CONDITIONS FOR SELF-ORGANIZED MODULATION

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Conditions for the creation of a limit cycle, which provide the transition of a nonequilibrium system into a self-organized modulation mode, have been studied. An approach, which allows one to replace the equations of self-consistent evolution for a pair of real-valued variables by a single equation of motion for a complex-valued order parameter, is proposed. The optimum basis has been found, in which the evolution of the complex-valued order parameter is described by the Ginzburg–Landau equation characterized by a complex-valued non-linearity only. Conditions for the system to transit into the self-organized modulation mode are determined.

1. Introduction

Synergetics is an interdisciplinary branch of science, which allows one to describe the self-organization in open systems considered in physics, chemistry, biology, sociology, and other sciences, as well as their applications [1–4]. As a rule, such systems are studied in the framework of nonequilibrium kinetics methods which are based on the second principle of thermodynamics; the latter postulates that entropy either is invariant or grows in time. However, such a behavior is inherent to closed statistical ensembles, whereas synergetics deals with open systems, in which a considerable deviation from the equilibrium state can result in an entropy reduction, although the entropy is a measure of statistical disorder (chaos). Due to the self-organization, there emerges an ordered state which corresponds to a minimum of the effective potential; in such a state, the system can dwell as long as the influence of environment allows.

Unlike thermodynamic transformations, the self-organization process can result not only in the formation of statistical states which correspond to time-invariant local minima of the effective potential, but also to the formation of temporal, spatial, or spatio-temporal dissipative structures, the evolution of which is governed by an attractive set much more complex than a single point.

Very often, the formation of such structures follows the scenario of the Hopf bifurcation emergence, which gives rise to the appearance of a stable limit cycle [5]. Since, in this case, the evolution of the system is reduced to a periodic variation of quantities that parametrize its behavior, one may assert that the creation of the limit cycle means the transition of the system into a *self-organized modulation* mode.

Such a mode has been studied in the framework of the simplest approach which allows both phase transitions and the formation of dissipative structures to be analyzed from a common viewpoint [6]. This approach is based upon the Lorentz–Haken synergetic model, in which the self-consistent behavior of the system is characterized by the evolution of the order parameter, conjugate field, and control parameter [7]. In contrast to conventional synergetic systems, a manifestation of self-organized modulation means that a pair of degrees of freedom rather than a single one have the largest scale of time variation. If this pair includes the order parameter, the variation rate of which linearly depends on other parameters, the creation of a limit cycle turns out impossible, so that the behavior of the system does not differ from a thermodynamic transformation [8].

In connection with the aforesaid, the study of limit cycle creation conditions, which would provide a transition of the system into the self-organized modulation mode, seems challenging. Such conditions were considered in monography [5]; however, owing to its redundant formalization, this source is rather difficult for understanding. Moreover, it turns out that the criterion of limit cycle creation found in work [5] is so cumbersome that its application is practically impossible even for the simplest systems. Therefore, we intend to carry out a consecutive consideration of conditions needed for the system to transit into the self-organized modulation mode. The consideration is illustrated by the numerical solution of the equations of motion, which confirms the reliability of the criterion found.

The structure of the paper is as follows. In Section 2, we expound a method which allows the equations of self-consistent evolution for a pair of real-valued variables to be reduced to a single equation of motion for a complex-valued variable with an isolated linear term (in the mathematical literature, such an equation is classified to the Poincaré canonical form). Section 3 is devoted to the choice of an optimum basis of the complex-valued order parameter, in which the evolution of the latter is described by the Ginzburg–Landau equation with only a cubic non-linearity. The conditions for the limit cycle formation are found in Section 4. In Section 5, they are analyzed for a Lorentz system which undergoes a discontinuous transformation.

2. Complex representation of the motion equations

According to the central manifold theorem [5], the evolution of a system with n degrees of freedom ($n > 2$) can be represented by periodic dependences of a pair of real variables $X_1(t)$ and $X_2(t)$. Therefore, our task is reduced to finding the conditions, under which the equations of motion for these variables

$$\dot{X}_1 = F^{(1)}, \quad F^{(1)} = F^{(1)}\left(X_1, X_2\right);$$

$$\dot{X}_2 = F^{(2)}, \quad F^{(2)} = F^{(2)}\left(X_1, X_2\right) \tag{1}$$

have a solution that corresponds to a limit cycle. Hereafter, the dotted symbol means the differentiation of the corresponding quantity with respect to time. The right-hand sides of Eqs. (1) are generalized forces conjugate to the corresponding degrees of freedom. The simplest example of the limit cycle is given by the system of linear Lotka–Volterra equations

$$\dot{X}_1 = \lambda X_1 - \omega X_2,$$

$$\dot{X}_2 = \omega X_1 + \lambda X_2 \tag{2}$$

which is governed by the real-valued parameters λ and ω. Really, the division of either of those equations by the other followed by integration under the condition $\lambda = 0$ brings about the equation for a circle, $X_1^2 + X_2^2 = \text{const}$, where the constant determines the circle radius. Whence, it is evident that, in the general case, Eqs. (1) produce a limit cycle, if their linear component can be reduced to system (2). This means that ($X_1 = X_{10}$ and $X_2 = X_{20}$) in the stationary state, where the time dependence disappears, and the Jacobi matrix

$$\Lambda_{\alpha\beta} \equiv \left.\frac{\partial F^{(\alpha)}}{\partial X_\beta}\right|_{X_\beta = X_{\beta 0}}; \qquad \alpha, \beta = 1, 2, \tag{3}$$

acquires a canonical form

$$\hat{\Lambda} = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} \tag{4}$$

which, despite its simplicity, is general. Really, a condition is imposed upon the diagonal components of matrix (4) that they should be simultaneously equal to zero at the critical point, which corresponds to the creation of a limit cycle. Therefore, they can differ from each other by only a numerical factor which can be easily got rid due to a proper choice of measurement units. As to the non-diagonal components, the equivalence of their absolute values is a consequence of the Onsager requirement of the symmetry of kinetic coefficients in Eqs. (2), whereas the choice of their opposite signs ensures the self-organization in the system.

Equations (1) can be conveniently expressed in the vector form

$$\dot{\mathbf{X}} = \mathbf{F}, \tag{5}$$

making use of the notations

$$\mathbf{X} \equiv \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \mathbf{F} \equiv \begin{pmatrix} F^{(1)} \\ F^{(2)} \end{pmatrix}. \tag{6}$$

In the framework of such a representation, the characteristic values Λ and $\bar{\Lambda}$ and the characteristic vectors e_α and e^+_α of the Jacobi matrix (3) – they are defined by the conjugate equations

$$\sum_{\beta=1}^{2}\Lambda_{\alpha\beta}e_\beta=\Lambda e_\alpha,\quad \sum_{\beta=1}^{2}e^+_\beta\Lambda_{\beta\alpha}=\bar{\Lambda}e^+_\alpha \tag{7}$$

– play an outstanding role. The Lyapunov parameter

$$\Lambda\equiv\lambda+\mathrm{i}\omega \tag{8}$$

is determined by the increment λ and the frequency ω. In the canonical case (4), the characteristic value of quantity (8) corresponds to the vectors

$$\mathbf{e}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix},\qquad \mathbf{e}^+=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\end{pmatrix} \tag{9}$$

which satisfy the normalization conditions

$$\mathbf{e}^+\mathbf{e}\equiv(e|e)\equiv\sum_{\alpha=1}^{2}e^+_\alpha e_\alpha=1. \tag{10}$$

To express Eqs. (5) in the canonical form, let us introduce a pseudovector

$$\mathbf{x}\equiv\mathbf{X}-\mathbf{X}_0 \tag{11}$$

reckoned from the stationary state $\mathbf{X}_0$ and a nonlinear component of the force

$$\mathbf{f}\equiv\mathbf{F}-\hat{\Lambda}\mathbf{x}. \tag{12}$$

Then, Eq. (5) reads

$$\dot{\mathbf{x}}=\hat{\Lambda}\mathbf{x}+\mathbf{f}(\mathbf{x}). \tag{13}$$

Next, we define the complex-conjugate variables

$$z\equiv(e|x)=\sum_{\alpha=1}^{2}x_\alpha e^+_\alpha=\frac{1}{\sqrt{2}}\left(x_1+\mathrm{i}x_2\right),$$

$$\bar{z}\equiv(x|e)=\sum_{\alpha=1}^{2}\bar{x}_\alpha e_\alpha=\frac{1}{\sqrt{2}}\left(\bar{x}_1-\mathrm{i}\bar{x}_2\right) \tag{14}$$

which are the projections of the state vector (11) onto the characteristic vectors (9) of matrix (4). They are obtained if the projection operator

$$\widehat{P}\equiv|e)\,(e|=\frac{1}{2}\begin{pmatrix}1 & \mathrm{i}\\-\mathrm{i} & 1\end{pmatrix} \tag{15}$$

is applied to the initial pseudo-vector (11) expressed in the component form (6):

$$\widehat{P}\mathbf{x}=\frac{1}{2}\begin{pmatrix}1 & \mathrm{i}\\-\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\frac{x_1+\mathrm{i}x_2}{2}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix}\equiv z\mathbf{e}. \tag{16}$$

Multiplying this equation by $\mathbf{e}^+$, we obtain a matrix expression for definitions (10):

$$z=\mathbf{e}^+\widehat{P}\mathbf{x}\equiv\left(e|\widehat{P}|x\right),$$

$$\bar{z}=\mathbf{x}^+\widehat{P}\mathbf{e}\equiv\left(x|\widehat{P}|e\right), \tag{17}$$

The latter relation is obtained by applying the complex conjugation operator to the former one and taking the Hermitian character of the operator $\hat{P}$ into account. Using expressions (15), (9), and (4), one can easily verify the main properties of the projection operator:

$$\widehat{P}^2=\widehat{P};\quad \widehat{P}\mathbf{e}=\mathbf{e},\quad \mathbf{e}^+\widehat{P}=\mathbf{e}^+;\qquad \widehat{P}\hat{\Lambda}=\hat{\Lambda}\widehat{P}. \tag{18}$$

By applying operator (15) to Eq. (13), taking the last of properties (18) into account, multiplying the result by $(\vec{e}|\equiv\mathbf{e}^+$, and summing up the components, we obtain the canonical Poincaré form

$$\dot{z}=\Lambda z+v\left(z,\bar{z}\right),\quad v\left(z,\bar{z}\right)\equiv\left(e|\,\widehat{P}\,|f\right)=\mathbf{e}^+\widehat{P}\mathbf{f}, \tag{19}$$

where the linear component of the generalized force is separated.

3. Presentation of the Equations of Motion in the Ginzburg–Landau Form

It is easy to check up that the transition from the component representation

$$\dot{x_1}=F^{(1)}(x_1,x_2),$$

$$\dot{x_2}=F^{(2)}(x_1,x_2) \tag{20}$$

to the canonical equation of motion for the complex-valued variable $z\equiv\frac{1}{\sqrt{2}}\left(x_1+\mathrm{i}x_2\right)$ defined by Eqs. (14) and (17) is achieved through the representation of the right-hand side of Eq. (19) as a complex-valued force

$$F=F(z,\bar{z})\equiv$$

$$\equiv\frac{1}{\sqrt{2}}\left[F^{(1)}\left(x_1(z,\bar{z}),x_2(z,\bar{z})\right)+\mathrm{i}F^{(2)}\left(x_1(z,\bar{z}),x_2(z,\bar{z})\right)\right]. \tag{21}$$

Its nonlinear component $v \equiv F - \Lambda z$ can be expanded in a series[1]

$$v(z,\bar{z}) = \sum_{2\le m+n\le 3} \frac{v_{mn}}{m!n!} z^m \bar{z}^n + O(|z|^4), \quad |z|^2 \equiv z\bar{z} \tag{22}$$

with the coefficients

$$v_{mn} \equiv \left.\frac{\partial^{m+n} v(z,\bar{z})}{\partial z^m \partial \bar{z}^n}\right|_{z,\bar{z}=0} = \left.\frac{\partial^{m+n} F(z,\bar{z})}{\partial z^m \partial \bar{z}^n}\right|_{z,\bar{z}=0}. \tag{23}$$

Taking equality (21) into account, as well as the relations

$$\frac{\partial}{\partial z} = \frac{\partial x_1}{\partial z}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial z}\frac{\partial}{\partial x_2} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right),$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x_1}{\partial \bar{z}}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \bar{z}}\frac{\partial}{\partial x_2} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right) \tag{24}$$

between the derivatives, expression (23) looks like

$$v_{mn} = 2^{-\frac{1+m+n}{2}} \left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right)^m \left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right)^n \times$$

$$\times \left[F^{(1)}(x_1,x_2) + \mathrm{i}F^{(2)}(x_1,x_2)\right]_{x_1,x_2=0}. \tag{25}$$

For various m and n, this formula gives

$$v_{11} = 2^{-3/2}\left[\left(F^{(1)}_{11} + F^{(1)}_{22}\right) + \mathrm{i}\left(F^{(2)}_{11} + F^{(2)}_{22}\right)\right], \tag{26}$$

$$\begin{pmatrix} v_{20} \\ v_{02} \end{pmatrix} = 2^{-3/2}\left[\left(F^{(1)}_{11} - F^{(1)}_{22} \pm 2F^{(2)}_{12}\right) + \right.$$

$$\left. + \mathrm{i}\left(F^{(2)}_{11} - F^{(2)}_{22} \mp 2F^{(1)}_{12}\right)\right], \tag{27}$$

$$v_{21} = \frac{1}{4}\left\{\left[\left(F^{(1)}_{111} + F^{(1)}_{122}\right) + \left(F^{(2)}_{112} + F^{(2)}_{222}\right)\right] + \right.$$

$$\left. + \mathrm{i}\left[\left(F^{(2)}_{111} + F^{(2)}_{122}\right) - \left(F^{(1)}_{112} + F^{(1)}_{222}\right)\right]\right\}, \tag{28}$$

where the notations

$$F^{(\delta)}_{\alpha\beta} \equiv \left.\frac{\partial^2 F^{(\delta)}}{\partial x_\alpha \partial x_\beta}\right|_{\mathbf{x}=0}, \quad F^{(\delta)}_{\alpha\beta\gamma} \equiv \left.\frac{\partial^3 F^{(\delta)}}{\partial x_\alpha \partial x_\beta \partial x_\gamma}\right|_{\mathbf{x}=0};$$

$$\alpha,\beta,\gamma;\delta = 1,2, \tag{29}$$

are used.

Now, let us consider the complex-valued order parameter ϕ which is connected with the z-variable by the relations

$$z = \phi + \chi(\phi,\bar{\phi}), \quad \chi \equiv \sum_{2\le m+n\le 4} \frac{\chi_{mn}}{m!n!}\phi^m \bar{\phi}^n. \tag{30}$$

The coefficients χ_{mn} are so defined that Eq. (19) should get the Ginzburg–Landau form (using the mathematical terminology, the normal Poincaré form without quadratic terms):

$$\dot{\phi} = \Lambda\phi + C|\phi|^2\phi + O(|\phi|^4), \quad |\phi|^2 \equiv \phi\bar{\phi}. \tag{31}$$

Now, our task is to express the parameter of nonlinearity C in terms of the structural constants v_{mn} which are defined by equalities (23).

Differentiation of the complex function $z(t) = \phi(t) + \chi\left(\phi(t),\bar{\phi}(t)\right)$ gives rise to

$$\dot{z} = \dot{\phi} + \chi_\phi\dot{\phi} + \chi_{\bar{\phi}}\dot{\bar{\phi}}; \qquad \chi_\phi \equiv \frac{\partial\chi}{\partial\phi}, \quad \chi_{\bar{\phi}} \equiv \frac{\partial\chi}{\partial\bar{\phi}}. \tag{32}$$

Subtracting Eq. (31) from Eq. (19), we come to the equation

$$\chi_\phi\dot{\phi} + \chi_{\bar{\phi}}\dot{\bar{\phi}} =$$

$$= \Lambda\chi(\phi,\bar{\phi}) + \left[v(z,\bar{z}) - C|\phi|^2\phi\right] + O(|\phi|^5), \tag{33}$$

the left-hand side of which follows from Eqs. (32), whereas the first of equalities (30) is taken into account in its right-hand side. Expressing the time derivatives from Eq. (31), we obtain – to the cubic-term accuracy – the equation

$$\phi\Lambda\chi_\phi + \bar{\Lambda}\bar{\phi}\chi_{\bar{\phi}} - \Lambda\chi =$$

$$= \sum_{2\le m+n\le 3} \frac{v_{mn}}{m!n!}\left(\phi+\chi\right)^m\left(\bar{\phi}+\bar{\chi}\right)^n -$$

$$- \left(C\phi + C\chi_\phi\phi + \bar{C}\chi_{\bar{\phi}}\bar{\phi}\right)|\phi|^2, \tag{34}$$

[1]We do not discuss the choice for the upper limits of summation in Eqs. (22) and (30), considering it as intuitively clear. The mathematical explanation of such a choice can be found in work [5].

where expansion (22) is made allowance for. According to the last of expressions (30), the left-hand side of this equation looks like

$$\phi\Lambda\chi_\phi + \bar{\Lambda}\bar{\phi}\chi_{\bar{\phi}} - \Lambda\chi =$$

$$= \sum_{2\le m+n\le 4} \frac{\chi_{mn}\left(m\Lambda + n\bar{\Lambda} - \Lambda\right)}{m!n!}\phi^m\bar{\phi}^n. \quad (35)$$

In order to determine the coefficients χ_{mn} with $m+n=2$, consider Eq. (34), where the terms of the second order with respect to ϕ and $\bar{\phi}$ are retained. In this case, according to Eq. (35), the left-hand side reads

$$\sum_{m+n=2} \frac{\chi_{mn}\left(m\Lambda + n\bar{\Lambda} - \Lambda\right)}{m!n!}\phi^m\bar{\phi}^n =$$

$$= \frac{\chi_{20}}{2}\Lambda\phi^2 + \chi_{11}\bar{\Lambda}\phi\bar{\phi} + \frac{\chi_{02}}{2}\left(2\bar{\Lambda} - \Lambda\right)\bar{\phi}^2. \quad (36)$$

Comparing the multipliers at ϕ^2-, $\phi\bar{\phi}$-, and $\bar{\phi}^2$-terms with the corresponding factors in the expression

$$\frac{v_{20}}{2}\phi^2 + v_{11}\phi\bar{\phi} + \frac{v_{02}}{2}\bar{\phi}^2, \quad (37)$$

which the left-hand side of Eq. (34) is reduced to, we find that

$$\chi_{20} = \frac{v_{20}}{\Lambda}, \quad \chi_{11} = \frac{v_{11}}{\bar{\Lambda}}, \quad \chi_{02} = \frac{v_{02}}{2\bar{\Lambda} - \Lambda}. \quad (38)$$

Further, we have to separate those terms in Eq. (34) which contain $|\phi|^2\phi$. In so doing, only the term that corresponds to $(m=2, n=1)$ gives the contribution to sum (35), so that the left-hand side of Eq. (34) looks like

$$\chi_{21}\frac{2\Lambda + \bar{\Lambda} - \Lambda}{2!1!}\phi^2\bar{\phi} =$$

$$= \frac{\chi_{21}}{2}\left(\Lambda + \bar{\Lambda}\right)|\phi|^2\phi = \chi_{21}\Re\Lambda|\phi|^2\phi. \quad (39)$$

Accordingly, the right-hand side contains the sum

$$\frac{v_{20}}{2}\left(\phi+\chi\right)^2 + v_{11}\left(\phi+\chi\right)\left(\bar{\phi}+\bar{\chi}\right) +$$

$$+\frac{v_{02}}{2}\left(\bar{\phi}+\bar{\chi}\right)^2 + \frac{v_{21}}{2}\left(\phi+\chi\right)^2\left(\bar{\phi}+\bar{\chi}\right), \quad (40)$$

resulting in the $|\phi|^2\phi$-prefactor

$$v_{20}\chi_{11} + v_{11}\left(\chi_{20} + \bar{\chi}_{11}\right) + v_{02}\bar{\chi}_{02} + \frac{v_{21}}{2} =$$

$$2v_{11}v_{20}\frac{\Re\Lambda}{|\Lambda|^2} + \frac{|v_{11}|^2}{\Lambda} + \frac{|v_{02}|^2}{2\Lambda - \bar{\Lambda}} + \frac{v_{21}}{2}, \quad (41)$$

where the last transformation takes the form of coefficients (38) into account. Since the terms in Eq. (34) that contain $\chi_\phi|\phi|^2\phi$ or $\chi_{\bar{\phi}}|\phi|^2\bar{\phi}$ make a contribution, the order of which is higher than that of $|\phi|^2\phi$, they can be omitted. Then, equalities (38), (39), and (41) express the parameter of non-linearity in Eq. (31) as follows:

$$C = 2v_{11}v_{20}\frac{\Re\Lambda}{|\Lambda|^2} + \frac{|v_{11}|^2}{\Lambda} + \frac{|v_{02}|^2}{2\Lambda - \bar{\Lambda}} + \frac{v_{21}}{2} - \chi_{21}\Re\Lambda. \quad (42)$$

4. Conditions for Limit Cycle Formation

The found expression (42) is incomplete, because it contains the unknown coefficient χ_{21}, the determination of which requires that not only the terms proportional to $|\phi|^2\phi$, but also the cubic terms ϕ^3, $|\phi|^2\bar{\phi}$, and $\bar{\phi}^3$ be taken into consideration [5]. However, one should bear in mind that the parameter of non-linearity is contained only in the term of higher order in series (31), where the real value of the Lyapunov parameter $\Lambda \equiv \lambda + \mathrm{i}\omega$ is put to be equal to zero. Then, Eq. (42) yields the expression for the Floquet parameter $\Phi \equiv C|_{\lambda=0}$,

$$\Phi = \frac{1}{2}v_{21} - \frac{\mathrm{i}}{\omega_0}\left(|v_{11}|^2 + \frac{1}{3}|v_{02}|^2\right), \quad (43)$$

where the characteristic frequency $\omega_0 \equiv \omega|_{\lambda=0}$ is introduced. The stationary point that generates the limit cycle becomes unstable under the condition $\Re\Phi < 0$. As a result, the condition for limit cycle formation $\Re v_{21} < 0$ reads

$$\left(F^{(1)}_{111} + F^{(1)}_{122}\right) + \left(F^{(2)}_{222} + F^{(2)}_{112}\right) < 0, \quad (44)$$

where equality (28) is taken into account.

Note that a quite another expression was obtained in monography [5] instead of Eq. (43):

$$\Phi = \frac{1}{2}v_{21} - \frac{\mathrm{i}}{\omega_0}\left(|v_{11}|^2 + \frac{1}{6}|v_{02}|^2 - \frac{1}{2}v_{20}v_{11}\right). \quad (45)$$

However, the equation $\dot{\mathbf{y}} = \vec{\mathcal{F}}(\mathbf{y})$ rather than Eq. (20) was used at that in definitions (26)–(28) of structural constants $v_{\alpha\beta}$. In this equation, the deviation from the stationary state $\mathbf{X}_0$ was determined by equality (11) combined with the subsequent rotation

$$\mathbf{y} \equiv \mathcal{P}\mathbf{x}, \quad \mathbf{x} \equiv \mathbf{X} - \mathbf{X}_0. \tag{46}$$

The corresponding matrix

$$\widehat{\mathcal{P}} \equiv \begin{pmatrix} 1 & 0 \\ \frac{\Lambda_{11}}{\omega_0} & -\frac{\Lambda_{12}}{\omega_0} \end{pmatrix} \tag{47}$$

brings the complex force (21) to the form

$$\mathcal{F} \equiv \mathcal{P}F. \tag{48}$$

As a result, the criterion of limit cycle formation found in work [5] is expressed by rather a cumbersome inequality

$$2\omega_0 \left[\left(\mathcal{F}^{(1)}_{111} + \mathcal{F}^{(1)}_{122} \right) + \left(\mathcal{F}^{(2)}_{222} + \mathcal{F}^{(2)}_{112} \right) \right] <$$

$$< \left(\mathcal{F}^{(1)}_{11} + \mathcal{F}^{(1)}_{22} \right) \left(\mathcal{F}^{(2)}_{11} - \mathcal{F}^{(2)}_{22} - 2\mathcal{F}^{(1)}_{12} \right) +$$

$$+ \left(\mathcal{F}^{(2)}_{11} + \mathcal{F}^{(2)}_{22} \right) \left(\mathcal{F}^{(1)}_{11} - \mathcal{F}^{(1)}_{22} + 2\mathcal{F}^{(2)}_{12} \right). \tag{49}$$

Here, in contrast to expression (44), the subscripts denote the differentiation with respect to the components of the transformed coordinate, $\mathbf{y}$, rather than the initial one, $\mathbf{x}$.

5. Self-organized Modulation of the Lorentz System

To confirm the eligibility of criterion (44), let us analyze the Lorentz system, the behavior of which is determined by the evolution equations for the radiation field strength E, the medium polarization P, and the difference S between the level occupations [1]. In this case, the self-organized modulation mode is possible only provided that the equation for the field strength acquires a nonlinear contribution, which is caused by the dispersion of its relaxation time [6, 8]. As a result, the behavior of the system is described by the equations

$$\tau_E \dot{E} = -E + a_E P - \varphi(E), \ \varphi(E) \equiv \frac{\kappa E}{1 + E^2/E_\tau^2};$$

$$\tau_P \dot{P} = -P + a_P ES;$$

$$\tau_S \dot{S} = (S_e - S) - a_S EP. \tag{50}$$

Here, τ_E, τ_P, and τ_S are the variation scales for quantities indicated by the subscripts; a_E, a_P, and a_S are the corresponding positive coupling constants; $\kappa > 0$ is the parameter of non-linearity; E_τ is a typical strength value; and S_e is the pumping parameter. The quantities t, E, P, and S are measured in units of τ_E, $E_s = (a_P a_S)^{-1/2}$, $P_s = (a_E^2 a_P a_S)^{-1/2}$, and $S_s = (a_E a_P)^{-1}$, respectively. Then, Eqs. (50) look like

$$\dot{E} = -E + P - \varphi(E), \ \varphi(E) \equiv \frac{\kappa E}{1 + E^2/E_\tau^2};$$

$$\sigma \dot{P} = -P + ES,$$

$$\varepsilon \dot{S} = (S_e - S) - EP, \tag{51}$$

where the ratios $\sigma \equiv \tau_P/\tau_E$ and $\varepsilon \equiv \tau_S/\tau_E$ between characteristic times were introduced.

Provided that the adiabatic condition $\tau_P \ll \tau_E$ holds true, the left-hand side of the second equation in (51) can be put equal to zero, and the polarization is expressed by the equality $P = ES$. Substituting this relation in other equations in (51), we obtain the system

$$\dot{E} = -E(1 - S) - \varphi(E),$$

$$\dot{S} = \varepsilon^{-1} \left[S_e - S(1 + E^2) \right]. \tag{52}$$

Stationary states correspond to either a disordered, $(0, S_e)$, or ordered, (E_0, S_0), one, where

$$E_0 = E_{00} \left[1 + \sqrt{1 + \frac{E_\tau^2}{E_{00}^4}(S_e - S_c)} \right]^{1/2},$$

$$S_0 = \frac{(1 + E_{00}^2) - \sqrt{(1 + E_{00}^2)^2 - (1 - E_\tau^2)S_e}}{1 - E_\tau^2};$$

$$E_{00}^2 \equiv \frac{1}{2} \left[(S_e - 1) - (1 + \kappa) E_\tau^2 \right], \quad S_c \equiv 1 + \kappa. \tag{53}$$

If the external influence is confined by an interval (S_c^0, S_c) with the lower limit

$$S_c^0 = 1 + E_\tau^2(\kappa - 1) + 2E_\tau \sqrt{\kappa(1 - E_\tau^2)}, \tag{54}$$

the effective potential has a barrier, which gives rise to a discontinuous transformation; whereas, at the supercritical pumping, $S_e > S_c$, the phase transition becomes continuous. Such a scenario is implemented, if

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the parameter of non-linearity exceeds the value $\kappa_{\min} = E_\tau^2/(1-E_\tau^2)$.

The further analysis demonstrates that the behavior of the system is governed by the derivatives of the function $\varphi(E)$ in the first equation in (51):

$$\varphi'(E) = \kappa\frac{1-E^2/E_\tau^2}{(1+E^2/E_\tau^2)^2},$$

$$\varphi''(E) = -\frac{2\kappa}{E_\tau}\frac{(E/E_\tau)(3-E^2/E_\tau^2)}{(1+E^2/E_\tau^2)^3},$$

$$\varphi'''(E) = -\frac{6\kappa}{E_\tau^2}\frac{1-6(E/E_\tau)^2+(E/E_\tau)^4}{(1+E^2/E_\tau^2)^4}. \tag{55}$$

In so doing, the elements of the Jacobi matrix look like

$$\Lambda_{11} = S - \Lambda_c,\ \Lambda_c \equiv 1+\varphi'(E); \quad \Lambda_{12} = E;$$

$$\Lambda_{21} = -2\varepsilon^{-1}SE; \quad \Lambda_{22} = -\varepsilon^{-1}(1+E^2), \tag{56}$$

Here, the quantities E and S play the role of the variables X_1 and X_2; they should be put equal to 0 and S_e, respectively, in the disordered case and to (53) in the ordered one. In the former case,

$$\Lambda_{11} = S_e - S_c, \quad \Lambda_{12} = \Lambda_{21} = 0, \quad \Lambda_{22} = -\varepsilon^{-1}. \tag{57}$$

Equation (7) for eigenvalues gives the increment

$$\lambda = \frac{1}{2}\left[(S-\Lambda_c) - \varepsilon^{-1}\left(1+E^2\right)\right] \tag{58}$$

and the characteristic frequency

$$\omega = \frac{1}{2}\sqrt{8\varepsilon^{-1}E^2S - \left[(S-\Lambda_c)+\varepsilon^{-1}(1+E^2)\right]^2}. \tag{59}$$

The stationary state becomes unstable under the condition

$$\varepsilon(S-\Lambda_c) > 1+E^2, \tag{60}$$

and the oscillatory behavior appears if

$$8\varepsilon E^2 S > \left[\varepsilon(S-\Lambda_c)+(1+E^2)\right]^2. \tag{61}$$

For the disordered state $(0, S_e)$, condition (60) reads

$$S_e > S_c + \varepsilon^{-1}, \tag{62}$$

whereas condition (61) is satisfied. This means that the disordered state cannot generate a limit cycle, so that only the ordered state will be considered below.

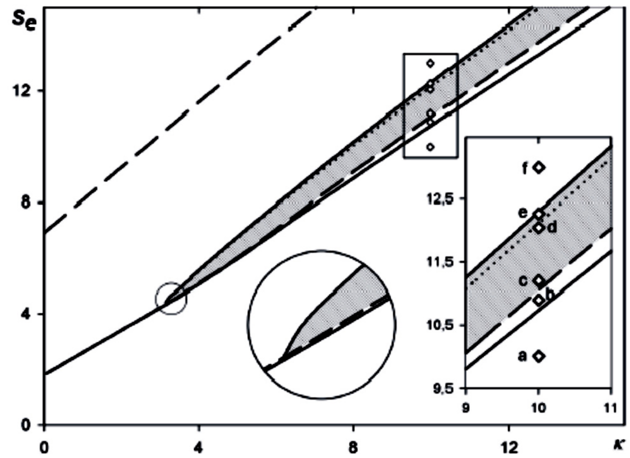

Fig. 1. Phase diagram which determines the parameter range for a stable limit cycle (hatched). The lower solid line is given by the condition $\Phi = 0$, the upper one by the equation $C(\lambda > 0) = 0$. Dashed lines correspond to the condition $\Im\omega = 0$, the dotted curve to the equation $\lambda = 0$

For the verification of the cycle creation criterion (44), let us list the nonzero derivatives of generalized forces included into it:

$$F_{11}^{(1)} = -\varphi''(E_0), \quad F_{12}^{(1)} = 1;$$

$$F_{11}^{(2)} = -2\varepsilon^{-1}S_0, \quad F_{12}^{(2)} = -2\varepsilon^{-1}E_0;$$

$$F_{111}^{(1)} = -\varphi'''(E_0); \quad F_{112}^{(2)} = -2\varepsilon^{-1}. \tag{63}$$

Then, taking Eq. (55) into account, inequality (44) is transformed into

$$3\frac{\varepsilon\kappa}{E_\tau^2}\frac{1-6(E_0/E_\tau)^2+(E_0/E_\tau)^4}{(1+E_0^2/E_\tau^2)^4} < 1. \tag{64}$$

The values of the parameters S_e and κ, at which this condition holds true, are shown in Fig. 1. It is evident that the limit cycle is formed at a substantial pumping S_e and large values of non-linearity parameter κ (according to the results of work [6], the growth of the strength E_τ eigenvalue expands the range of self-organized modulation). It is significant that the values of the parameters S_e, κ, and E_τ, at which the self-organized modulation mode is implemented, are not always determined by the condition $\Phi = 0$, which – in accordance with Eq. (64) – gives a criterion when the ordered state losses its stability (in Fig. 1, the lower continuous line corresponds to this condition). At large

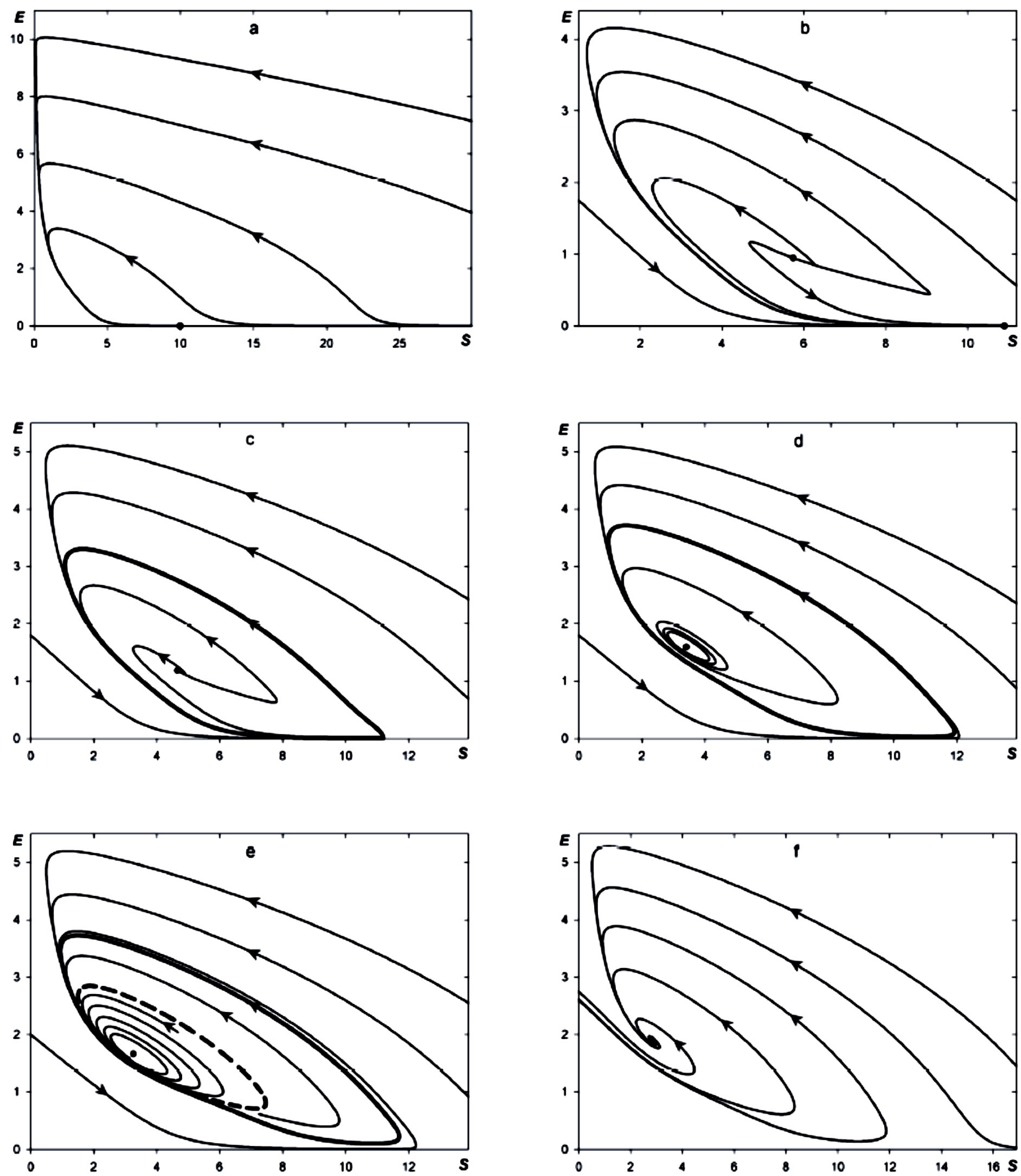

Fig. 2. Phase portraits of the system at points a to f in Fig. 1 (thick solid curves correspond to the stable limit cycle, the dashed curve to the unstable one)

κ-values, the lower boundary of the limit cycle creation (dashed line) is determined by the condition $\Im\omega = 0$. This provides the oscillatory behavior of the system with frequency (59). On the other hand, despite that the ordered state with the increment $\lambda > 0$ is stable, the self-organized modulation mode can be implemented in the range above the dotted line; the ordered system transits into this mode following the scenario of the first-kind phase transition. In this case, the upper boundary of the limit cycle is determined by the condition $C = 0$, provided that $\lambda > 0$, which is imposed upon the parameter of non-linearity (42) at a positive value of increment (58).

To confirm the validity of inequality (64), we numerically solved Eqs. (52) with the parameter values that correspond to points a to f in Fig. 1. The choice of these points is determined by the following conditions ($\kappa, E_\tau = \text{const}$):

point a: $\lambda < 0$, $\Im\omega \neq 0$, $\Phi > 0$;
point b: $\lambda > 0$, $\Im\omega \neq 0$, $\Phi < 0$;
point c: $\lambda > 0$, $\Im\omega = 0$, $\Phi < 0$;
point d: $\lambda > 0$, $\Im\omega = 0$, $\Phi < 0$;
point e: $\lambda < 0$, $\Im\omega = 0$, $C < 0$;
point f: $\lambda < 0$, $\Im\omega \neq 0$, $C > 0$.
According to Fig. 2, the limit cycle is formed at points c, d, and e, and is absent from states a, b, and f. In accordance with Fig. 1, this circumstance completely confirms the validity of criteria (44), (60), and (61) for the Lorentz system (51).

Now, consider condition (49) which was found in work [5]. According to Eqs. (46)–(48), its form is determined by transformed coordinates

$$y_1 = x_1, \quad y_2 = \frac{1}{\omega_0} \left(\Lambda_{11} x_1 - \Lambda_{12} x_2\right) \tag{65}$$

and forces

$$\mathcal{F}^{(1)} = F^{(1)}, \quad \mathcal{F}^{(2)} = \frac{1}{\omega_0} \left(\Lambda_{11} F^{(1)} - \Lambda_{12} F^{(2)}\right). \tag{66}$$

Such a relation between the initial coordinates (x_1, x_2) and forces $(F^{(1)}, F^{(2)})$ and the final coordinates (y_1, y_2) and forces $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})$ brings about too cumbersome expressions for the derivatives of generalized forces. For instance, the expression for $\mathcal{F}^{(2)}_{222}$ contains more than ten terms, none of them being simpler than fractions (55). If we substitute these derivatives into inequality (49), we get so cumbersome expressions that it is practically impossible for anybody to take advantage of them, even for the numerical researches of limit cycle formation conditions. It is clear that, at the passage to objects that are more complicated than the Lorentz system, the situation becomes even worse, so that the application of the compact criterion (44) seems more acceptable.

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УМОВИ САМООРГАНІЗОВАНОЇ МОДУЛЯЦІЇ

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Р е з ю м е

Досліджено умови народження граничного циклу, які забезпечують перехід нерівноважної системи у режим самоорганізованої модуляції. Викладено схему, використання якої дозволяє представити рівняння самоузгодженої еволюції пари дійсних змінних одним рівнянням руху комплексного параметра порядку. Знайдено оптимальний базис, у якому його еволюція описується рівнянням Гінзбурга–Ландау, що має тільки комплексну нелінійність. Визначено умови переходу у режим самоорганізованої модуляції.