in the region where the safety factor exceeds unity. It is also satisfied in W7-AS discharges with $\iota \ll 1$. In contrast to this, $\mathfrak{S} \gg 1$ in Wendelstein 7-X (W7-X), where $\iota \lesssim 1$, $\epsilon/\epsilon_t \gtrsim 2.3$, and $\delta_0 \sim 1.5$. This implies that GAM will not exist in W7-X.

We recall that Eq. (6) was obtained by assuming that $\epsilon^{(\mu,0)}_B = 0$ for $\mu \geq 2$. Taking into account all the terms in Eq. (5) leads to the infinite number of solutions. However, only low frequency solutions with $k_{mn} = 0$ can satisfy the requirement that the frequency must be beyond the Alfvén-sound gaps. A solution of this kind with $\omega > \iota c_s/R + \Delta_{\rm gap}$, where $\Delta_{\rm gap}$ is the half-width of the lowest Alfvén-sound gap, represents ω_G. This solution qualitatively agrees with Eq. (6).

Now we consider a particular Alfvén continuum branch with $k_{mn} \neq 0$. It follows from Eq. (5) and (3) that the usual equation for the local Alfvén resonance, $\omega = k_\parallel v_A$, takes place when $F(\omega) \ll 1$, where

$$F(\omega) = \frac{1}{2}\sum_{\mu\nu}\frac{\widehat{\epsilon}^{\,2}_{\mu\nu}}{\widetilde{\omega}^2 - (\widetilde{k}_{mn} + \widetilde{k}_{\mu\nu})^2}, \quad (8)$$

$\widehat{\epsilon}_{\mu\nu} = \sqrt{2/\delta_0}\ \mu\epsilon^{(\mu\nu)}_B/\epsilon$, $\widetilde{\omega} = \omega R/c_s$, $\widetilde{k}_{\mu\nu} = \mu\iota - \nu N$. This condition is satisfied at high frequencies, $\omega^2 \gg \omega_G^2$, provided that ω is not close to Alfvén-sound gaps. In addition, it is satisfied at very low frequencies, $\omega^2 \ll \iota^2 c_s^2/R^2$ provided that $\mathfrak{S} \gg 1$, i.e., when the geodesic acoustic frequency makes no sense. It is interesting to note that Eq. (5) has a solution with $\omega \propto k_\parallel v_A$ at very low frequencies, $\omega \ll \omega_G$, even when $\mathfrak{S} \lesssim 1$ (we used the fact that $F(\omega) \approx -\mathfrak{S}^{-1}$ for $\omega \to 0$):

$$\omega \approx \frac{k_\parallel v_A}{\sqrt{1+\mathfrak{S}^{-1}}}\,. \tag{9}$$

It follows from the foregoing that the character of the Alfvén continuum depends on the sound parameter. Figure 1, where a particular Alfvén continuum branch ($\omega_{A,mn}$) in the presence of the Alfvén-sound gap is shown for $\mathfrak{S} \gg 1$ and $\mathfrak{S} \ll 1$, demonstrates this.

When $\omega_{G1,2}$ weakly depends on ω, and the modes are well-localized, using the transformation suggested in [15], one can present Eq. (1) in the form of a Schrödinger equation

$$\frac{d^2Z}{d\zeta^2} + \left[\mathcal{E}_1 - V(\zeta)\right] Z = 0, \tag{10}$$

where

$$\mathcal{E}_1 = g - \frac{1}{4}, \tag{11}$$

$Z = Z(\Phi_{mn})$, ζ is a coordinate variable, and $V(\zeta) > 0$, which implies that a necessary condition for the existence of solutions of Eq. (10) is

$$g > 1/4. \tag{12}$$

The quantity g is different in stellarators and tokamaks. In particular, when compressibility effects are weak and the plasma cross section is circular, we have

$$g^S = -\frac{1}{2}\left(1 + \frac{\iota'' r}{\iota'}\right)\Bigg|_{r*} g^T,$$

$$g^T = \frac{2}{r_*}\left(\frac{\rho''}{\rho'} - \frac{\rho'}{2\rho} - \frac{\iota''}{\iota'}\right)^{-1}\Bigg|_{r_*}, \tag{13}$$

where the superscripts T/S mean that the magnitude is relevant to tokamaks / currentless stellarators, r_* is the radius, around which a mode is localized, and the prime means the radial derivative.

We assume first that the ι-profile is monotonic and is given by $\iota = \iota_0\left(1 + \alpha r^2/a^2\right)$, where a is the plasma radius, and α is a parameter, $\alpha > -1$. Then we obtain $g^S = -g^T$. Therefore, in the considered approximation, the GAE modes in stellarators exist only for those plasma parameters, for which GAEs are absent in tokamaks and vice versa.

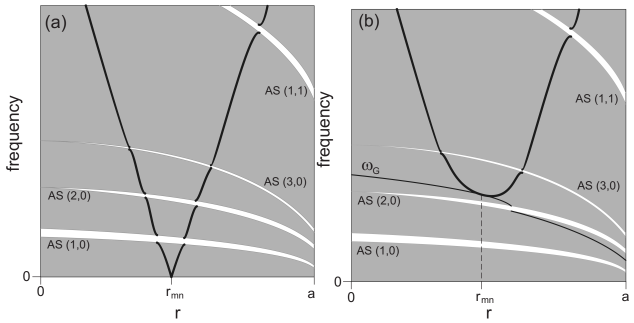


Fig. 1. Alfvén continuum branch: (a) $\mathfrak{S} \gg 1$ (W7-X case); (b) $\mathfrak{S} \ll 1$ (W7-AS case). Notations: gray, continuum region, white, Alfvén-sound gaps, r_{mn} is the ι_{mn} radius; AS(μ,ν) denote the Alfvén-sound gaps caused by the Fourier harmonics $\epsilon_B^{(\mu\nu)}$

Let us assume now that $\iota(r)$ is non-monotonic, and r_* is localized close to the point r_m, where $\iota(r)$ has an extremum. One can see that the latter condition can be satisfied when the plasma is homogeneous in the region of the mode localization, and/or the mode frequency lies well below the TAE-gap in the Alfvén continuum. Then we conclude from Eq. (13) that $g^S \to 1$, whereas $g^T \to 0$. This means that the condition given by Eq. (12) is well satisfied in stellarators, and, thus, there exist eigenmodes there; they can be either GAEs (when the Alfvén continuum branch, ω_A, has a minimum) or NGAEs (when ω_A has a maximum). In addition, the obtained result ($g^T = 0$) explains why taking into account additional factors (such as the presence of energetic ions, the toroidicity, and the plasma density gradient) are necessary to calculate the Alfvén cascade (AC) modes or reversed shear Alfvén eigenmodes (RSAE) [16] in tokamaks.

Note that the low-frequency GAE/NGAE modes with $m \gg 1$ are much more sensitive to the magnitude of the rotational transform than the gap modes; in particular, their frequencies are more sensitive than those of the TAEs by a factor of $2m$. One can see that, when $\omega = k_\parallel v_A$,

$$\Delta\omega = m\Delta\iota\frac{v_A}{R}, \tag{14}$$

where $\Delta\omega$ is the change of ω or the uncertainty of the GAE/NGAE mode frequency associated with $\Delta\iota$, which is a change of ι or the ι-error bar, and ι and v_A are taken at the point, where the mode amplitude is maximum. For instance, $\Delta\iota = 0.001$ leads to $\Delta\omega = 3$ kHz in a hydrogen plasma with a density of 5×10^{13} cm^{-3}, $B_0 = 2.5$ T, $R = 200$ cm, and $m = 5$.

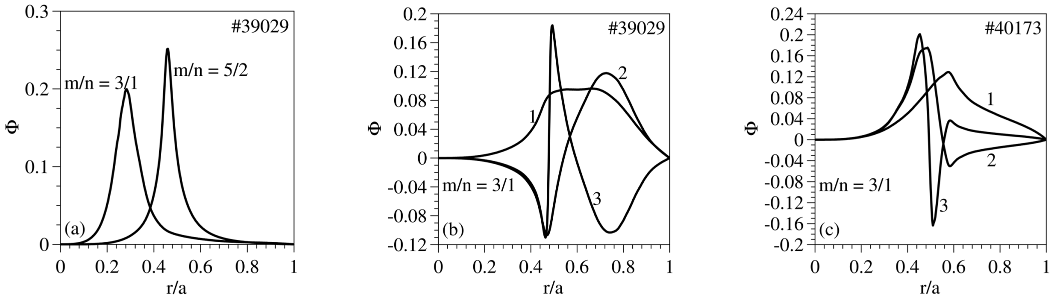
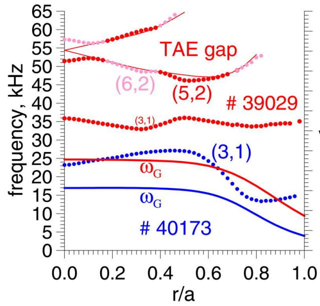


Fig. 2. Calculated AEs in the W7-AS shots #39029 and #40173: (*a*) GAE in AIs with the lowest frequency (33 kHz) and the highest frequency (46 kHz), respectively, in shot #39029; (*b*) NGAE in AIs with intermediate frequencies in shot #39029; curve *1*, AI with the frequency of 38 kHz; curve 2, 35.3 kHz; curve *3*, 35 kHz; (*c*) NGAEs relevant to AI in shot #40173



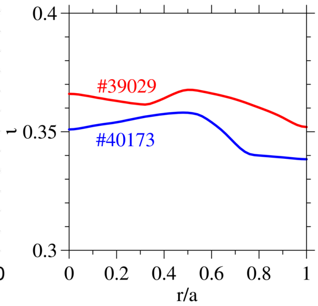


Fig. 3. Continuum branches with given (m, n) and the geodesic acoustic frequency, ω_G

Fig. 4. Reconstructed profiles of $\iota(r)$. These profiles of ι are in plausible ranges, given the VMEC calculation without currents and the expected effects due to the Ohmic current used to compensate bootstrap and NBI driven currents

3. Modeling of Low-Frequency Alfvénic Activity in Wendelstein 7-AS

We selected two W7-AS shots, where the steady-state or quasisteady-state low-frequency Alfvénic activity was observed:

Using the Alfvén continuum calculated by the code COBRAS and Alfvén eigenmodes calculated by the code BOA-fe, as well as equilibria calculated by the code VMEC, we reconstructed the ι-profiles in shots #39029 and #40173 (see Figs. 2–4). In addition, we found that the fraction of hydrogen in shot #39029 should be about 60% to fit the experimental data. However, these results neglect the plasma rotation and the concomitant Doppler frequency shift because of the lack of data on the rotation. Therefore, the reconstruction of $\iota(r)$ for shots #39029 and #40173 should be considered as a demonstration of the possibility to use the destabilization of GAE/NGAE modes for plasma diagnostics rather than finding the precise ι in these shots.

4. Summary and Conclusions

The obtained results can be summarized as follows.

(i) The conditions of existence of GAE/NGAE modes depend on the nature of the rotational transform; i.e., these conditions in currentless stellarators (where $\iota(r)$ is produced by external coils) and tokamaks (where the same $\iota(r)$ is produced by the plasma current) are different. NGAE in stellarators exists even in the approximation of cold plasma of ideal MHD.

(ii) Plasma compressibility strongly affects the GAE/NGAE modes when the "sound parameter" ($\mathfrak{S}$) is small. The modes with the geodesic acoustic frequency exist only when $\mathfrak{S} \ll 1$. From this point of view, the stellarators W7-AS and W7-X are qualitatively different devices ($\mathfrak{S} \ll 1$ in W7-AS, but $\mathfrak{S} \gg 1$ in W7-X).

(iii) The GAE/NGAE modes are very sensitive to the magnitude of the rotational transform (much more than gap modes). Therefore, the experimental data on low-frequency Alfvénic activity can be used for the reconstruction of the profile of $\iota(r)$. Our work demonstrates this for particular shots of W7-AS. Furthermore, a possibility to use the observation of AI

Shot N	B(T)	$P_{\rm inj}$ (MW)	$n_e(0)$ (cm^{-3})	$T_e(0)$ (eV)	ι_a	Species	ω (kHz) obs.	m obs.	Moment of time (s)
39029	2.53	0.44	1.1×10^{14}	540	0.355	D-plasma	46	5	0.45
						H-beam	33, 35, 38	3	
40173	2.53	0.357	6.6×10^{13}	435	0.34	D-plasma	16	3	0.35
						D-beam	28		

to determine the fraction of hydrogen in a plasma consisting of the mixture of deuterium and hydrogen is demonstrated.

(iv) Specific calculations modelling low-frequency Alfvén instabilities in three selected W7-AS discharges are in a reasonable agreement with observations.

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ПРО НИЗЬКОЧАСТОТНІ АЛЬФВЕНІВСЬКІ НЕСТІЙКОСТІ В СТЕЛАРАТОРАХ

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Р е з ю м е

Розвинено теорію низькочастотних альфвенівських власних мод [ґлобальної альфвенівської власної моди (GAE) та незвичайної ґлобальної альфвенівської власної моди (NGAE)] у стелараторах з урахуванням стисливості плазми. Показано, що умови існування GAE/NGAE мод в стелараторах є іншими, ніж у токамаках, а також що геодезична акустична мода (GAM) може не існувати в стелараторах. Проведено числове моделювання низькочастотних альфвенівських нестійкостей у стелараторі Wendelstein 7-AS.