# ROBUSTNESS OF NOISE-PRESENT BELL'S INEQUALITY VIOLATION BY ENTANGLED STATE

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UDC 537.311


The robustness of Bell's [in the Clauser–Horne–Shimony–Holt (CHSH) form] inequality violation for an entangled state under the simultaneous presence of colored and white noises in the system is studied. The two-photon polarization state is modeled by a two-parameter density matrix. By choosing the parameters, one can set a relative fraction of pure entangled Bell's state, as well as the fractions of white and colored noises. The analysis of the dependence of Bell's operator on the parameters is made. Computational results are compared with experimental data [9] and with those computed within the one-parameter density matrix [8] which is a special case of the model considered in this work.

## 1. Introduction

After the famous work by Einstein, Podolsky, and Rosen (EPR) [1], where they expressed the concept of quantum description incompleteness of the physical reality, there were numerous attempts to build more thorough theories which would not violate the causality principle in the classical meaning.

By accepting EPR as the working hypothesis, Bell formalized it [2] in the deterministic manner in terms of the local hidden variable model (LHVM) based on the following principles: 1) measurement results are determined by properties the particles possess prior to and regardless of the measurement ("realism"); 2) results obtained at one location are independent of any actions performed at spacelike separations ("locality"); 3) the local devices are independent of the hidden variables which determine the local results ("free will") [3].

Bell showed that the above assumptions impose some constraints on statistical correlations in experiments involving bipartite systems. Such constraints were formulated in the form of the nowadays well-known Bell's inequalities. Then Bell showed that the corresponding correlations, which one can obtain by quantum mechanical rules, violate these inequalities for some quantum mechanical states called entangled. In this way, the entanglement is such a feature of the quantum formalism that gives specific purely quantum correlations that can't be simulated within any classical model. Later on, Bell's inequalities were reformulated in the form suitable for experimental verification or confutation of their violation.

In 1982, Aspect's group [4] performed a verification experiment for the possible violation of Bell's inequalities in the CHSH form [5], where a correlation measurement of two-photon polarization states was executed. The measurement results corresponded well to the predictions of quantum mechanics. Experimental data give the Bell's inequality violation by five standard deviations. Numerous later experiments showed that their results are in adequacy with the quantum mechanical description of the nature.

Thus, specific quantum correlations got the reality status, and entangled states, which provide such correlations, became an object of intensive researches. It turned out that the entanglement can play the role of a basically new resource in such scientific fields as quantum cryptography, quantum teleportation, quantum communication, and quantum computation. This became a great stimulus for researching the methods of creation, accumulation, distribution, and broadcasting this resource.

## 2. Noise-Present Entanglement Detection

One of the most important questions in the topic under study concerns ways of the identification of the entanglement presence in one or another realistic quantum mechanical state. Since entangled states violate Bell's inequalities, thus, the violation of Bell's inequalities can be a basic tool to detect the entanglement. In realistic applications, pure entangled states become mixed states due to different types of noise. Thus, the question about the robustness of Bell's inequalities violation against the noise arises. In other words, one wants to know under which proportion of an entangled state and noise in a realistic mixed state the entanglement presence can be revealed.

For understanding the difference between *white and colored noise*, we will first recall some well-known features. In quantum theory, we assume that physical systems can be in a pure state or in a mixed state. Pure states are described by the state vectors (wave functions) $|\Psi\rangle$. Such a state can be written in the form of a coherent superposition of some set of states:

$$|\Psi\rangle = \alpha_1|\Psi_1\rangle + \alpha_2|\Psi_2\rangle + \ldots + \alpha_n|\Psi_n\rangle,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are some complex numbers. Thus, some phase relations exist between states $|\Psi_i\rangle$. A pure state counterpart in optics is the coherent light.

In contrast to a pure state, a mixed state can't be described by one state vector. A mixed state can be given, for example, by the set of states $|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle$, which enter the mixture with the corresponding weight coefficients $W_1, W_2, \ldots, W_n$, where $W_i$ are real nonnegative numbers and $\sum_i W_i = 1$. Therefore, there is no phase relation between states $|\Psi_i\rangle$ in this case. A mixed state counterpart in optics, generally speaking, is the partially coherent light.

The most economical way (without superfluous parameters) to represent mixed states is to use a density matrix. The density state operator, which is given by the set of $|\Psi_i\rangle$ and the corresponding numbers $W_i$, can be written as

$$\rho = \sum_i W_i|\Psi_i\rangle\langle\Psi_i|.$$

If a state of the physical system is represented in the $N$-dimensional space, then, by selecting some new complete set of $N$ orthonormal basis vectors $|\chi_\alpha\rangle$ in this space, one can build the density matrix for the given state:

$$\rho_{\alpha\beta} = \langle\chi_\alpha|\hat{\rho}|\chi_\beta\rangle.$$

The dimension of the density matrix depends only on the space dimension and doesn't depend on the number of functions $|\Psi_i\rangle$ which enter a mixture.

In the present work, we consider two-photon polarization states. The dimension of the corresponding space is $N = 4$. Thus, any polarization state (pure or mixed) can be represented in some basis in the form of a mixture of four orthonormal states $|\chi_\alpha\rangle$, $\alpha = \overline{1,4}$. If all $W_\alpha$, except one, are equal to zero, then the state is pure. If $W_1 = W_2 = W_3 = W_4 = \frac{1}{4}$, then the density matrix is proportional to the unit matrix: $\rho = \frac{1}{4}\hat{I}$. The state corresponding to this matrix is called *white noise*, since, first, there is no specific phase relation between the basic states and, second, all basic states have equal weight coefficients. The unit matrix is the same after any unitary transformation of the basic states.

Therefore, the density matrix of white noise doesn't depend on the selection of basis states.

From the viewpoint of quantum communication, the density matrix of white noise is a matrix which corresponds to the maximal entropy, and thus, it contains the minimal possible information.

Any state that is different from both a pure state and white noise can be called *colored noise*. Such a state is a noise, since it is not pure and, thus, is not completely coherent. This state is called colored, because the basic states in it have different weight coefficients, i.e. there is the preponderance of some basic states over another ones. We mention that the considered states are called a noise in those cases where they are added to some pure state.

The most reliable source of the two-particle entanglement is polarization-entangled photons created by the parametric down-conversion process (PDC) [6].

An entangled singlet two-photon state from the PDC process can be described as a spherically symmetric function which is one of the known Bell's states:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right), \tag{1}$$

where $|0\rangle$ and $|1\rangle$ are two mutually orthogonal photon polarization states. The density matrix for the two-photon state in the presence of white noise is the following:

$$\hat{\rho}_W = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\hat{I}, \tag{2}$$

where $\hat{I}$ is the $4 \times 4$ identity matrix. These states are called Werner states [7].

Usually, Werner states were used for the Bell's inequality violation test with polarization entangled photons from the PDC. But the experimental evidence and physical arguments show that the colorless noise model is not good for the description of states obtained in the PDC process. A more realistic description is given by the alternative one-parameter noise model, where the singlet state is mixed with decoherence terms which are called "colored noise" [8, 9]:

$$\hat{\rho}_C = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{2}(|01\rangle\langle01| + |10\rangle\langle10|), \qquad (3)$$

In the polarization density matrices (2) and (3), one can model different relative proportions for the pure entangled state $|\Psi^-\rangle$ and the noise by varying the parameter $p$ from 0 to 1.

Bell's inequality in the CHSH form looks as

$$|\beta| \le 2, \qquad (4)$$

where

$$\beta = -\langle A_0B_0\rangle - \langle A_0B_1\rangle - \langle A_1B_0\rangle + \langle A_1B_1\rangle \qquad (5)$$

is called the Bell operator.

To analyze the maximal violation of the Bell's inequality (4), the following one-qubit observables were taken separately for states with white (2) and colored (3) noises [8]:

$$A_0 = \sigma_z; \qquad (6)$$

$$A_1 = \cos(\theta)\sigma_z + \sin(\theta)\sigma_x; \qquad (7)$$

$$B_0 = \cos(\phi)\sigma_z + \sin(\phi)\sigma_x; \qquad (8)$$

$$B_1 = \cos(\phi-\theta)\sigma_z + \sin(\phi-\theta)\sigma_x. \qquad (9)$$

The parameters $\theta$ and $\phi$ in (6)–(9) are related to the orientation of analyzers in experimental devices, and $\sigma_x$ and $\sigma_z$ are the common Pauli matrices. Computations showed that, for the Werner state (2), the maximal value of $\beta$ as a function of the $p$ parameter is

$$\beta_{\max}(p) = 2\sqrt{2}p$$

and, for all values of $p$, is attained at $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4}$.

Thus, Bell's inequality (4) is violated only for $p > 1/\sqrt{2} \approx 0.707$. This implies that, in the case where the entangled state $|\Psi^-\rangle$ is distorted only by white noise, the entanglement presence can be detected if the fraction of noise is less than $\sim 29\%$.

In the case with colored noise (3), the maximal value of $\beta$ for different values $p$ is attained at different values of angles $\theta$ and $\phi$. The most interesting fact is that state (3) violates the CHSH inequality for all values $0 < p \le 1$. Thus, Bell's inequality violation is extremely robust against colored noise.

The experimental verification of the previously mentioned predictions concerning the CHSH inequality robustness against colored noise was carried in work [9]. Some crystal (beta-barium borate) was irradiated by a laser operating in the pulsed mode. As a result, photon pairs in polarization-correlated states were created in the PDC process. These states correspond to the following polarization density matrix:

$$\hat{\rho} = p|\Phi^+\rangle\langle\Phi^+| + \frac{1-p}{2}(|00\rangle\langle00| + |11\rangle\langle11|), \qquad (10)$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is one of four entangled Bell's states. The state $|1\rangle$ conforms to the ordinary ray polarization, and the state $|0\rangle$ does to the extraordinary ray polarization in the uniaxial crystal.

The one-qubit observables in the Bovino's experiment [9] satisfied expressions (6)–(9). Computations showed that the $p$-parameter dependence of the $\beta_{\max}$ value in state (10) is just the same as that in the state (3). The experimental setup made possible to regulate the fraction of colored noise, that is, the parameter $p$ was varied from zero to almost one. Particular cases in (10) are the pure state $|\Phi^+\rangle$ ($p = 1$) and just only noise ($p = 0$).

The comparison of the experimental values of $\beta_{\max}(p)$ and the theoretical predictions showed that the polarization state model (10) generally appropriately describes the two-photon state from the PDC process. But, for all $0 < p < 1$, experimental values of $\beta_{\max}$ were found to be a little smaller than the corresponding theoretical values. Due to this fact, in concordance with experimental data, the CHSH inequality is violated only for $p \gtrsim 0.2$, but not for all values of $p$, as follows from computations. The reason for such a discrepancy can be the presence of some portion of white noise besides colored noise in a realistic polarization state.

In the present work, we carry on the theoretical analysis for the robustness of Bell's inequality (in the CHSH form) violation with the simultaneous presence of colored and white noises. The density matrix for a

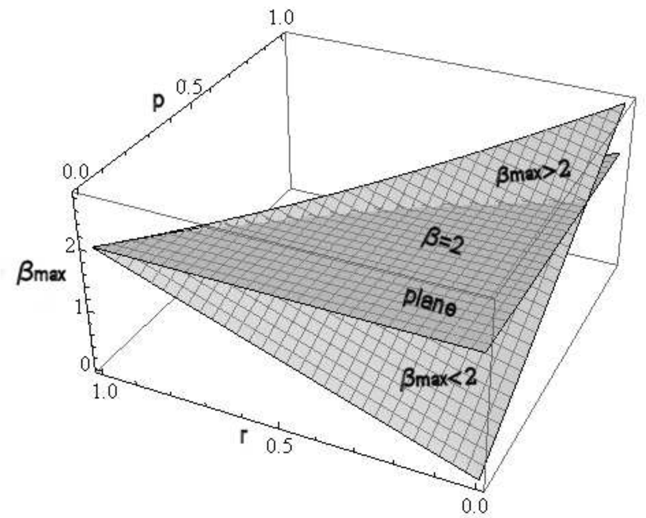


Fig. 1. 3D plot for the maximal Bell operator values and the $\beta_{CW}=2$ surface that corresponds to the classical limit

two-photon polarization state in such generalized model can be expressed in the form

$$\hat{\rho}_{CW}=p|\Phi^+\rangle\langle\Phi^+|+\frac{r}{2}(|00\rangle\langle 00|+|11\rangle\langle 11|)+$$

$$+\frac{1-(p+r)}{4}\hat{I}. \qquad (11)$$

Varying the parameter $p$ in the interval from 0 to 1, one can change the pure state $|\Phi^+\rangle$ fraction in (11), and, by setting $r$ from 0 to $(1-p)$ with a fixed $p$, one can change the relative fractions of colored and white noises. For $r=0$, we have the particular case (2) (colored noise absence), and, for $r=1-p$, we have (3) (white noise absence).

In state (11), the quantity $\beta$ which corresponds to the one-qubit observables (6)–(9) is a four-parameter function

$$\beta_{CW}(p,r,\theta,\phi)=\cos(\phi)[(2p+r)(\sin^2(\theta)+\cos(\theta))+$$

$$+r\cos(\theta)]-\sin(\phi)(2p+r)[\cos(\theta)-1]\sin(\theta). \qquad (12)$$

In the absence of colored noise ($r=0$), we have

$$\beta_W(p,\theta,\phi)=2p\{\cos(\phi)[\sin^2(\theta)+\cos(\theta)]-$$

$$-\sin(\phi)[\cos(\theta)-1]\sin(\theta)\},$$

whereas, in the absence of white noise ($r=1-p$),

$$\beta_C(p,\theta,\phi)=\cos(\phi)[(1+p)\sin^2(\theta)+2\cos(\theta)]-$$

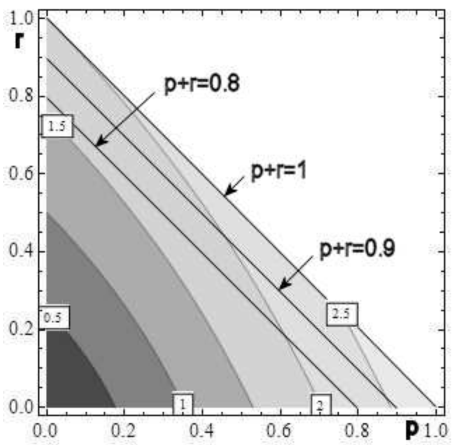


Fig. 2. Contour plot for $\beta^{\max}(p,r)$ =const – maximal Bell operator values on the coordinate plane $(p,r)$

$$-\sin(\phi)(1+p)[\cos(\theta)-1]\sin(\theta).$$

For fixed values of the parameters $p$ and $r$, expression (12) is a function of $\theta$ and $\phi$. Solving the problem on the extremum of a function of two variables, one can find the maximal values $\beta_{CW}^{\max}(p,r)$, as well as the angles $\theta$ and $\phi$ that ensure the maximal $\beta_{CW}(p,r)$.

In Fig. 1, the shaded surface graphically displays the function $\beta_{CW}^{\max}(p,r)$ of two variables $p$ and $r$. For comparison, we show the plane $\beta=2$ which is the limiting value of Bell's inequality. The surface part above the plane $\beta=2$ is the CHSH inequality violation area.

In Fig. 2, the projections of the lines of intersection of planes $\beta=$ const with the surface $\beta_{CW}^{\max}(p,r)$ on the $(p,r)$ plane are represented. From the figure, one can see that the straight line $p+r=1$ (white noise absence) fully lies in the $\beta^{\max}>2$ area, which corresponds to the above conclusion that Bell's inequality violation is robust against colored noise. For $r=0$ (colored noise absence), Bell's inequality is violated only for $p>1/\sqrt{2}$. For any fixed $p$ (pure entangled state weight factor), $\beta_{CW}^{\max}$ value decreases with increase in the fraction of white noise. Thus, as expected, by adding some amount of white noise to colored one, we can reach a better agreement of theoretically computed $\beta^{\max}$ with experimental ones. The Bell's inequality violation is unsteady under the increasing fraction of white noise at a fixed total amount of noise (white and colored).

In Fig. 3,*a* and 3,*b*, we show the surfaces which graphically display the angles $\theta(p,r)$ and $\phi(p,r)$ that ensure maximal values of $\beta$ as a function of $p$ and $r$.

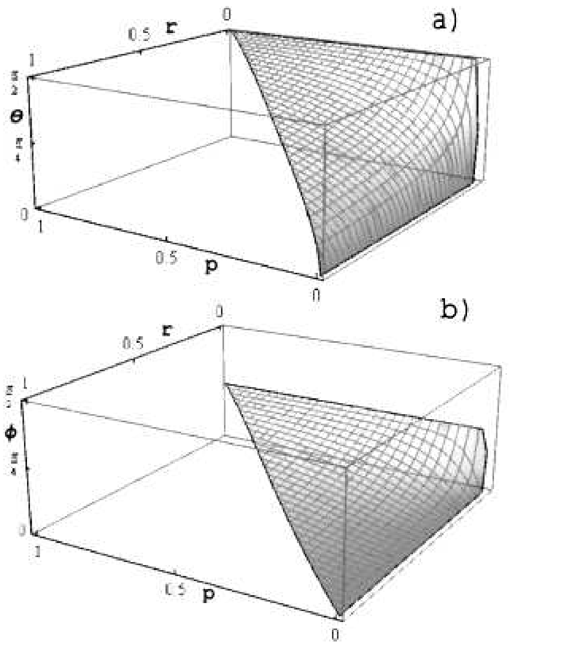


Fig. 3. Values of the parameters $\theta$ ($a$) and $\phi$ ($b$) that ensure the maximal values of $\beta_{CW}$ for the corresponding values of the parameters $p$ and $r$

In Fig. 4, we present the curves which are the lines of intersection of the vertical planes $r+p=1$ and $r=0$ with the surface $\beta_{CW}^{\max}(p,r)$ in the case where there is no white noise in the photon polarization state, and the dashed line corresponds to the case without colored noise. The plots of the limiting cases for the dependence of $\beta^{\max}$ on $p$ and $r$ coincide with those in work [8].

In Fig. 5, we show the values of the angles $\theta$ and $\phi$ that ensure maximal values of the Bell operator. Two solid curves correspond to the case where white noise is absent in the two-photon polarization state (11) ($p+r=1$), and two dashed lines correspond to the case where colored and white noises enter into expression (11) with the same weight $r=(1-p)/2$. Solid curves coincide with the plots in work [8]. From the figure, one can see that the values of the angles $\theta$ and $\phi$ for a fixed pure entangled state fraction ($p$ is constant) depend on the distribution of weighting coefficients of white and colored noises. Hence, in order to obtain the maximal values of $\beta$, the orientation of analyzers should depend on the distribution of noise between white and colored components.

In Fig. 6, the points represent the experimental maximal values of $\beta$ from work [9]; the dashed curve displays the theoretical predictions for the maximal values of $\beta$ within the one-parameter colored noise model

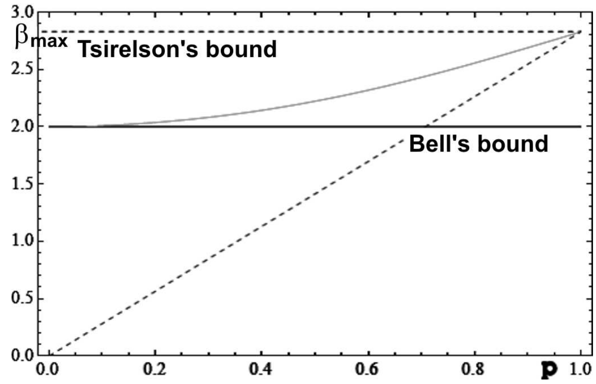


Fig. 4. Maximal Bell operator values in the case $r=1-p$ – no white noise (top curve) and $r=0$ – no colored noise (bottom dashed straight line). Classical bound is 2. Tsirelson's bound [10] is $2\sqrt{2}=2.83$

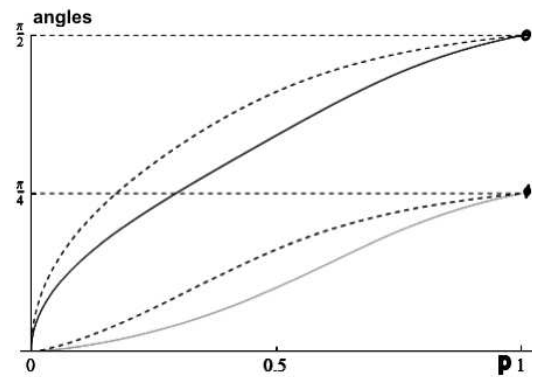


Fig. 5. The values of the parameters $\theta$ and $\phi$ that correspond to the maximal Bell operator values (two solid curves correspond to the case $r=1-p$ – no white noise; two dashed curves correspond to $r=(1-p)/2$ — equal weight coefficients for white and colored noises)

[8]; the solid curve shows the results of theoretical calculations within the two-parameter (generalized) noise model with a white noise fraction of 3.5% of the total noise amount in the system. In the figure, we can see that, for such a noise proportion, the experimental data better correspond to theoretical predictions, i.e. the generalized (two-parameter) noise model is more correct than the one-parameter one in the description of realistic states. It is seen that, in this case, some experimental points lie upper or lower than the theoretical curve. According to the two-parameter model, this is explained by the fact that, by moving from one point to another one, not only the total amount of noise in the system changes, but also the relative fractions of white and colored noises.

This kind of interpretation is absolutely logical, because, for each measurement, the experimental setup

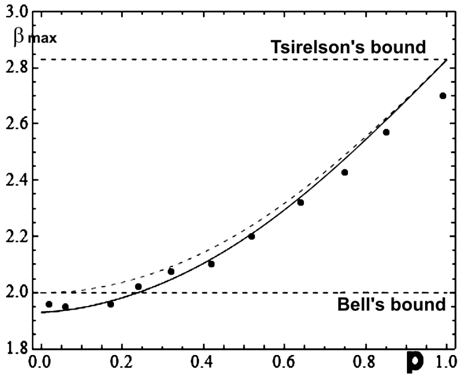


Fig. 6. Points present experimental maximal values of $\beta$ from work [9]; the dashed curve is the theoretical prediction for the maximal values of $\beta$ within the one-parameter colored noise model [8]; the solid curve shows theoretical calculations for the two-parameter (generalized) noise model with a white noise fraction of 3.5% of the total noise amount in the system

is tuned up in a new way (in particular, one has to change the orientation of analyzers in space). Remaining within the frame of the theoretical model which is considered in this work and choosing the corresponding parameter $r$ for each experimental point (for fixed $p$), one can fully conform the results of theoretical computations to the experimental data. We recall that the preselected values of the parameters $p$ and $r$, according to our model, determine the pure entangled state fraction and the relative noise fractions. The percentage for white and colored noise fractions that give coincidence between theoretical values $\beta_{\max}$ and experimental data, is represented in the Table. Experimental data was taken from figure from the work [9].

## 3. Conclusions

For the adequate modeling of the two-photon polarization state created in the parametric down-conversion process (PDC of type II), one should take the presence of colored noise, as well as white one, into account. While the Bell's inequality violation is extremely robust against colored noise (Bell's inequality is violated for all $0 < p \leq 1$), the violation is unsteady under white noise. The presence of white noise, which is characterized by the weighting coefficient of just 0.1 ($p + r = 0.9$), as one can see in Fig. 2, leads to the Bell's inequality violation only for $p \gtrsim 0.5$. Taking simultaneously both colored and white noises into account gives a possibility to conform the results of theoretical computations to experimental data. Taking $p$ and $r$ as adjustable parameters, one can determine the fractions of colored and white noises by comparing the results of theoretical calculations with experimental data. The best model is the one which explains the experimental values of $\beta^{\max}$, as well as the angles $\theta$ and $\phi$ that ensure these values.

**Table for noise proportions in the system. Correspondence to experimental points in Fig. 6**

| N | $p$ | $1-p$ | White, % | Colored, % | $r$ |
|---|---|---|---|---|---|
| 1 | 0.02 | 0.98 | 2 | 98 | 0.96 |
| 2 | 0.06 | 0.97 | 3 | 97 | 0.92 |
| 3 | 0.17 | 0.83 | 4 | 96 | 0.80 |
| 4 | 0.24 | 0.76 | 2 | 98 | 0.75 |
| 5 | 0.32 | 0.68 | 2 | 98 | 0.67 |
| 6 | 0.42 | 0.58 | 5 | 95 | 0.55 |
| 7 | 0.52 | 0.48 | 5 | 95 | 0.46 |
| 8 | 0.64 | 0.36 | 7 | 93 | 0.40 |
| 9 | 0.75 | 0.25 | 15 | 85 | 0.21 |
| 10 | 0.85 | 0.15 | 15 | 85 | 0.13 |

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Received 28.02.08

СТІЙКІСТЬ ПОРУШЕННЯ НЕРІВНОСТІ БЕЛЛА ЗАПЛУТАНИМ СТАНОМ ЩОДО КОЛЬОРОВОГО І БІЛОГО ШУМІВ

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Р е з ю м е

Досліджено стійкість порушення нерівності Белла (представленої в формі CHSH) заплутаним станом за одночасної наявності як кольорового, так і білого шуму в системі. Двофотонний поляризаційний стан моделюється двопараметричною матрицею густини. Вибором значень параметрів можна задавати відносну частку чистого заплутаного стану Белла, а також частки кольорового та білого шумів. Проведено аналіз залежності оператора Белла від параметрів. Результати обчислень порівнюються з експериментальними даними роботи [9] і з розрахунками на основі однопараметричної матриці густини [8], яка є частковим випадком моделі, розглянутої в даній роботі.