A JOINT SPACE-FREQUENCY DISTRIBUTION OF OPTICAL SIGNALS

YU.M. KOZLOVSKII

Institute of Physics of Condensed Systems, Nat. Acad. Sci. of Ukraine
(1, Sventsytskyi Str., Lviv 79011, Ukraine; e-mail: nesh@ph.icmp.lviv.ua)

UDC 532:533: 536

A joint space-frequency distribution, the special cases of which are the Ville and Wigner distributions, has been proposed. The frequency representation of the joint distribution has been obtained for the first time, which enabled us to remove features that arose in the coordinate representation. An expression for the joint distribution of a rectangular pulse has been calculated. A relation between the Ville and Wigner distributions has been found. In particular, it was demonstrated that the Wigner distribution is formed by rotating the Ville one on the information diagram of the conjugated coordinates (x, p) by an angle that is proportional to the joint parameter t. The results of numerical calculations of the joint space-frequency distribution of a rectangular pulse at various values of the joint parameter have been presented.

1. Introduction

Since the middle of the last century, space-frequency distributions have occupied a special place in the description of optical systems for information processing. The distributions of such a type have been successfully applied for the first time while considering quantum-mechanical problems by E. Wigner in 1932 [1] and by J. Ville in 1948 [2].[1] In those seminal works, the fundamentals of the distribution theory have been formulated, and the main properties of distributions have been studied. Later on, various modifications of the Wigner and Ville distributions were analyzed; nevertheless, the first attempt to systematize space-frequency distributions was made by L. Cohen [3]. In that work, the goals, problems, and the ways of their solution have been accurately formulated for the first time.

The main purpose of constructing the space-frequency distribution consists in finding a certain general function depending on the coordinate and the frequency $W(x, \omega)$. This function must simultaneously describe the intensity of the signal $f(x)$ at a definite coordinate and a definite frequency. The generalized formulation of the space-frequency distribution, which satisfies the aforementioned conditions, has been proposed by L. Cohen [3]. This distribution is described by the formula

$$\mathcal{C}(x,\omega;\Phi) = \frac{1}{2\pi} \int\limits_{-\infty}^{-\infty} \int\limits_{-\infty}^{-\infty} \int\limits_{-\infty}^{-\infty} f\left(y + \frac{x_0}{2}\right) f^*\left(y - \frac{x_0}{2}\right) \times$$

$$\times \Phi(x,\omega) \exp -i(\omega x_0 - \omega_0 x + \omega_0 y) dy dx_0 d\omega_0, \qquad (1)$$

where $f(x)$ is the signal, and $\Phi(x, \omega)$ is the so-called kernel function. Various members belonging to the Cohen class can be obtained from formula (1) by substituting different kernel functions $\Phi(x, \omega)$. For instance, if $\Phi(x, \omega) = 1$, formula (1) transforms into the Wigner distribution function; whereas, at $\Phi(x, \omega) = \delta(x - x_0)\delta(\omega - \omega_0)$, it becomes the Ville one. Since our research deals just with the Ville and Wigner distributions, we consider these two distributions in more details, because they are widely applicable in the space-frequency analysis, in particular, in the optical systems of information processing.

The Ville distribution can be written down in the following form [2]:

$$\mathcal{A}_{f_1 f_2^*}(x_0;\omega_0) = \int f_1\left(x + \frac{x_0}{2}\right) f_2^*\left(x - \frac{x_0}{2}\right) \times$$

[1] In a number of works, along with the term “the Ville distribution” , the term “the uncertainty function” was used. But, taking into account the fact that the given distribution was suggested by Ville, the former term is used throughout the work.

$$\times \exp(-i\omega_0 x)dx. \tag{2}$$

The Wigner distribution is one of the most popular coordinate-frequency distributions and is used for the space-frequency analysis [4,5]. By definition, the Wigner distribution looks like

$$\mathcal{W}_{f_1 f_2^*}(x;\omega) = \int f_1\left(x + \frac{x_0}{2}\right) f_2^*\left(x - \frac{x_0}{2}\right) \times$$

$$\times \exp(-i\omega x_0)dx_0. \tag{3}$$

Although the use of the Ville distribution somewhat simplifies calculations, while restoring the distribution of the input signal intensity, the majority of authors uses, nevertheless, the Wigner distribution in their works. A logical step in this direction was made in work [6], where the generalized Wigner function was introduced. This function is constructed in such a manner that the transition from the Wigner formalism to the Ville one can be carried out by varying a definite parameter θ. Taking into account the ample opportunities which are offered by the implication of the Ville distribution in the theory of optical information processing (see work [7]), we propose a method of construction of a generalized coordinate-frequency distribution using the Ville function as a basic one. Both approaches mentioned above are equivalent; however, the use of the latter allows one to find a whole class of new distributions and to restore the intensity of the input signal which is experimentally registered at the optical system output [8].

2. A Joint Space-frequency Distribution of Two Signals

2.1. Theoretical background

A challenging problem in the distribution construction is the design of a distribution of the extended type with a certain extra parameter $0 \leq t \leq 1$; a new distribution must coincide with the Ville and Wigner distributions at its limiting t-values, being a more general construction in the intermediate range $0 < t < 1$. The first attempt to build up a distribution of the extended type with the parameter θ was made in work [6], by generalizing the well-known Wigner distribution function. The results obtained in that work confirmed the fruitfulness of such an approach. Bearing in mind that the work mentioned was executed in the framework of quantum mechanics, a question arises of whether such an approach is applicable to classical systems.

Taking into account the fact that the Ville and Wigner distributions were also first introduced to describe quantum-mechanical phenomena but turned out fruitful for describing the classical systems as well, we can hope for that a similar state of affairs would take place in the situation concerned. In work [7], a distribution of the extended type with the parameter t has been introduced by generalizing the Ville distribution function. The application of the Ville function has a basic advantage in comparison with that of the Wigner distribution function, which consists in a simplified scheme for the restoration of the input-signal intensity distribution. Let the generalized distribution of two signals, $f_1(x)$ and $f_2(x)$, be represented by the following formula:

$$\mathcal{K}^{(t)}_{f_1 f_2^*}(x;p) = C_t \int\int dx_0 d\omega_0 \mathcal{A}_{f_1 f_2^*}(x_0;\omega_0) \times$$

$$\times \exp\left\{-i\frac{(x-x_0)^2 + (p-\omega_0)^2}{\tan(\theta/\ 2)}\right\} \times$$

$$\times \exp\left\{i\left[x_0 p - \omega_0 x\right]\right\}. \tag{4}$$

The constant C_t and the joint distribution parameter t are defined by the relations [7]

$$C_t = \frac{2}{\pi}\frac{1}{1-\exp i\theta}, \qquad t = \frac{\theta}{\pi}. \tag{5}$$

In what follows, distribution (4) will be referred to as a joint space-frequency distribution or, briefly, a joint distribution in the coordinate representation.

One can easily see that expression (4) is a generalization of the ordinary Ville distribution function with the parameter t. The limiting cases of distribution (4) are: at $t = 0$, the Ville distribution (2), and, at $t = 1$, the Wigner one (3). Hence, as an addition to two known distributions (2) and 3), we suggest a whole series of distributions, every of which being associated with a definite value of the parameter t.

The coordinate representation of the joint distribution (4) has features in the vicinity of the point $t = 0$. In order to exclude the corresponding uncertainties, we suggest to use, along with the coordinate representation, the frequency one for the joint distribution which is described by the formula:

$$\widetilde{\mathcal{K}}^{(t)}_{f_1 f_2^*}(x;p) = \widetilde{C}_t \int\int dx_0 d\omega_0 \mathcal{W}_{f_1 f_2^*}(x_0;\omega_0) \times$$

$$\times \exp\left\{ i\frac{1}{4}\tan\frac{\theta}{2}\left[(x+x_0)^2+(p+\omega_0)^2\right]\right\}\times$$

$$\times \exp\left\{ i\left[-x_0 p+\omega_0 x\right]\right\}, \tag{6}$$

where the constant $\widetilde{C}_t$ is determined by the relation

$$\widetilde{C}_t = \frac{(1-i)^2}{1-\exp i\theta}\tan\frac{\theta}{2}. \tag{7}$$

It is evident that, in the case of the frequency representation for the joint distribution, the peculiarities at the point $t = 0$ and in its vicinity disappear. However, there emerge the peculiarities in the vicinity of the point $t = 1$. Therefore, the pair of representations – the coordinate and the frequency ones – mutually complement each other and completely describe the joint space-frequency distribution on the interval $t = [0, 1]$.

2.2. Limiting cases

This work aims at revealing the mechanism of redistribution between the Wigner and Ville distributions and at studying the properties of joint distributions which describe the range of the distribution parameter $0 < t < 1$. Therefore, essentially important are the researches of the limiting cases of the joint distributions (4) and (6).

2.2.1. Case $t = 1$

In order to describe this case, we take advantage of the coordinate representation for the generalized uncertainty function (4). By putting $t = 1$ (or $\theta = \pi$) in Eq. (4), we obtain the following result:

$$\mathcal{K}^{t=1}_{f_1 f_2^*}(x;p) = \frac{2}{\pi}\int\int dx_0 d\omega_0 \mathcal{A}_{f_1 f_2^*}(x_0;\omega_0)\times$$

$$\times \exp(ix_0 p - i\omega_0 x). \tag{8}$$

Hence,

$$\mathcal{K}^{t=1}_{f_1 f_2^*}(x;p) = \frac{2}{\pi}\mathcal{W}_{f_1 f_2^*}(x;p). \tag{9}$$

Expression (9) corresponds to the Wigner distribution function.

2.2.2. Case $t = 0$

To describe this case, we take advantage of the frequency representation of the generalized uncertainty function (6). In this case, $t = 0$ (or $\theta = 0$), and we obtain

$$\mathcal{K}^{t=0}_{f_1 f_2^*}(x;p) = \int\int dx_0 d\omega_0 \mathcal{W}_{f_1 f_2^*}(x_0;\omega_0)\times$$

$$\times \exp(i\omega_0 x - ix_0 p). \tag{10}$$

Hence,

$$\mathcal{K}^{t=0}_{f_1 f_2^*}(x;p) = \mathcal{A}_{f_1 f_2^*}(x;p). \tag{11}$$

Expression (11) corresponds to the Ville distribution function. Therefore, in the limiting cases, the joint distribution (4) is precisely equal to either the Wigner or the Ville distribution function. Thus, the introduced distribution describes the known ones at the limiting values of the parameter t. The study of the set of intermediate distributions comprises the subject of our further research.

3. Distribution of a Rectangular Pulse

The use of a rectangular pulse function in optics is important, because this function makes it possible to describe an isolated slit [9]. On the other hand, the rectangular pulse function plays the role of an element in optical images. Hence, the image can be considered as an arranged set of rectangular slits characterized by different contrasts. Therefore, a detailed research of the properties of the rectangular pulse function forms the basis for constructing various optical images. A rectangular pulse is represented by the well-known formula

$$\mathrm{rect}\left(\frac{x}{2a}\right) = \begin{cases} 1 & |x| < a, \\ 1/2 & |x| = a, \\ 0 & |x| > a, \end{cases}$$

where $2a$ is the pulse width. The Fourier transform of this function looks like

$$F(\omega) = \hat{\mathcal{F}}\left[\mathrm{rect}\left(\frac{x}{2a}\right)\right] = \int\limits_{-a}^{a} e^{-i\omega x}dx = 2a\frac{\sin(\omega a)}{\omega a}, \tag{12}$$

where $\mathrm{sinc}\frac{x}{\pi} = \frac{\sin x}{x}$ is the sampling function.

3.1. Distribution of a rectangular pulse in the Ville region

The Ville distribution of a rectangular pulse can be obtained either directly from definition (2) or calculated analytically [9]:

$$\mathcal{A}_{f_1 f_2^*}(x_0;\omega_0) = 2\frac{\sin[\omega_0(a+\frac{x_0}{2})]}{\omega_0}\mathrm{rect}\left(\frac{a+x_0}{2a}\right)+$$

$$+2\frac{\sin[\omega_0(a-\frac{x_0}{2})]}{\omega_0}\mathrm{rect}\left(\frac{a-x_0}{2a}\right). \qquad (13)$$

According to its definition, this distribution is confined in space by the width $2a$ of the rectangular pulse function, but has no frequency limit. In the case $\omega_0 = 0$, the distribution becomes degenerate into a triangular pulse, and, at $x_0 = 0$, into the Fourier spectrum of the rectangular pulse function, i.e. the sampling function.

3.2. Distribution of a rectangular pulse in the Wigner region

The Wigner distribution for a rectangular pulse can be calculated directly from definition (3):

$$\mathcal{W}_{f_1 f_2^*}(x;\omega) = 2\frac{\sin[2\omega(a+x)]}{\omega}\mathrm{rect}\left(\frac{a+2x}{2a}\right)+$$

$$+2\frac{\sin[2\omega(a-x)]}{\omega}\mathrm{rect}\left(\frac{a-2x}{2a}\right). \qquad (14)$$

The Wigner distribution for a rectangular pulse is confined in space by the width a and, similarly to the Ville distribution, has no finite frequency limit. In the case $\omega = 0$, the distribution degenerates into a triangular pulse and, at $x = 0$, into the Fourier spectrum of the rectangular pulse function.

4. Analytical Calculations of the Joint Distribution for a Rectangular Pulse

4.1. Joint distribution for a rectangular pulse

It is worth noting that, although the external profiles of the Ville and Wigner distributions for a rectangular pulse are similar, the signals restored with their help are substantially different. Therefore, the main goal of this work is a detailed study of the properties of the joint distribution of a rectangular pulse. On the basis of theoretical formulas, we try to obtain a joint distribution of the rectangular pulse function, which has a wide application to the description of the optical systems of information processing. By substituting the expression for the Ville distribution of a rectangular pulse (13) into the definition of joint distribution (4), we obtain the following expression for the distribution:

$$\mathcal{K}^{(t)}_{rr^*}(x;p) = 2C_t\int\limits_{-\infty}^{\infty}d\omega_0\int\limits_{-2a}^{0}dx_0\frac{\sin\left[\omega_0\left(a+\frac{x_0}{2}\right)\right]}{\omega_0}\times$$

$$\times\exp\left\{i\left[x_0 p-\omega_0 x\right]\right\}\times$$

$$\times\exp\left\{-i\frac{(x-x_0)^2+(p-\omega_0)^2}{\tan(\frac{\theta}{2})}\right\}+$$

$$+2C_t\int\limits_{-\infty}^{\infty}d\omega_0\int\limits_{0}^{2a}dx_0\frac{\sin\left[\omega_0\left(a-\frac{x_0}{2}\right)\right]}{\omega_0}\times$$

$$\times\exp\left\{i\left[x_0 p-\omega_0 x\right]\right\}\times$$

$$\times\exp\left\{-i\frac{(x-x_0)^2+(p-\omega_0)^2}{\mathrm{tg}(\frac{\theta}{2})}\right\}. \qquad (15)$$

The preliminary calculations can be carried out already by this formula, but the numerical calculation of the integral with infinite limits is complicated, and the results are not sufficiently accurate. Therefore, it is expedient either to transform expression (15) in order to remove the infinite limits of integration and make it as simple as possible for numerical calculations or to obtain the formula for the joint distribution of a rectangular pulse analytically. The calculation of the joint distribution of a rectangular pulse (15) can be carried out following two alternative ways. Let us consider each of them in more details.

4.2. Method of reduced formulas

The task is to reduce formula (15) down to a single term and to simplify it as much as possible. Taking into account the fact that the rectangular pulse function is even and carrying out a number of mathematical transformations, we obtain the coordinate representation for the joint distribution for a rectangular pulse:

$$\mathcal{K}^{(t)}_{rr^*}(x;p) = 2C_t\int\limits_{-\infty}^{\infty}d\omega_0\int\limits_{0}^{2a}dx_0\frac{\sin\left[\omega_0\left(a-\frac{x_0}{2}\right)\right]}{\omega_0}\times$$

$$\times \exp\left\{-i\frac{x^2+x_0^2}{\tan(\frac{\theta}{2})}\right\}\exp\left\{-i\frac{(p-\omega_0)^2}{\tan(\frac{\theta}{2})}\right\}\times$$

$$\times \exp\left\{-i\left[\omega_0 x\right]\right\}\cos\left\{px_0+2\frac{xx_0}{\tan(\frac{\theta}{2})}\right\}. \quad (16)$$

An analogous formula can be deduced for the frequency representation:

$$\widetilde{\mathcal{K}}_{rr^*}^{(t)}(x;p)=2\widetilde{C}_t\int_{-\infty}^{\infty}d\omega_0\int_0^{2a}dx_0\frac{\sin\left[2\omega_0\left(a-x_0\right)\right]}{\omega_0}\times$$

$$\times\cos\left\{px_0-\frac{1}{2}\tan\left(\frac{\theta}{2}\right)xx_0\right\}\exp\left\{i\left[\omega_0 x\right]\right\}\times$$

$$\times\exp\left\{i\frac{1}{4}\tan\left(\frac{\theta}{2}\right)(p+\omega_0)^2\right\}\times$$

$$\times\exp\left\{i\frac{1}{4}\tan\left(\frac{\theta}{2}\right)(x^2+x_0^2)\right\}. \quad (17)$$

Formulas (16) and (17) describe well the joint distribution of a rectangular pulse in the interval $t=[0,1]$. Nevertheless, these formulas contain the integrals over ω_0 with infinite limits; and this circumstance does not allow one to calculate three-dimensional plots, which constitute the basis of the space-frequency analysis, with a sufficient accuracy. Therefore, making use of expression (16), the following formula can be obtained:

$$\mathcal{K}_{rr^*}^{(t)}(x;p)=C_t\sqrt{\pi|\tan\left(\frac{\theta}{2}\right)|}\times$$

$$\times\exp\left\{-i\frac{\pi}{4}\mathrm{sign}\left(\tan\left(\frac{\theta}{2}\right)\right)\right\}\exp\left\{-i\frac{p^2}{\tan\left(\frac{\theta}{2}\right)}\right\}\times$$

$$\times\int_0^{2a}dx_0A\exp\left\{-i\frac{x^2+x_0^2}{\tan\left(\frac{\theta}{2}\right)}\right\}\cos\left\{px_0+2\frac{xx_0}{\tan\left(\frac{\theta}{2}\right)}\right\}\times$$

$$\times\int_{-1}^{1}dt\exp\left\{i\left(B\left[At-2\left(\frac{x}{2}-\frac{p}{\tan\left(\frac{\theta}{2}\right)}\right)\right]\right)^2\right\}, \quad (18)$$

where $A=a-\frac{x_0}{2}$ and $B=\frac{1}{2}\sqrt{\tan\left(\frac{\theta}{2}\right)}$. Formula (18) precisely reproduces the joint distribution of a rectangular pulse and has a number of advantages in comparison with formulas (15), (16), and (17). First, the joint distribution in form (18) includes only integrals with finite limits, which essentially simplifies numerical calculations and enhances their accuracy. Second, the formula describes well the joint distribution of a rectangular pulse within the whole open interval $t=(0,1)$ and allows one to closely approach the limiting cases $t=0$ and $t=1$. In this work, formula (18) is a key one for epy corresponding numerical calculations.

4.3. Method of Fresnel integrals

Making use of special functions – namely, Fresnel integrals – allows the formula for calculating the joint distribution of a rectangular pulse to be analytically reduced to single integrals. For this purpose, let us express Eq. (18) as follows:

$$\mathcal{K}_{rr^*}^{(t)}(x;p)=C_t\sqrt{\pi|\tan\left(\frac{\theta}{2}\right)|}\times$$

$$\times\exp\left\{-i\frac{\pi}{4}\mathrm{sign}\left(\tan\left(\frac{\theta}{2}\right)\right)\right\}\exp\left\{-i\frac{p^2}{\tan\left(\frac{\theta}{2}\right)}\right\}\times$$

$$\times\int_0^{2a}dx_0\exp\left\{-i\frac{x^2+x_0^2}{\tan\left(\frac{\theta}{2}\right)}\right\}\cos\left\{px_0+2\frac{xx_0}{\tan\left(\frac{\theta}{2}\right)}\right\}\times$$

$$\times\frac{2}{\sqrt{\tan\left(\frac{\theta}{2}\right)}}\int_{\eta_1}^{\eta_2}d\eta e^{i\eta^2}, \quad (19)$$

where

$$\eta_1=\frac{1}{2}\sqrt{\tan\left(\frac{\theta}{2}\right)}\left(-A-\left[x-\frac{2p}{\tan\left(\frac{\theta}{2}\right)}\right]\right),$$

$$\eta_2=\frac{1}{2}\sqrt{\tan\left(\frac{\theta}{2}\right)}\left(A-\left[x-\frac{2p}{\tan\left(\frac{\theta}{2}\right)}\right]\right). \quad (20)$$

By fulfilling a number of mathematical transformations and taking the definition of Fresnel integrals,

$$S(x)=\int_0^x dxe^{ix^2}, \quad (21)$$

into account, we obtain the final formula for calculation of the joint rectangular pulse distribution in terms of the Fresnel integrals:

$$\mathcal{K}^{(t)}_{rr^*}(x;p) = C_t\sqrt{\pi|\tan\left(\frac{\theta}{2}\right)|}\times$$

$$\times\exp\left\{-i\frac{\pi}{4}\mathrm{sign}\left(\tan\left(\frac{\theta}{2}\right)\right)\right\}\exp\left\{-i\frac{p^2}{\tan\left(\frac{\theta}{2}\right)}\right\}\times$$

$$\times\int\limits_0^{2a}dx_0\exp\left\{-i\frac{x^2+x_0^2}{\tan\left(\frac{\theta}{2}\right)}\right\}\cos\left\{px_0+2\frac{xx_0}{\tan\left(\frac{\theta}{2}\right)}\right\}\times$$

$$\frac{2}{\sqrt{\tan\left(\frac{\theta}{2}\right)}}(S(\eta_1)+S(\eta_2)). \qquad (22)$$

The representation obtained is essentially important while carrying out numerical calculations, because it involves a single integral. In addition, the application of Fresnel functions allows accurate calculations of the asymptotes of such a distribution to be made, which is essentially important for exact calculations of limiting cases.

5. Results of Numerical Calculations of the Joint Distribution of a Rectangular Pulse

In the previous section, we derived a number of important formulas for calculating the joint distribution of a rectangular pulse. In this section, we analyze the results obtained on the basis of numerical calculations. The main aim of carrying out the numerical calculations is to interpret the redistribution of the joint distribution of a rectangular pulse between the planes of Ville and Wigner distributions. The analytical formulas obtained demonstrate that such a redistribution occurs, when the joint parameter t changes. At $t = 0$, we have the Ville distribution, while at $t = 1$, the Wigner one. By varying the parameter t, we come into the region, which, in this work, is proposed to be referred to as a common one, because it contains the information common for both distributions. Numerical calculations were carried out on the basis of formula (18).

5.1. Limiting cases for the formation of the joint rectangular pulse distribution

First of all, we study the formation of the joint distribution of a rectangular pulse in the limiting cases, in order to compare the obtained results with their known analogs. So, let us consider each of them separately.

5.1.1. Ville distribution

In the classical case $t = 0$, in order to obtain a complete set of data on the input signal at a definite coordinate, it is necessary to sum up all spatial frequencies [10]. In this case, if $x = 0$, the distribution of a rectangular pulse in formula (18) is identical to its Fourier transform. At $p = 0$, the distribution acquires the shape of a triangular pulse. Therefore, the joint distribution of a rectangular pulse really becomes degenerate into the well-known Ville distribution at the value of the joint parameter $t = 0$ (Fig. 1,*a*).

5.1.2. Wigner distribution

Another limiting case of distribution (18) – at $t = 1$ – is the Wigner one. In this case, for the restoration of the input signal from its distribution, which is confined in space, it is also necessary to make allowance for all spatial frequencies. As is seen from Fig. 1,*f*, the distribution corresponds to the Wigner one. Hence, we may assert that the Wigner and Ville distributions are really the limiting cases of the joint distribution (18).

5.1.3. Joint distribution

Consider now the common region. By varying the joint parameter t, we change, in that way, the distribution of the input signal. Therefore, even if the variation of the parameter t is not large, the distribution of a rectangular pulse becomes rotated on the information diagram. In the course of rotation, both the coordinate and frequency cross-sections of the distribution change. Similar effects are also observed in the range of the fractional Fourier transformation [11, 12] and while studying the Fresnel diffraction at an isolated slit [10]. Hence, the rotation of the rectangular pulse distribution in the common region – clockwise and by an angle proportional to the joint parameter t – is observed, while the redistribution from the Ville plane to the Wigner one occurs. In the case of the motion in the opposite direction, i.e. from the Wigner plane to the Ville one, there occurs the counter-clockwise rotation.

The results of such a redistribution are demonstrated in Figs. 1 and 2 for some selected values of the joint parameter. The mechanism of redistribution can be evidently illustrated by giving, as an example, the cross-

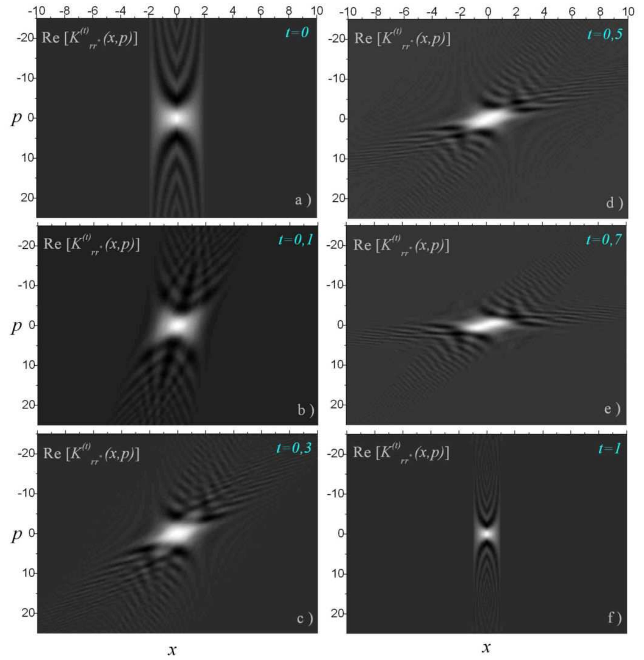

Fig. 1. Formation of the real part of the joint distribution for a rectangular pulse for various values of the parameter t

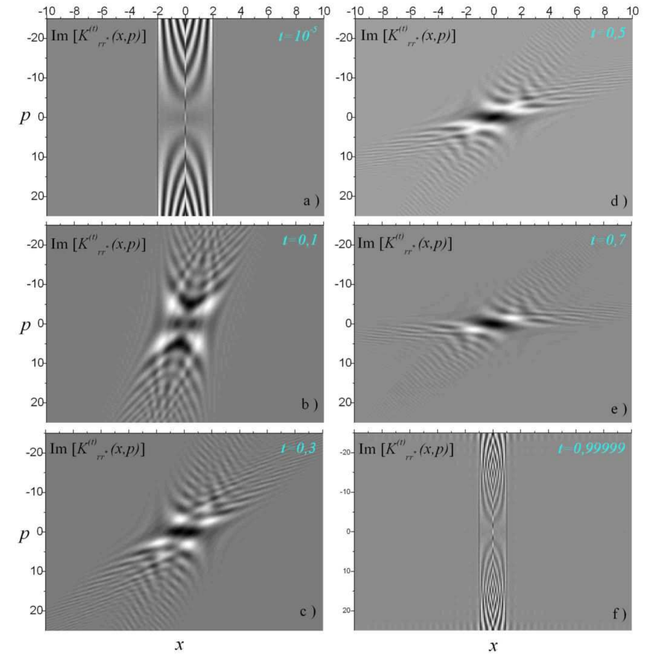

Fig. 2. Formation of the imaginary part of the joint distribution for a rectangular pulse for various values of the parameter t

section $x = 0$ of the joint distribution of a rectangular pulse (see Fig. 3). It is important to point out that, in the limiting cases ($t = 0$ and $t = 1$), the imaginary part of distribution is equal to zero, i.e. the distribution is real-valued in those cases. Minor changes of the joint parameter t give rise to the appearance of its imaginary part, so that the distribution in the common region is complex-valued. It testifies that the redistribution occurs exclusively through the complex plane. It should also be noted that, while entering into the common region, the rectangular pulse distribution becomes frequency-confined, i.e. every coordinate is associated with a definite frequency. It is a key distinction from the limiting cases, where every coordinate is associated with an infinite number of frequencies, which makes the process of input signal restoration from its distribution essentially complicated.

Now, consider the formation of a rectangular pulse distribution in the common region. For this purpose, we analyze some characteristic values of the joint parameter t in detail.

T h e i n t e r v a l $t = 0 \div 10^{-5}$. Let us start the analysis from the limiting case $t = 0$ (the Ville distribution). In this case, the distribution is real, and its cross-sections are as follows: in the case $x = 0$, the well-known triangular pulse (see Fig. 3, $t = 0$), and, in the case $p = 0$, the Fourier spectrum of a rectangular pulse (see Fig. 3, $t = 0$). The complete distribution of a rectangular pulse in the case $t = 0$ is presented in Figs. 1,*a* and 2,*a*. As was mentioned above, in order to restore the input signal in this case, all spatial frequencies given within the frequency interval $-\infty < \omega < \infty$ have to be taken into account.

T h e v i c i n i t y o f $t = 0.1$. By increasing the value of the joint parameter t to 0.1, we make a first step into the mixed region. Here, the situation changes drastically. The distribution becomes complex, i.e. both the real and imaginary parts are responsible for the formation of the joint distribution, and those parts should be analyzed separately. On the information diagram, the real (Fig. 1,*b*) and imaginary (Fig. 2,*b*) parts of the rectangular pulse distribution rotate by an angle proportional to the joint parameter t. The joint distribution starts to redistribute at the zero coordinate, owing to which it changes (Fig. 3). There appears an imaginary part, while the real one becomes definite within the fixed frequency limits. It testifies that, for the restoration of input signal, it is sufficient to consider a finite number of frequencies.

T h e v i c i n i t y o f $t = 0.3$. In this region, the rotation of both the real and imaginary parts of the input signal distribution continues (Figs. 1,*c* and 2,*c*). It should be noted that, in this region, the amplitude of the real part decreases, and that of the imaginary part increases. Such a process is natural, because the Wigner and Ville distributions are normalized in such a manner that the Wigner distribution, by amplitude, is twice as large as the Ville one. Therefore, the "transformation" from the former to the latter is accompanied by a reduction of the distribution amplitude. The cross-section of the joint distribution at zero coordinate (Fig. 3, $t = 0.2$) indicates a tendency for the band of finite frequencies, for which the input signal can be restored unambiguously, to grow.

C a s e $t = 0.5$. This case is of special interest within the scope of this research, because the joint distribution is located exactly in the middle between the Wigner and the Ville one (Figs. 1,*d* and 2,*d*). If the redistribution is associated with a rotation of the input signal distribution, then this case corresponds to the distribution rotation by the angle $\varphi = 45°$; the Ville distribution corresponds to the rotation angle $\varphi = 0°$, and the Wigner one to $\varphi = 90°$. The characteristic feature of the distribution formed in this case stems from the fact that $\tan 45° = 1$, so that any other features disappear both in the frequency and coordinate representations. The partial case of the joint distribution at the value of the joint parameter $t = 0.5$ requires a more detailed analysis, which goes beyond the scope of this work.

T h e v i c i n i t y o f $t = 0.7$. At such values of the joint parameter, the redistribution between the real and the imaginary parts of the input signal distribution continues due to the distribution rotation (Figs. 1,*e* and 2,*e*). The cross-section of the joint distribution at zero coordinate (Fig. 3, $t = 0.8$) is completely confined to the band of finite frequencies, for which the input signal can be restored unambiguously.

T h e i n t e r v a l $t = 0.9 \div 1.0$. In this region, the distribution of the input signal becomes ultimately rotated by the angle $\varphi = 90°$, which corresponds to the Wigner distribution (Fig. 1,*f*). The imaginary part of the distribution disappears (Fig. 3, $t = 1$), and, in the cross-section at zero coordinate, there emerge infinite frequencies. Nevertheless, at the large value of the joint parameter, $t = 0.99999$, the imaginary part is visually similar to the Wigner distribution, although being 10^5 times weaker than the latter by intensity. The previous researches allow us to draw an important conclusion that there is an unambiguous relation between the Ville and Wigner distributions. As was established above, the Wigner distribution is formed by rotating the Ville distribution on the information diagram by the angle $\varphi = 90°$.

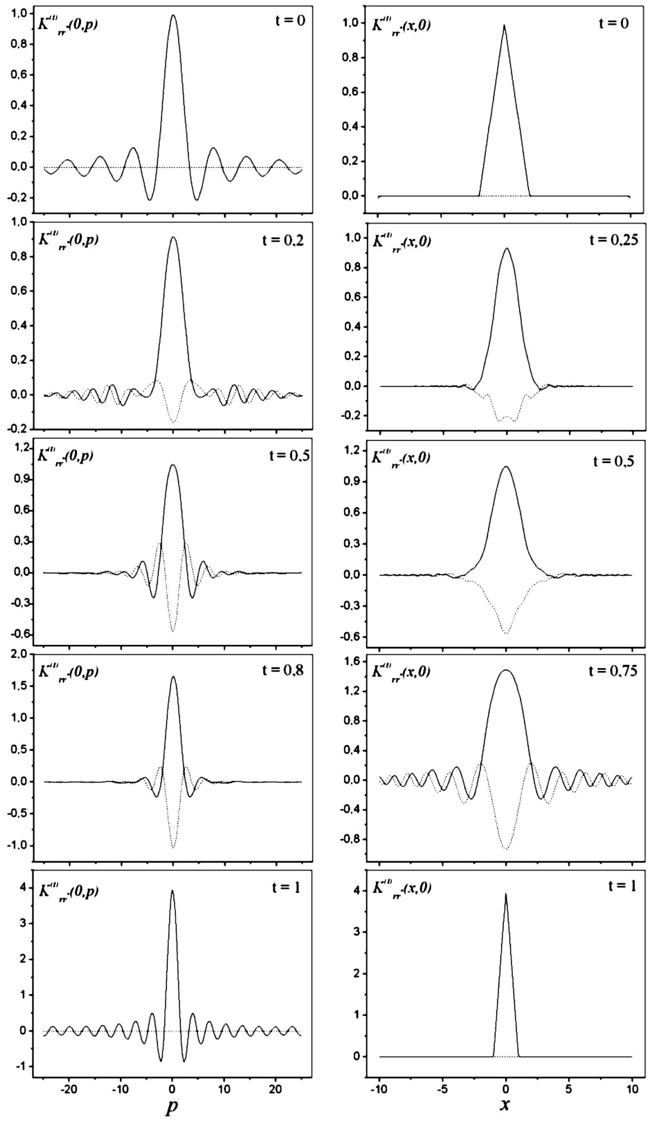

Fig. 3. Formation of the cross-sections $x = 0$ (the left column) and $p = 0$ (the right column) of the joint distribution of a rectangular pulse at various values of the parameter t. Solid curves – the real part, and the dotted curves – the imaginary one

6. Conclusions

The work is devoted to the consideration of the properties of the joint space-frequency distribution introduced by the author as a generalization of the well-known Ville and Wigner distributions. Numerical and analytical calculations have been carried out to study the mechanism of redistribution between the distributions mentioned above. The Ville and Wigner distributions have been demonstrated to be the limiting cases of the joint distribution. The region between those two limiting cases has been studied in detail, and a continuous relation between them has been revealed as a rotation on the information diagram of conjugated coordinates by an angle proportional to the joint parameter t.

For the description of the joint distribution to be complete, the frequency representation has been proposed for the first time, which made it possible to calculate its limiting cases – a task, for which the coordinate representation is inapplicable. Bearing in mind that the study of the joint distribution is aimed at its application to describe optical systems, the distribution of a rectangular pulse function, as an element of optical images, has been investigated in detail. A number of analytical formulas for the calculation of the joint rectangular pulse distribution has been derived for the first time.

The results of numerical calculations confirmed our theoretical assumptions and allowed us to establish the unequivocal relation between the Ville and Wigner distributions, as well as the mechanism of redistribution between them. The main advantage of using the joint distribution consists in a possibility to restore the intensity of the input signal, which is experimentally registered at the output of the optical system. The preliminary conclusions about the fruitfulness of applying the joint distribution for the description of optical systems can be made on the basis of that fact that the Ville distribution, which is its limiting case, is successfully used in solving the similar problems.

It is important also to note that the joint distribution combines two distributions which are widely used for the description of optical systems aimed at the information processing. Therefore, there are all reasons to assume that the joint space-frequency distribution will be used for the description of modern optical systems.

The author expresses his sincere gratitude to the President of Ukraine for granting (the order of the President of Ukraine No. 18/2007-rp), which provided a financial support to carry out this work.

1. E.P. Wigner, Phys. Rev. **40**, 749 (1933).
2. J. Ville, Cables et Transmissions **2A**, 61 (1948).
3. L. Cohen, Proc. IEEE **77**, 941 (1989).
4. A.W. Lohmann, J. Opt. Soc. Am. A **10**, 2181 (1993).
5. H.M. Ozaktas, M.A. Kutay, and Z. Zalevsky, *The Fractional Fourier Transform with Applications in Optics and Signal Processing* (Wiley, Chichester, 2000).
6. S. Chountasis, A. Vourdas, and C. Bendjaballah, Phys. Rev. A **60**, 3467 (1999).
7. Yu.M. Kozlovskii, *Preprint ICMP-06-27U* (Institute of Physics of Condensed Systems, Lviv, 2006) (in Ukrainian).
8. Yu.M. Kozlovskii, Ukr. J. Phys. Opt. **3**, 124 (2003).
9. M.V. Shovgenyuk, *Preprint ICMP-92-25U* (Institute of Physics of Condensed Systems, Lviv, 1992) (in Ukrainian).
10. M.V. Shovgenyuk, *Preprint ICMP-92-21U* (Institute of Physics of Condensed Systems, Lviv, 1992) (in Ukrainian).
11. M.V. Shovgenyuk and Yu.M. Kozlovskii, *Preprint ICMP-01-05U* (Institute of Physics of Condensed Systems, Lviv, 2001) (in Ukrainian).
12. M.V. Shovgenyuk and Yu.M. Kozlovskii, Dopov. Nats. Akad. Nauk Ukr. **6**, 92 (2000).

Received 25.10.07.
Translated from Ukrainian by O.I. Voitenko

СПІЛЬНИЙ ПРОСТОРОВО-ЧАСТОТНИЙ РОЗПОДІЛ ОПТИЧНИХ СИГНАЛІВ

Ю.М. Козловський

Р е з ю м е

Запропоновано спільний просторово-частотний розподіл, частковими випадками якого є розподіли Вейля та Вігнера. Вперше отримано частотне представлення спільного просторово-частотного розподілу, використання якого дало можливість усунути особливості, які мали місце при використанні координатного представлення. Розраховано вираз спільного розподілу для прямокутного імпульсу. Встановлено взаємозв'язок між розподілами Вейля та Вігнера. Показано, що розподіл Вігнера формується при повороті розподілу Вейля на інформаційній діаграмі спряжених координат $(x; p)$ на кут, пропорційний спільному параметру t. Наведено чисельні розрахунки спільного просторово-частотного розподілу прямокутного імпульсу при різних значеннях спільного параметра.