A MATHEMATICAL MODEL OF PLANE GLOW DISCHARGE AND HOLLOW CATHODE EFFECT

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A non-local mathematical model for glow discharge processes, which are significant for main structures of glow discharge (cathode dark space, negative glow, and positive glow), is proposed. The model includes a possibility of the electron pendulum effect in the hollow cathode plane geometry. An integral equation for a source of ionization is derived, and its solution is calculated. A self-sustained discharge condition is formulated. A self-consistent system of differential equations and boundary conditions for the electric field and the electron and ion current densities is deduced and solved. The current voltage characteristic illustrating the hollow cathode properties is obtained. The origin of the difference between a hollow cathode discharge (HCD) and a simple glow discharge (SGD) is investigated.

1. Introduction

It is well known [1, 2] that the difference between a SGD in a plane capacitor and a HCD in a double capacitor with two parallel cathode plates and a grid anode in the middle is stipulated with the absence or presence of pendulum oscillations of ionizing electrons inside a discharge gap. The essence of the effect is similar to properties of a harmonic oscillator: if forces of friction are small in comparison with an elastic force, the oscillator reveals damped oscillations, otherwise it damps down without oscillations.

A hollow cathode (HC) device was invented by Pashen [3] more than 90 years ago, but a theory of hollow cathode discharge stays on the primary stage as far. The first Engel–Shtenbek's mathematical model of glow discharge [4] was a theory of the cathode dark space (CDS), in which the ionization source was determined with a local value of the first Townsend ionization factor. Such a model of the ionization source was valid for a quasiuniform electric field only, when the free path of an electron was small in comparison with the scale of a spatial variation of the electric field. So, primary local HC-theories [2, 5, 6] could not do more than to approximate experimental results with the use of empirical factors of the current amplification – the HC effect. Later on (1972–1985), a special part of fast ionizing electrons was realized, and the attempts to describe their kinetics were done [7–10], but a significance of CDS was ignored. When a non-local CDS was taken into account (1985, 1992), there were constructed the first current voltage characteristics of HC discharge [11, 12] in a one-dimensional geometry, but these models had strong assumptions about the small damping of electron oscillations, so it was impossible to compare SGD and HCD in the frame of a single approach.

We can mention the multidimensional models [13, 14] with a rather complicated geometry, but mathematical complications with angle distributions and Monte Carlo calculations do not contribute much into a theory. Here, we construct a simplest unified mathematical model of glow discharge, which enables to demonstrate the difference between SGD and HCD with calculations.

2. Model Description

A motion of an ionizing electron is considered as one-dimensional: only in the direction normal to electrodes. It obeys the dynamics of some anharmonic oscillator. The elastic force of the oscillator is the force of the electrostatic field. The force of friction must include all losses in the energy of motion in the direction considered. Most significant are the losses on impact ionization of neutral gas atoms and on their excitation. As a simplification, the averaged energy losses can be substituted instead of the pulsed losses at every collision. The elastic scattering of an electron by an atom leads to the conversion of its energy of motion in the mentioned direction into the energy of chaotic motion in the direction which is parallel to electrode plates, so it is a loss in the energy of a motion in the direction under consideration. Because of additional, but physically inessential, mathematical complications, elastic collisions are not included into the model. The electron and ion densities are considered to be small in comparison with the neutral atom density. The neutral density is guessed to be uniform, and a discharge –

to be stationary and uniform in the direction parallel to electrodes.

3. Integral Equation for a Source of Ionization

Under these assumptions, a stationary kinetic equation for ionizing electrons looks as

$$v\frac{\partial f_e}{\partial x}+\frac{\partial}{\partial v}\left(\dot{v}\left(x,v\right)\ f_e\right)=s\left(x\right)\delta\left(v\right), \qquad (1)$$

where $f_e=f_e\left(x,v\right)$ is the electron distribution function, $s\left(x\right)$ is the number of ionization acts per unit volume and unit time, the Dirac delta-function $\delta\left(v\right)$ assumes secondary electrons to be born with negligibly small energy, and the acceleration $\dot{v}\left(x,v\right)$ is described by the equation

$$\dot{v}=-\frac{e}{m_e}\left(E\left(x\right)+\mathrm{sgn}\left(v\right)L\left(\frac{m_e v^2}{2e}\right)\right),$$

$$|x|\le x_c,\quad -\infty<v<+\infty,$$

$$L\left(w\right)\equiv N_a\left(\varepsilon_{\rm ion}\sigma_{\rm ion}\left(w\right)+\varepsilon_{\rm ex}\sigma_{\rm ex}\left(w\right)\right). \qquad (2)$$

Here, the first and second terms are the contributions, respectively, of the elastic force of the electric field and the force of friction, x_c is the width of the cathode-anode gap, N_a is the gas density, $\varepsilon_{\rm ion},\sigma_{\rm ion}\left(w\right),\varepsilon_{\rm ex},\sigma_{\rm ex}\left(w\right)$ are the ionization and excitation energies and cross-sections.

The choice of the dynamical model for ionizing electrons (2) needs some substantiation. Consider the stationary electron motion in a uniform electric field E = const, which corresponds to dynamics (2). We obtain that the kinetic energy w of an electron is a root of the equation

$$N_a\left(\varepsilon_{\rm ion}\sigma_{\rm ion}\left(w\right)+\varepsilon_{\rm ex}\sigma_{\rm ex}\left(w\right)\right)=E.$$

On the other hand, for the ionization factor, we have

$$\alpha=N_a\sigma_{\rm ion}\left(w\right).$$

Joining both equations into a system, we obtain a parametric definition of the function $\alpha=\alpha\left(E\right)$ in the case of the motion of an ionizing electron along a uniform electric field with constant velocity. We can compare such a dependence with the experimental one and with the widely used empiric Townsend formula $\alpha=AP\exp\left(-\frac{BP}{E}\right)$. The direct numerical comparison of curves for argon is shown on Fig. 1. It is seen that the dynamical model chosen gives a quite good coincidence

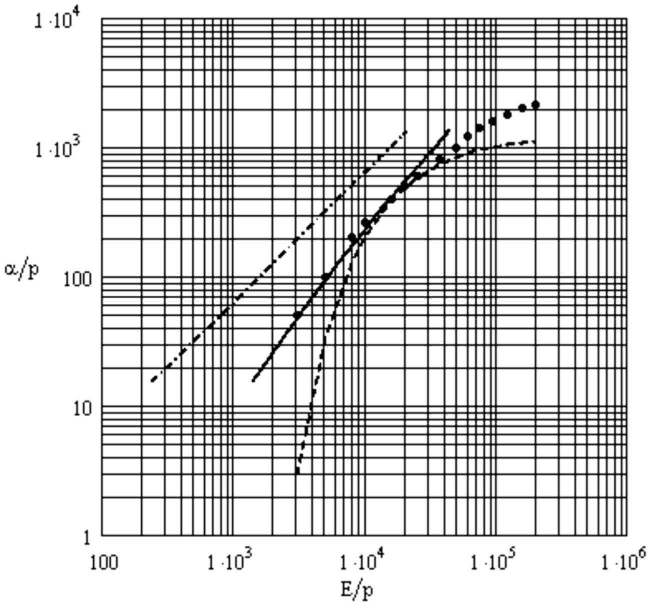

Fig. 1. Comparison of the ionization model and experimental data. Great points – experimental dependence of the first Townsend ionization factor for argon. Dotted line – empiric Townsend exponent formula. Solid line – model chosen in the present work. The dashed line corresponds to the same model but at the neglect of losses in the electron energy on the atom excitation

with experimental values for the first Townsend ionization coefficient for argon. This is, in our opinion, an argument in favor of the dynamical model chosen here.

A solution of Eq. (1) can be obtained analytically (see [15]) as

$$f_e\left(x,v\right)=f_e\left(-x_c\,\mathrm{sgn}\left(V_0\left(x,v\right)\right),V_0\left(x,v\right)\right)\times$$

$$\times\exp\left(\int\limits_0^{T(x,v)}dt'\left(|v|\frac{dL}{dw}\right)\left(V\left(t',V_0\left(x,v\right)\right)\right)\right)+$$

$$+\int\limits_0^{T(x,v)}dt''s\left(X\left(t'',V_0\left(x,v\right)\right)\right)\delta\left(V\left(t'',V_0\left(x,v\right)\right)\right)\times$$

$$\times\exp\left(\int\limits_{t''}^{T(x,v)}dt'\left(|v|\frac{dL}{dw}\right)\left(V\left(t',V_0\left(x,v\right)\right)\right)\right). \qquad (3)$$

Here, $v_0 = V_0(x,v)$ is the initial velocity of an electron, which passes through a phase point x,v, on the cathode; $t = T(x,v)$ is the electron motion duration from the cathode to a phase point x,v; $x = X(t,v_0)$ and $v = V(t,v_0)$ are the time-dependent coordinate and velocity of the electron which define its trajectory; and $\left(|v|\frac{dL}{dw}\right)(v) \equiv |v|\frac{dL}{dw}(w)$, $w = \frac{m_e v^2}{2e}$. The delta-function in the integral can be eliminated by the variable transformation, and the solution gets a form

$$f_e(x,v) = f_e(-x_c \mathrm{sgn}(V_0(x,v)), V_0(x,v)) \times$$

$$\times \exp\left(\int_0^{T(x,v)} dt' \left(|v|\frac{dL}{dw}\right)(V(t',V_0(x,v)))\right) +$$

$$+\frac{m_e}{e}\sum_{n:\,0<t_n(V_0(x,v))<T(x,v)} \frac{s(x_n(V_0(x,v)))}{|E_n(V_0(x,v))|}\times$$

$$\times \exp\left(\int_{t_n(V_0(x,v))}^{T(x,v)} dt' \left(|v|\frac{dL}{dw}\right)(V(t',V_0(x,v)))\right). \quad (4)$$

Here, $t_n = t_n(v_0)$, $n = 1,2,...$ $t_1 < t_2 < ...$ are roots of the equation $V(t,v_0) = 0$ (namely, they are the instant times, at which an electron of the cathode with the initial velocity v_0, "returns back" and then moves in the opposite direction), $x_n(v_0) \equiv |X(t_n(v_0),v_0)|$ are the (right-hand) coordinates ($0 < x_n \le x_c$) of turning points, and $E_n(v_0) \equiv E(x_n(v_0))$ are the electric field strength at these points.

In another way, from its physical origin, the electron source density (it is also ionization source density) is defined by the expression

$$s(x) = \int_{-\infty}^{+\infty} dv\, N_a |v| \sigma_{\mathrm{ion}}\left(\frac{m_e v^2}{2e}\right) f_e(x,v). \quad (5)$$

From the pair of equations (4) and (5), one can exclude a source and obtain an integral equation for the distribution function. But we do quite the contrary: we exclude the proper distribution function $f_e(x,v)$. Then, by transforming the variables in integrals for simplification, we obtain the integral equation for a source of ionization $s(x)$:

$$s(x) = \int_{-\infty}^{+\infty} dv_0 |v_0| f_e(-x_c \mathrm{sgn}(v_0), v_0) \times$$

$$\times \sum_k N_a \sigma_{\mathrm{ion}}(w_k(x,v_0)) +$$

$$+\int_x^{x_c} dx' s(x') \Big(\sum_k N_a \sigma_{\mathrm{ion}}(w_k(x, V_0(-x',0)))\times$$

$$\times \theta(t_k(x,V_0(-x',0)) - T(-x',0)) +$$

$$+\sum_k N_a \sigma_{\mathrm{ion}}(w_k(x,V_0(x',0)))\times$$

$$\times\theta(t_k(x,V_0(x',0)) - T(x',0))\Big), \quad 0 < x < x_c, \quad (6)$$

where $\theta(z)$: $\theta(z) = 0$, $z \le 0$; $\theta(z) = 1$, $z > 0$ is the Heaviside function, t_k, $k = 1,2,...$ are the instsnt times when the electron is at a positive coordinate x: $X(t_k,v_0) = x$; $v_k(x,v_0) \equiv V(t_k,v_0)$ (see Fig. 2).

If the anode is made of wire meshes and, therefore, is transparent for ionizing electrons (HCD geometry), then the sums over k in (6) might have several non-zero summands. It is a consequence of electron oscillations, which is named the pendulum effect.

But if the anode is made of a solid metal plate (SGD geometry), it is not transparent for any electron. Then hollow cathode electron oscillations are absent. So we can consider only the right-hand side of the device, where every trajectory of a cathode electron has its origin at $x = +x_c$, $v < 0$, and every trajectory of a secondary electron has its origin in the nearest crossing of a whole trajectory solution with the abscissa axis. Then the integration over the variable v_0 in the first item of (6) should be restricted to negative values only; k in the sum in (6) with argument $-x'$ has no value, and, in the sum with argument x', it has a single value $k = 1$. So expression (6) in SGD gets a form

$$s(x) = \int_{-\infty}^{0} dv_0 |v_0| f_e(x_c,v_0) N_a \sigma_{\mathrm{ion}}(w(x,v_0)) +$$

$$+\int_x^{x_c} dx' s(x') N_a \sigma_{\mathrm{ion}}(w(x,V_0(x',0))), \; 0 < x < x_c. \quad (7)$$

Let the cathode electrons be monochromatic (have a single value of initial velocity $v = -v_{ec}$ at $x = +x_c$ and $v = v_{ec}$ at $x = -x_c$):

$$f_e(\pm x_c, v) = n_{ec}\delta(v \pm v_{ec}) = \frac{J_{ec}}{e v_{ec}}\delta(v \pm v_{ec}),$$

$J_{ec} \equiv J_e(x_c)$.

Then (6) yields

$$S(x) = \sum_k N_a \sigma_{\text{ion}}(w_k(x, -v_{ec})) +$$

$$+ \sum_k N_a \sigma_{\text{ion}}(w_k(x, v_{ec})) +$$

$$+ \int_x^{x_c} dx' S(x') \Big(\sum_k N_a \sigma_{\text{ion}}(w_k(x, V_0(-x', 0))) \times$$

$$\times \theta(t_k(x, V_0(-x', 0)) - T(-x', 0)) +$$

$$+ \sum_k N_a \sigma_{\text{ion}}(w_k(x, V_0(x', 0))) \times$$

$$\times \theta(t_k(x, V_0(x', 0)) - T(x', 0)) \Big), \quad 0 < x < x_c, \quad HCD \tag{8}$$

where $S(x) \equiv s(x) e / J_{ec}$ is the number of ionization acts per unit path, which corresponds to one cathode electron.

In the case of a simple glow discharge without the hollow cathode effect, (8) yields

$$S(x) = N_a \sigma_{\text{ion}}(w(x, -v_{ec})) +$$

$$+ \int_x^{x_c} dx' S(x') N_a \sigma_{\text{ion}}(w(x, V_0(x', 0))),$$

$$0 < x < x_c. \quad SGD. \tag{9}$$

If we guess the local dependence $\sigma_{\text{ion}}(x) = \sigma_{\text{ion}}(w(x))$ by neglecting a dependence on the second argument, $V_0(x', 0)$, in the integrand in (9), the equation can be simplified further:

$$S(x) = \alpha(x) \Big(1 + \int_x^{x_c} dx' S(x') \Big), \quad \alpha(x) \equiv N_a \sigma_{\text{ion}}(w(x)).$$

As the electron current density is equal to

$$J_e(x) = J_{ec} \Big(1 + \int_x^{x_c} dx' S(x') \Big), \quad \frac{dJ_e}{dx} = -J_{ec} S(x),$$

Eq. (9) becomes equivalent to the well-known continuity equation $dJ_e/dx = -\alpha(x) J_e$ from the local Engel–Shtenbek's cathode dark space theory [1, 4]. Thus, Eqs. (8) and (9) give a non-local extension of the classical ionization theory.

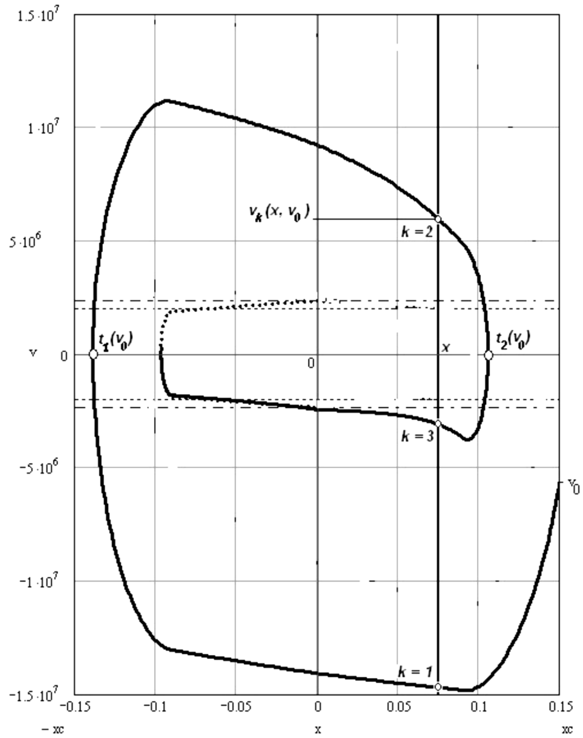

Fig. 2. In pendulum oscillations in the HCD geometry, an electron can reach the same anode distance several times

4. Equations for the Self-consistent Problem

Beyond the cathode dark space, the source of ionization $S(x)$ must be compensated with recombination processes, because the quasineutral condition for plasma together with the drift relations for ions and slow electrons, as well as the electric current continuity, define the ion and electron current densities identically. So the effective source, which is the right-hand side of the continuity ion current equation, is described by

$$S' = S - r\, n_e n_i, \tag{10}$$

where $r = eR/J_{ec}$, and R is the recombination constant.

Let $j_e = J_e/J_{ec}$, $j_i = J_i/J_{ec}$ be the dimensionless electron and ion current densities. The ion current density obeys the following continuity equation with boundary conditions at the anode and cathode plates:

$$\frac{d}{dx} j_i = S'(x), \quad j_i(0) = 0, \quad j_i(x_c) = \frac{1}{\gamma}. \tag{11}$$

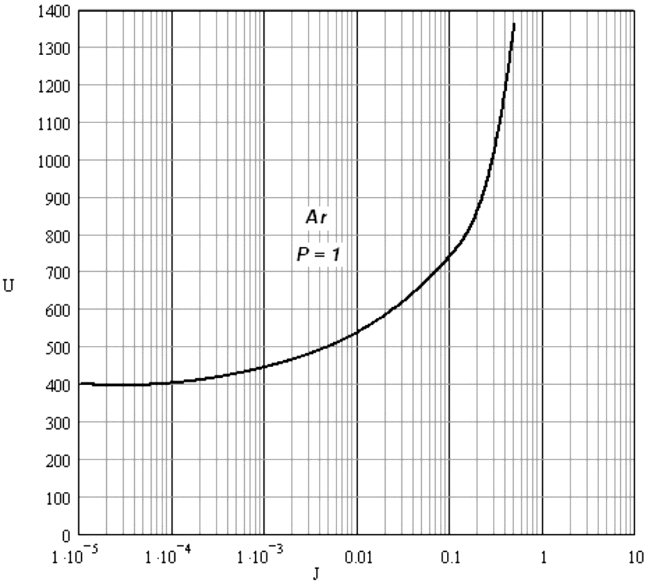

Fig. 3. Current voltage characteristics for HCD and SGD coincide in argon at the pressure $P = 1$ Torr. The anode-cathode distance is equal to 15 cm. On the abscissa axis – current density, A/m^2; on the ordinate axis – voltage, V

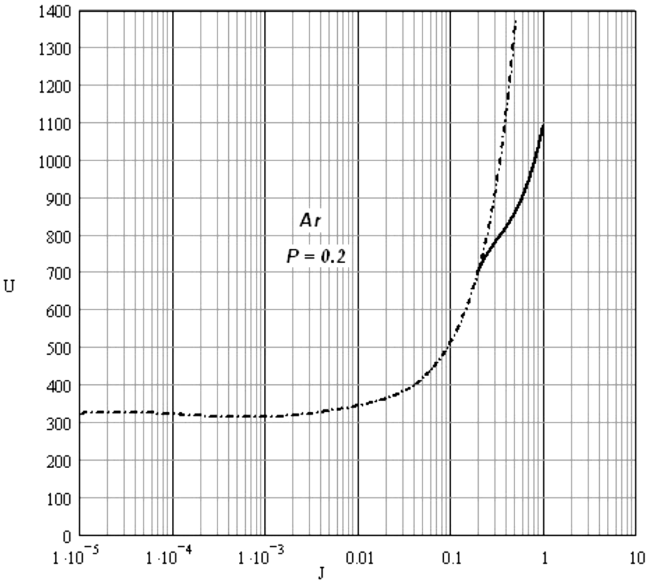

Fig. 4. Current voltage characteristics for HCD (solid) and SGD (dotted) in argon at the pressure $P = 0.2$ Torr have a small region of divergence at the current density $J > 0.2$ A/m^2

Here, the boundary conditions correspond to a self-sustained discharge condition

$$j_i(x_c) - j_i(0) = \int_0^{x_c} S'(x)\,dx = \frac{1}{\gamma}.$$

The electron current density at $0 \le x \le x_c$ is a solution of the Cauchy problem with initial condition in the cathode:

$$\frac{d}{dx} j_e = -S'(x), \quad j_e(x_c) = 1. \tag{12}$$

By summing (11) and (12) and integrating over x, we obtain

$$j_e(x) = 1 + \frac{1}{\gamma} - j_i(x). \tag{13}$$

According to the drift approach for ions, the ion density is equal to

$$n_i(x) = \frac{J_{ec} j_i(x)}{e b_i E(x)}. \tag{14}$$

(Here, b_i is the mobility ion factor in the drift approach.)

The drift approach for electrons

$$n_e(x) = \frac{J_{ec} j_e(x)}{e b_e E(x)} \tag{15}$$

with the mobility electron factor b_e is not reliable much, as it should be related only to the slow electron component. But the density of ionizing electrons is sufficiently small in comparison with the ion density to make significant errors in the Poisson equation

$$\frac{dE}{dx} = \frac{e}{\varepsilon_0}\left(n_i(x) - n_e(x)\right). \tag{16}$$

Substituting the expressions for n_e, n_i from (14) and (15), as well as j_e from (13) and S' from (10), into (10) and (16) and using the relation $J_{ec} = \frac{\gamma}{1+\gamma} J$, where J is the total electric current density, we obtain the second-order nonlinear ODE system

$$\begin{cases} \frac{dE^2}{dx} = \frac{2J}{\varepsilon_0 b_i} \frac{\gamma}{1+\gamma} \left(1 + \frac{b_i}{b_e}\right)(j_i - j_{i0}), \\ \frac{dj_i}{dx} = S(x) - R \frac{J}{e b_e b_i} \frac{\gamma}{1+\gamma} \frac{1}{E^2} j_i \left(1 + \frac{1}{\gamma} - j_i\right). \end{cases} \tag{17}$$

$$j_{i0} \equiv \left(1 + \frac{1}{\gamma}\right) \frac{b_i}{b_e + b_i}.$$

The boundary conditions are

$$j_i(0) = 0, \quad j_i(x_c) = \frac{1}{\gamma}. \tag{18}$$

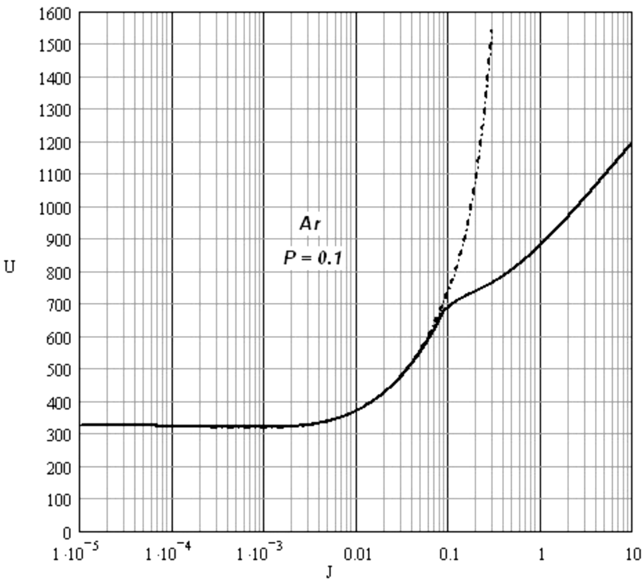

Fig. 5. Current voltage characteristics for HCD (solid) and SGD (dotted) in argon at the pressure $P = 0.1$ Torr. The divergence of curves is greater than the previous one

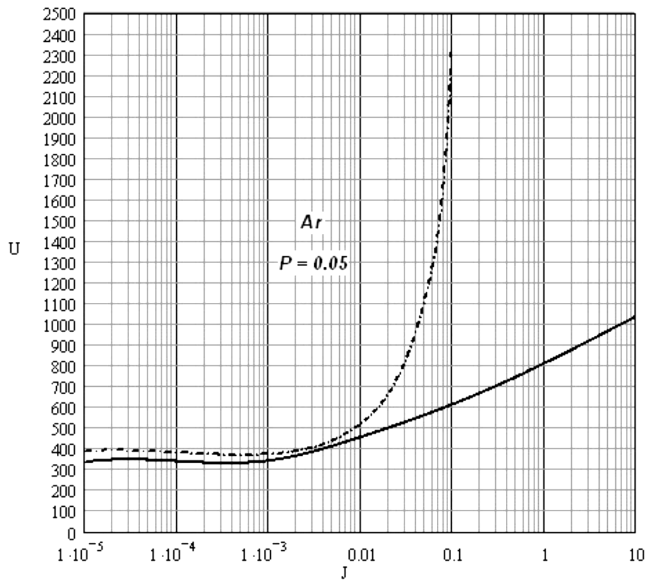

Fig. 6. Current voltage characteristics for HCD (solid) and SGD (dotted) in argon at the pressure $P = 0.05$ Torr. The divergence of curves is unquestionable

5. Calculations

Solving the self-consistent problem (8), (2), (17), (18) – for HCD or (9), (2), (17), (18) – for SGD in an argon extender gives the current voltage characteristics which are shown in Figs. 3–6.

At high pressures, the hollow cathode effect is absent (Fig. 3). It is the same reason, as pendulum oscillations are absent at great pendulum decrements. The less the pressure, the more the oscillations are possible, and the greater is the divergence of current voltage characteristics in SGD and HCD (the next figures).

At high current densities, the difference in SGD and HCD current voltage characteristics is stipulated by the presence of the pendulum effect of ionizing electrons in HCD, which run through anode meshes of HC and increase the source of ionization in comparison with a single passage through the discharge gap in SGD. But, as the integral of an ionization source is fixed by the self-sustained discharge condition, a compensation for this additional ionization is given through a decrease of the voltage in the cathode dark space of HCD. The latter decreases the energy of ionizing electrons, so the ionization cross-section falls, and this restores a "true" value of the ionization source.

Both curves in Figs. 3–6 are obtained in the common theoretical and calculation model with the use of the same algorithm. The simulation of a SGD comparison device was done with "switching off" the anode transparency and excluding a possibility of electron oscillations in SGD. As was explained above, the integral equation (8) for the ionization source in HCD gets a simplified form (9) for the SGD geometry.

6. Conclusions

The hollow cathode effect can be described by a relatively simple mathematical model, and it can be calculated not as an effect of *another physical processes*, but as an effect of *change in geometry* of the motion of ionizing electrons. The model includes the Poisson equation, the drift relation for the currents and the densities of ions and slow electrons, the self-sustained discharge condition, and the electron-ion recombination in the area of a small electric field. Also it is necessary to include a source of ionization which depends in a non-local manner on the electric field. The source must account the energy distribution of ionizing electrons and their dynamics. The dynamics must include a possibility of the pendulum effect and the ionization not only by cathode electrons, but secondary electrons as well.

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МАТЕМАТИЧНА МОДЕЛЬ ПЛАСКОГО ЖЕВРІЮЧОГО РОЗРЯДУ І ЕФЕКТУ ПОРОЖНИСТОГО КАТОДА

В.В. Горін

Р е з ю м е

Запропоновано нелокальну математичну модель процесів у жевріючому розряді, які є суттєвими для його основних структур: катодного шару, негативного світіння та позитивного стовпа. Модель припускає можливість електронного маятникового ефекту у порожнистому катоді в пласкій геометрії. Виведено інтегральне рівняння для джерела іонізації і розраховано його розв'язок. Сформульовано умову самопідтримки розряду. Складено та вирішено у розрахунках задачу про самоузгоджену систему диференціальних рівнянь та граничних умов для електричного поля, густин електронного та іонного струмів. Отримано вольт-амперну характеристику, яка ілюструє властивості порожнистого катода. Досліджено походження різниці між розрядом у порожнистому катоді та простим жевріючим розрядом.