T-MODELS AND KANTOWSKI–SACHS MODELS

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The spherically symmetric solutions of the Einstein equations for a T-region (T-solutions) are considered. It is known that such solutions have a set of common unusual properties. It is shown that, in spite of their community, the T-solutions can be separated into two different classes which have different physical nature and admit different geometrical interpretations.

1. Introduction

The spherically symmetric metrics describe the space-time which may contain, in common case, both R- and T-regions separated by the event horizon surfaces. For the metric

$$ds^2 = e^{\nu(R,\tau)}d\tau^2 - e^{\lambda(R,\tau)}dR^2 - r^2(R,\tau)d\sigma^2, \quad (1)$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2$, the event horizon is determined by the equation

$$e^{-\nu/2}\frac{\partial r}{\partial \tau} = e^{-\lambda/2}\frac{\partial r}{\partial R} \quad (2)$$

or equivalently by

$$m(R,\tau) = r(R,\tau), \quad (3)$$

where the mass function

$$m(R,\tau) = r(R,\tau)\left(1 + e^{-\nu}\left(\frac{\partial r}{\partial \tau}\right)^2 - e^{-\lambda}\left(\frac{\partial r}{\partial R}\right)^2\right). \quad (4)$$

The coordinate system (1) describes a T-region alone under the following condition:

$$e^{-\lambda}\left(\frac{\partial r}{\partial R}\right)^2 = 0. \quad (5)$$

In this case, $\frac{\partial r}{\partial R} = 0$, so the mass function either depends on only τ or is constant:

$$m(\tau) = r(\tau)\left(1 + e^{-\nu}\left(\frac{\partial r}{\partial \tau}\right)^2\right). \quad (6)$$

Correspondingly, for the coordinate systems describing an R-region alone, $\frac{\partial r}{\partial \tau} = 0$, and the mass function depends on R only:

$$m(R) = r(R)\left(1 - e^{-\lambda}\left(\frac{\partial r}{\partial R}\right)^2\right). \quad (7)$$

Thus, the spherically symmetric metric describing a T-region alone has the following general form:

$$ds^2 = e^{\nu(\tau)}d\tau^2 - e^{\lambda(R,\tau)}dR^2 - r^2(\tau)d\sigma^2. \quad (8)$$

In this metric, only the coefficient e^{λ} can depend both on R and τ. It is possible to change the coordinate system in (8) into a synchronous one:

$$ds^2 = dt^2 - X^2(r,t)dr^2 - Y^2(t)d\sigma^2. \quad (9)$$

For the first time, the solution for a single T-region was obtained by Novikov through the direct solution of the Einstein equations [1]. This was the solution for the empty space:

$$ds^2 = \frac{dt^2}{\frac{r_g}{t} - 1} - \left(\frac{r_g}{t} - 1\right)dr^2 - t^2 d\sigma^2. \quad (10)$$

Three years later, Kantowski and Sachs obtained a T-solution for the dust matter [2]. At present, it is commonly appropriated to call the metrics describing a T-region alone as Kantowski–Sachs metrics.

Ruban investigated the general characteristics of T-models [3]. The T-models possess a number of exotic properties like non-statics, homogeneity, time-like singularity at the center $t = 0$, finite time extent, non-Euclidean hypercylindrical structure of spatial sections $t =$const, etc.

The interest in T-solutions arises due to the possibility to use them in studying the early Universe. Nowadays, there is a collection of new T-models which are proposed and investigated, for example, in [4–8].

Studying T-solutions, we can separate the manifolds described by them into two different classes.

2. Manifolds Described by *R*- and *T*-solutions

In general relativity, there are such manifolds which can be described by the metric in curvature coordinates in the R-region, where all the coefficients depend on only the spatial coordinate, as well as by the metric with all the coefficients depending only on time in the T-region.

Let us consider first the solutions with $e^{\nu} = e^{-\lambda}$. These solutions are, for example, the Schwarzschild metric [9]

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\sigma^2; \tag{11}$$

the Reissner–Nordström metric [9]

$$ds^2 = \frac{(R - r_1)(R - r_2)}{r^2} dt^2 - $$
$$- \frac{r^2 dr^2}{(R - r_1)(R - r_2)} - r^2 d\sigma^2, \tag{12}$$

where $r_{1,2} = \frac{r_g}{2} \pm \sqrt{\frac{r_g^2}{4} - q^2}$;
the de Sitter solution [9]

$$ds^2 = \left(1 - \frac{r^2}{a_\Lambda^2}\right) dt^2 - \frac{dr^2}{1 - \frac{r^2}{a_\Lambda^2}} - r^2 d\sigma^2, \tag{13}$$

where $a_\Lambda^2 = \frac{3}{\Lambda}$, and Λ is the cosmological constant. The similar properties are inherent in the Köttler [11] and Reissner–Nordström solutions with the cosmological constant.

All the listed coordinate systems are incomplete and possess the coordinate singularity. The Schwarzschild solution for $r > r_g$ describes the R-region of a manifold, and, for $r < r_g$, there is the T-solution obtained in [1]. The same situation is observed for the other metrics (12) and (13).

For solution (12) with real r_1 and r_2, there is a R-solution for $r > r_1$ and $r < r_2$, and a T-solution is available for $r_1 > r > r_2$.

In the de Sitter solution (13), there are an R-solution for $r < a_\Lambda$ and a T-solution for $r > a_\Lambda$.

Both the curvature coordinate system and the T-system are incomplete, because they do not describe the whole manifold. But, for the metrics under consideration, it is possible to turn to a synchronous coordinate system which is the complete system for the Schwarzschild and Reissner–Nordström solutions.

For the de Sitter solution, the complete coordinate system will be one of three known synchronous systems. The embedding of the de Sitter manifold into a flat 5-dimensional space-time represents a hyperboloid of rotation. R- and T-solutions describe only parts of this hyperboloid, as well as incomplete synchronous hyperbolic and parabolic systems do [12]. The complete coordinate system which describes the whole hyperboloid is the elliptic one:

$$ds^2 = d\tau^2 - a_\Lambda^2 e^{2\tau/a_\Lambda} (d\chi^2 + \sin^2 \chi d\sigma^2). \tag{14}$$

The transformation from the curvature coordinate system to the synchronous one,

$$\begin{aligned} ds^2 &= A(r)dt^2 - B(r)dr^2 - r^2 d\sigma^2 = \\ &= d\tau^2 - \left(\frac{\partial r(R,\tau)}{\partial \tau}\right)^2 \frac{AB}{f^2(R)} dR^2 - r^2(R,\tau) d\sigma^2, \end{aligned} \tag{15}$$

can be obtained from the equations

$$\left(\frac{\partial r(R,\tau)}{\partial \tau}\right)^2 = \frac{f^2(R)}{AB} - \frac{1}{B}, \tag{16}$$

where $f(R)$ is an arbitrary function. For the metrics with $A = B^{-1}$, one has

$$\left(\frac{\partial r(R,\tau)}{\partial \tau}\right)^2 = f^2(R) - A(r). \tag{17}$$

T-solutions of the class under consideration do not describe a new manifold, but only a part of the known one. In some sense, these solutions make our idea of the known manifold to be more profound.

Nevertheless, there is a lot of works, where such T-solutions are treated as new models. We mention, for example, work [13], where the Einstein equations are solved for metric (8) with the equation of state $\varepsilon + p = 0$. In this metric, all the coefficients depend only on the time. In this case, the solution obtained has the form

$$ds^2 = d\tau^2 - a_\Lambda^2 \sinh^2 \frac{\tau}{a_\Lambda} dr^2 - a_\Lambda^2 \cosh^2 \frac{\tau}{a_\Lambda} d\sigma^2. \tag{18}$$

Further, we will analyze properties of this metric. We draw conclusion that solution (18) turns eventually into the de Sitter one. In reality, the simple replacement $t = a_\Lambda \cosh \frac{\tau}{a_\Lambda}$ transforms (18) into the de Sitter solution

$$ds^2 = \frac{dt^2}{\left(\frac{t^2}{a_\Lambda^2} - 1\right)} - \left(\frac{t^2}{a_\Lambda^2} - 1\right) dr^2 - t^2 d\sigma^2. \qquad (19)$$

For $t > a_\Lambda$, this solution describes a T-region, and, for $t > a_\Lambda$, a R-region appears, and the solution coincides with (13). Solution (18) cannot pass to the de Sitter one in the course of time, because it describes the de Sitter manifold given by (13), and (19) is concerned with only a part of the whole manifold.

By the replacement $r = a_\Lambda \sin \frac{\rho}{a_\Lambda}$ in the de Sitter solution (13), one can pass to the coordinate system

$$ds^2 = \cos^2(\frac{\rho}{a_\Lambda})dt^2 - d\rho^2 - a_\Lambda^2 \sin^2 \frac{\rho}{a_\Lambda^2} d\sigma^2 \qquad (20)$$

which includes system (18), the coordinate singularities being inherent in the coordinate system but not in the manifold described by the de Sitter solution. We note that solution (20) turns into (18) after the replacement $\rho = \frac{\pi}{2} a_\Lambda - \imath\tau,\ t = r$.

Similarly for the Schwarzschild, Köttler, and Reissner–Nordström metrics, it is possible to find solutions in the form (9) which describe parts of the corresponding manifolds. The singularities of these solutions will remain the same as those of the original ones.

Thus, we have considered the case where, for solutions in the curvature coordinate system, the equation $e^\lambda = e^{-\nu}$ takes place. We now consider the case where $e^\lambda \neq e^{-\nu}$.

As follows from (7), $e_R^{-\lambda} = \frac{m}{r} + 1$ for a R-region in the curvature coordinates. For a T-region in the Novikov coordinates, $e_T^{-\nu} = \frac{m}{r} - 1$, where r plays the role of the time coordinate. If the mass function has the form such that the coefficient $e^{-\lambda}$ can be zero, then one will obtain the coordinate system describing both R- and T-regions by the coordinate transformation from the curvature system to the synchronous one according to (16). It is possible to turn to the T-solution with $e_T^{-\nu} = e_R^{-\lambda}$. But the rest metric coefficients (e_R^ν and e_T^λ) will be different because the equation of states is changed in the T-region.

For example, for the inner Schwarzschild solution in the curvature coordinates, $e^{-\lambda} = 1 - \frac{r^2}{a^2}$, the energy density $\varepsilon = \frac{3}{a^2}$ =const, and the pressure is a function of r. For the T-region in the Novikov coordinates, we have

$$ds^2 = \left(\frac{t^2}{a^2} - 1\right) dt^2 - e_T^\lambda dr^2 - t^2 d\sigma^2. \qquad (21)$$

The energy density for the T-solution depends on the time, whereas the pressure is constant: $p = \frac{3}{a^2}$.

For the class of manifolds under consideration, the mass function has the same form for the whole manifold both in the R- and T-regions independently of the coordinate system used for their description. So the mass function for the de Sitter solution $m = \frac{r^3}{a_\Lambda^2}$, where r^2 is the metric coefficient of the spherical part $d\sigma^2$, as well as the mass function for the Schwarzschild solution $m = r_g$, remains the same both in R- and T-regions. The situation is similar for all the metrics of the first class. These metrics describe only a part of the manifolds containing R- and T-regions, but there is the possibility to find a complete coordinate system which would describe the whole manifold.

3. Kantowski–Sachs Models

The Kantowski–Sachs solution [2] for the dust matter is an essentially new T-solution which cannot exist in a R-region at all and completely belongs to a T-region.

To better understand the physics of the model, let us consider the Tolman–Bondi solution for the dust matter in general relativity [15] in the elliptic case ($f^2(R) < 1$):

$$ds^2 = d\tau^2 - \left(\frac{\partial r}{\partial R}\right)^2 \frac{1}{f^2(R)} dR^2 - r^2(R,\tau) d\sigma^2, \qquad (22)$$

$$\begin{aligned} &\tau - \tau_0(R) = \tfrac{m(R)}{2(1-f^2(R))^{3/2}}(\alpha - \sin\alpha), \\ &r = \tfrac{m(R)}{1-f^2(R)} \sin^2 \tfrac{\alpha}{2}. \end{aligned} \qquad (23)$$

Solution (22) contains three arbitrary functions of integration: $m(R),\ f(R),\ and \tau_0(R)$, where $m(R)$ is the mass function which is the total mass of the dust including the gravitational interaction:

$$\frac{\partial m(R)}{\partial R} = \varepsilon r^2 \frac{\partial r}{\partial R}, \qquad (24)$$

where $\varepsilon(R,\tau)$ is the dust energy density. The dust matter mass itself (excluding the gravitational interaction) is

$$\frac{\partial \mu(R)}{\partial R} = e^{\frac{\lambda}{2}} r^2 \varepsilon, \qquad (25)$$

where $f(R)$ is the total energy of dust particles in a shell of radius R:

$$f^2(R) = e^\lambda \left(\frac{\partial r}{\partial R}\right)^2. \qquad (26)$$

We note that

$$\frac{\partial m(R)}{\partial R} = \frac{\partial \mu(R)}{\partial R} f(R). \tag{27}$$

Let's consider the special case $f(R) = 0$. It follows from (26) that, in the case under consideration, $\frac{\partial r}{\partial R} = 0$, i.e $r = r(\tau)$. According to (24), the mass function is constant. So we assume it to be $m = r_g$, where r_g is the gravitational radius. From (23), we have

$$r = r_g \sin^2 \frac{\alpha}{2}, \tau = r_g(\alpha - \sin\alpha). \tag{28}$$

If the energy density $\varepsilon = 0$, then solution (24) has the form

$$ds^2 = r_g^2 \sin^2 \frac{\alpha}{2} d\alpha^2 - \mathrm{ctg}^2 \frac{\alpha}{2} dR^2 - r_g^2 \sin^2 \frac{\alpha}{2} d\sigma^2 \tag{29}$$

which can be turned into the Novikov solution for the empty space by a coordinate transformation.

In the case of nonzero energy density $\varepsilon \neq 0$, the Einstein equations yield the expression for e^λ. Finally, one obtains the Kantowski–Sachs solution as

$$ds^2 = d\tau^2 - e^\lambda dR^2 - r^2 d\sigma^2, \tag{30}$$

where $r(\tau)$ is given by (28), and

$$e^\lambda = \mathrm{ctg}\frac{\alpha}{2} + \frac{\partial \mu(R)}{\partial R}(1 - \frac{\alpha}{2}\mathrm{ctg}\frac{\alpha}{2}). \tag{31}$$

The metric coefficient e^λ is the function of both the spatial and time coordinates. The energy density is also a function of R and t. Thus, the Kantowski–Sachs model can be considered as the limit case of the Tolman–Bondi solution [15], but not of the Friedman one.

As far as r is a function of only the time, the arbitrary function $\tau_0(R) = 0$ equals zero. There are only two arbitrary functions: the mass of the dust distribution $\mu(R)$ and the mass function $m = r_g$. It is worth noting that, in the general solution (22), $\mu(R)$ defines the type of motion through $f(R)$. If the mass function $m(R)$ grows with R more slowly than the dust mass $\mu(R)$, the elliptic type of motion takes place. In the limit ($f = 0$), the mass function is constant, while the dust mass is growing.

For the dust matter and a trapped magnetic field, the T-solution also belongs to the considered class of solutions.

In the synchronous coordinates, this solution has the form [16]

$$ds^2 = d\tau^2 - e^{\lambda(\tau)} dR^2 - r^2(\tau) d\sigma^2, \tag{32}$$

where the coordinate R is directed along the magnetic field. We have

$$\tau = r_g(\alpha - b\sin\alpha),\ r = \tfrac{r_g}{2}(1 - b\cos\alpha), \quad b^2 = 1 - \tfrac{4q^2}{r_g^2}, \tag{33}$$

where b =const. The metric coefficient

$$e^{\lambda/2} = \frac{b\sin\alpha}{1 - b\cos\alpha}(1 - \frac{\alpha}{2}\frac{\partial m(R)}{\partial R}) + \frac{\partial m(R)}{\partial R}\frac{1}{2b}(1 + \frac{1 - b^{-1}\cos^2\alpha}{1 - b\cos^2\alpha}), \tag{34}$$

and $\mu(R)$ is still the dust matter mass.

As far as $r = r(\tau)$ and $\frac{\partial r}{\partial R} = 0$, then it follows from (24) that $m = m(\tau)$. Thus, the mass function depends on τ only. For metric (32), relation (4) yields the expression for the mass function as

$$m(\tau) = r(\tau)(1 + (\frac{\partial r}{\partial \tau})^2) = r(\tau)(\frac{r_g}{r} - \frac{q^2}{r^2}). \tag{35}$$

The total mass of the dust matter is constant, as it is true for the dust without magnetic field. The second term in formula (35) is caused by the presence of a magnetic field. The function $f(R)$ also equals zero. In the limit of zero magnetic field ($q = 0$), solution (34) turns into the Kantowski–Sachs solution (30).

Besides the rest mass and the kinetic energy of the dust matter, the mass function for the dust distribution includes the negative gravitational potential energy. For the Tolman–Bondy dust models, the difference in the behaviors of the active mass $m(R)$ and the dust mass $\mu(R)$ defines the type of the dust distribution motion. In case of the T-dust, the active mass is constant during the infinite growing of the dust mass $\mu(R)$ because of the negative gravitational potential binding energy [3].

The mass function for the second class of solutions coincides in the R-region with the corresponding mass function of the empty space. For example, the mass function for the dust T-configuration in the R-region is equal to the Schwarzschild one, and the mass function for the dust T-configuration with a trapped magnetic field is the same as that for the Reissner–Nordström solution (35).

4. Conclusions

The spherically symmetric solutions of the Einstein equations with $d\sigma^2$ depending on time only are considered. These solutions characterize the T-region only. It is shown that they may describe the essentially different geometric manifolds. These solutions can be separated into two different classes.

The solutions of the first class are ones described by incomplete coordinate systems (all the metric coefficients depend only on time, including e^{λ}). It is shown that, for these solutions, it is possible to turn by a coordinate transformation to another incomplete coordinate system describing the R-region or to find such complete coordinate system which would describe both the R- and T-regions. For the manifolds of this class, the mass function has the same form for the whole manifold in any coordinate system and hence both in the R- and T-regions.

The second class of solutions characterizes the T-region only. The configurations described by these solutions cannot exist in the R-region, and there is no Newtonian limit for them. The mass function for the considered models of this class is found.

It is shown that the mass function for the second class of solutions coincides in the R-region with the mass function of the empty space. This also confirms the fact that such configurations can exist in the T-region only.

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T-МОДЕЛI I МОДЕЛI КАНТОВСЬКОГО–САКСА

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Р е з ю м е

Розглядаються сферично-симетричнi розв'язки рiвнянь Ейнштейна для T-областi (T-розв'язки). Вiдомо, що такi розв'язки мають ряд загальних нетривiальних властивостей. Показано, що, незважаючи на їх єднiсть, T-розв'язки можуть бути роздiленi на два рiзних класи розв'язкiв, котрi мають рiзну фiзичну i геометричну iнтерпретацiю.