NEW MODELS OF A QUANTUM OSCILLATOR

A.U. KLIMYK

Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03680, Ukraine; e-mail: aklimyk@bitp.kiev.ua)

UDC 539.12, 517.984

We construct new models of a quantum oscillator. As in the case of the Macfarlane–Biedenharn q-oscillator, these models are related to q-Hermite polynomials. The position and momentum operators in our models are appropriate representation operators for the quantum algebra $\mathrm{su}_q(1,1)$. As in the case of the standard harmonic oscillator in quantum mechanics, the position and momentum operators have continuous simple spectra. These spectra cover a finite interval on the real line which depends on a value of q. Eigenfunctions of these operators are explicitly found. Contrary to the case of the Macfarlane–Biedenharn q-oscillator, the position and momentum operators Q and P of our models satisfy the quantum mechanics relations $[H,Q]=-\mathrm{i}P$ and $[H,P]=\mathrm{i}Q$.

1. Introduction

There exist many algebraic constructions which can be used for building up different models of quantum oscillators. They are constructed on a base of different associative algebras or their deformations. For most of them, it is difficult to construct a complete theory of such an oscillator: the spectra of observables, the explicit form of the eigenfunctions of observables, a description of time evolution, etc. Only for some of such models, it is possible to develop a corresponding theory. In [1] and [2], the so-called q-oscillator, which is a q-deformed analog of the standard linear harmonic oscillator in quantum mechanics, was constructed. A theory of this oscillator was elaborated in detail. There are physical problems for which this q-oscillator is more adequate than the quantum harmonic oscillator does (see, for example, [3] and [4]). Unlike the quantum field theory constructed on the base of the standard quantum harmonic oscillator, the quantum field theory built on the base of the q-oscillator is free of some divergences. This q-oscillator has many useful properties which are not inherent in the common quantum harmonic oscillator (see, for example, [5] and [6]).

However, in the case of the Biedenharn–Macfarlane q-oscillator, the basic commutation relations

$$[H,Q]=-\mathrm{i}P\,, \qquad [H,P]=\mathrm{i}Q\,, \tag{1}$$

are broken. That is why the q-oscillator is not so attractive for many physicists.

For the quantum oscillators constructed in this paper, relations (1) are satisfied. We build our oscillators on the base of irreducible representations of the quantum algebra $\mathrm{su}_q(1,1)$ with lowest weights. Our models of the quantum oscillator are models obeying the dynamics of the harmonic oscillator, with the position and momentum operators and Hamiltonian being functions of elements of the quantum algebra $\mathrm{su}_q(1,1)$. The aim of this paper is to develop the theory of these oscillators by using the theory of irreducible representations of $\mathrm{su}_q(1,1)$.

In order to derive the properties of oscillators under discussion, we essentially use the theory of special functions and q-orthogonal polynomials. Namely, using the interrelation between self-adjoint operators representable by Jacobi matrices (in our case, they are the position and momentum operators) and orthogonal polynomials, we find spectra of the position and momentum operators and derive an explicit form of their eigenfunctions. We derive an explicit form of the evolution operator in the coordinate space. It is given as an integral operator. A kernel of this operator is given explicitly.

We employ the standard notations of the theory of basic hypergeometric functions and q-orthogonal polynomials (see, for example, [7]). We will use q-numbers defined as

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}$$

for any complex number a. The same notation $[A]_q$ is used for operators.

Everywhere below, it is assumed that q is a fixed real number such that $0 < q < 1$. Each fixed value of q gives a model of the quantum oscillator.

We believe that our oscillators can be useful for the application to quantum systems in the non-commutative space-time (for which the "motion group" is the quantum group $SU_q(2)$) and to quantum systems with the quantum algebra $\mathrm{su}_q(1,1)$ describing their dynamical symmetry. These oscillators can be considered (along with the well-known q-oscillator) as new non-trivial deformations of the standard quantum harmonic oscillator.

2. Representations of $\mathrm{su}_q(1,1)$ with the Highest Weight

The quantum algebra $\mathrm{su}_q(1,1)$ is defined as the associative algebra generated by the elements J_+, J_-, J_3 satisfying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_-, J_+] = \frac{q^{J_3} - q^{-J_3}}{q^{1/2} - q^{-1/2}} \equiv [2J_3]_q.$$

(Observe that here we have replaced J_- by $-J_-$ in the usual definition of the algebra $\mathrm{sl}_q(2)$; see [8], Chapter 3.)

Introducing the elements $J_1 = \frac{1}{2}(J_+ + J_-)$ and $J_2 = \frac{1}{2\mathrm{i}}(J_+ - J_-)$, we characterize the algebra $\mathrm{su}_q(1,1)$ by the relations

$$[J_3, J_1] = \mathrm{i}J_2, \quad [J_2, J_3] = \mathrm{i}J_1, \quad [J_1, J_2] = -\frac{\mathrm{i}}{2}[2J_3]_q. \quad (2)$$

The Casimir element of the algebra $\mathrm{su}_q(1,1)$ is given by the formula

$$C_q := [J_3 - 1/2]_q^2 + \tfrac{1}{2}\,[\,2J_3\,]_q - J_1^2 - J_2^2 - 1/4.$$

We are interested in irreducible representations of $\mathrm{su}_q(1,1)$ with the lowest weights. These irreducible representations will be denoted by T_l, where l is a lowest weight which may be any complex number (see, e.g., [9]).

In order to realize these representations, we consider the space $\mathcal{P}$ of all polynomials in one variable y. We fix l and introduce the monomials

$$e_n^l \equiv e_n^l(y) := c_n^l\, y^n, \quad c_n^l = q^{(1-2l)n/4}\frac{(q^{2l};q)_n^{1/2}}{(q;q)_n^{1/2}}, \qquad (3)$$

$n = 0, 1, 2, 3, \cdots,$

where $(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1})$. They form a basis in $\mathcal{P}$. The representation T_l is then realized by the operators $J_3 = y\frac{d}{dy}+l$, $J_\pm = y^{\pm 1}\,[J_3 \pm l]_q$. In this explicit realization, one has $J_3\, e_n^l = (l+n)\, e_n^l$ and

$$J_+\, e_n^l = \sqrt{[2l+n]_q[n+1]_q}\, e_{n+1}^l,$$

$$J_-\, e_n^l = \sqrt{[2l+n-1]_q[n]_q}\, e_{n-1}^l,$$

The basis functions $e_n^l(y)$ are eigenfunctions of the operators J_3 and C_q: $C_q\, e_n^l = \left([l-1/2]_q^2 - \frac{1}{4}\right) e_n^l$.

We may introduce a scalar product into $\mathcal{P}$, considering that the monomials $e_n^l(y)$, $n = 0, 1, 2, \dots$, are orthonormal, $\langle e_m^l, e_n^l\rangle = \delta_{mn}$. Closing the space $\mathcal{P}$ with respect to this scalar product, we obtain a Hilbert space which will be denoted as $\mathcal{H}_l$.

3. Description of the Models

Our models of the quantum oscillator are based on a fixed irreducible representations of the algebra $\mathrm{su}_q(1,1)$ with a highest weight. In order to describe the models, we fix a highest weight

$$l = \frac{\mathrm{i}\pi}{2h} + \frac{1}{2},$$

where h is determined by q: $q = \exp h$, and consider the representation T_l from the previous section. In this case, $q^l = \mathrm{i}q^{1/2}$ and the basis elements (3) are of the form

$$e_n^l = (-\mathrm{i})^{n/2}\frac{(-q;q)_n^{1/2}}{(q;q)_n^{1/2}}y^n.$$

We define the Hamiltonian H and the position and momentum operators Q and P in terms of the generators J_1, J_2, J_3 of this representation as

$$Q = q^{J_3/4}J_2q^{J_3/4}, \quad P = q^{J_3/4}J_1q^{J_3/4}.$$

$$H = J_3 - l + 1/2, \qquad (4)$$

Then, due to (2), for Q, P, and H, we have the commutation relations

$$[H, Q] = -\mathrm{i}\,P\,, \qquad [H, P] = \mathrm{i}\,Q\,, \qquad (5)$$

$$[Q,P] = \tfrac{\mathrm{i}}{2}q^{J_3/2}(q^{-\frac{1}{2}}J_+J_- - q^{\frac{1}{2}}J_-J_+)q^{J_3/2} = \mathrm{i}\,F_q(C_q, J_3)$$

$$= \mathrm{i}\{\tfrac{1}{2}e^{2hJ_3}\coth\tfrac{h}{2} - e^{hJ_3}[(C_q+\tfrac{1}{4})\sinh\tfrac{h}{2} + \tfrac{1}{2}\operatorname{csch}\tfrac{h}{2}]\}, \quad (6)$$

where $q := \exp h$ (the expression for $[Q, P]$ is calculated in the same way as in [10]). The operator $F_q(C_q, J_3)$ defined in (6) commutes with J_3 and therefore is also diagonal in the basis (3); in the irreducible representation T_l, $l = (1 + i\pi/h)/2$, we have

$$F_q\, e^j_m = \frac{e^{mh}\cosh(j+\frac{1}{2})h - e^{2mh}\cosh\frac{h}{2}}{2\sinh\frac{h}{2}}\, e^j_m.$$

The basis e^l_n, $n = 0, 1, 2, \cdots$, of the Hilbert space $\mathcal{H}_l$ consists of eigenfunctions of the Hamiltonian H:

$$H\, e^l_n = (n + 1/2)\, e^l_n, \qquad n = 0, 1, 2, \cdots,$$

that is, the spectrum of H coincides with the spectrum of the Hamiltonian of the standard quantum harmonic oscillator. Our models are similar to but do not coincide with models in [10] and [11]. Moreover, our models are more simple.

The time evolution of our system is the harmonic motion with

$$e^{i\tau H}\begin{pmatrix} Q \\ P \end{pmatrix} e^{-i\tau H} = \begin{pmatrix} Q(\tau) \\ P(\tau) \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}.$$

This is a group $U(1)$ of inner automorphisms of the algebra $\mathrm{su}_q(1,1)$ and of rotations of the phase-space surface. We have

$$\exp(i\tau H) = e^{-i(l-1/2)\tau}\exp(i\tau J_3)\,. \qquad (7)$$

Explicit form of the time evolution in the coordinate space will be derived below.

4. Spectrum and Eigenfunctions of the Momentum Operator

Since $P = q^{J_3/4} J_1 q^{J_3/4}$, the direct calculation shows that the momentum operator P in the basis of the Hamiltonian eigenfunctions e^l_n, $n = 0, 1, 2, \cdots$, has the form

$$P\, e^l_n = \frac{1}{2}\left(\frac{1-q^{2(n+1)}}{q^{-1/2}-q^{1/2}}\, e^l_{n+1} + \frac{1-q^{2n}}{q^{-1/2}-q^{1/2}}\, e^l_{n-1}\right).$$

Let us find the spectrum and eigenfunctions of this operator. If $\psi_p(y)$ is an eigenfunction of P, corresponding to the eigenvalue p, $P\,\psi_p(y) = p\,\psi_p(y)$, then

$$\psi_p(y) = \sum_{n=0}^{\infty} h_n(p)\, e^l_n(y), \qquad (8)$$

where $h_n(p)$ are coefficients depending on p.

In order to find an explicit form of eigenfunctions $\psi_p(y)$, we substitute expression (8) for $\psi_p(y)$ into the equation $P\,\psi_p(y) = p\,\psi_p(y)$ and obtain the relation

$$\frac{1}{2}\sum_{n=0}^{\infty} h_n(p)\left(\frac{\sqrt{1-q^{2(n+1)}}}{q^{-1/2}-q^{1/2}}\, e^l_{n+1} + \frac{\sqrt{1-q^{2n}}}{q^{-1/2}-q^{1/2}}\, e^l_{n-1}\right) =$$

$$= p\sum_{n=0}^{\infty} h_n(p)\, e^l_n.$$

Equating the coefficients of a fixed basis element e^l_n, we obtain a three-term recurrence relation for the coefficients $h_n(p)$:

$$2p\, h_n(p) =$$

$$= \frac{\sqrt{1-q^{2(n+1)}}}{q^{-1/2}-q^{1/2}}\, h_{n+1}(p) + \frac{\sqrt{1-q^{2n}}}{q^{-1/2}-q^{1/2}}\, h_{n-1}(p). \qquad (9)$$

It is clear from (9) that the coefficients $h_n(p)$ are uniquely determined up to a common constant factor. Since $h_{-1}(p) = 0$, we see, by setting $h_0(p) = 1$, that $h_n(p)$, $n = 1, 2, \cdots$, are evaluated uniquely. Moreover, relation (9) shows that $h_n(p)$ are polynomials in p of degree n.

To solve the recurrence relation (9), we make the substitution

$$h_n(p) = (q^2; q^2)_n^{-1/2} h'_n(p).$$

Then (9) turns into the equality

$$2(q^{-\frac{1}{2}} - q^{\frac{1}{2}})\, p\, h'_n(p) = h'_{n+1}(p) + (1 - q^{2n})\, h'_{n-1}(p). \qquad (10)$$

Comparing this relation with the recurrence relation

$$2xH_m(z|q) = H_{m+1}(z|q) + (1 - q^m)H_{m-1}(z|q)$$

(see formula (3.26.3) in [12]) for the continuous q-Hermite polynomials

$$H_m(z|q) = e^{in\theta}\,{}_2\phi_0(q^{-m}, 0;\;\; q;^m e^{-2i\theta}), \qquad z = \cos\theta,$$

we find that

$$h'_n(p) = H_n((q^{-1/2} - q^{1/2})p|q^2),$$

where $\cos\theta = (q^{-1/2} - q^{1/2})p$ and ${}_2\phi_0\ (q^{-m}, ...)$ is the basic hypergeometric polynomial. Consequently, for the coefficients in (8), we obtain that

$$h_n(p) = (q^2; q^2)_n^{-1/2} H_n((q^{-1/2} - q^{1/2})p|q^2). \qquad (11)$$

Thus, the eigenfunctions of the momentum operator P are

$$\psi_p(y) = \sum_{n=0}^{\infty}(q^2; q^2)_n^{-1/2} H_n(\cos\theta|q^2)\, e^l_n(y) +$$

$$= \sum_{n=0}^{\infty} \frac{\mathrm{i}^{-n/2}}{(q;q)_n} H_n(p(q^{-1/2} - q^{1/2})|q^2)\, y^n, \qquad (12)$$

where expression (3) for the basis elements has been taken into account.

The above expression (12) for the eigenfunctions $\psi_p(y)$ can be summed up by using the method of derivation of the generating function (23) [13]. Making this derivation, we conclude that *the eigenfunctions of the momentum operator are of the form*

$$\psi_p(y) = \frac{((-\mathrm{i})^{1/2} q^{1/2} y; q)_\infty (-(-\mathrm{i}q)^{1/2} y; q)_\infty}{(\mathrm{i}^{-1/2} e^{\mathrm{i}\theta} y; q)_\infty (\mathrm{i}^{-1/2} e^{-\mathrm{i}\theta} y; q)_\infty}, \qquad (13)$$

where, as before, $p = \cos\theta/(q^{-1/2} - q^{1/2})$.

The spectrum of the momentum operator P can be found by means of formula (12). Indeed, it is easy to verify that the operator P, coinciding with the operator $q^{J_3/4} J_1 q^{J_3/4}$ of the representation T_l with a fixed value of l, is bounded and self-adjoint. Moreover, P is representable in the basis $\{e_n^l\}$ by a Jacobi matrix, that is, by a tridiagonal matrix of the form

$$M = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad a_i \neq 0.$$

There exists a theory (see [14], Chapter VII; a short description of this theory can be found in [6]) which allows to connect the spectra of operators of such a type with the orthogonality measures for appropriate orthogonal polynomials. To employ this theory, we note that the eigenfunctions $\psi_p(y)$ are expressed in terms of the basis elements e_n^l by formula (8) with the *polynomial* coefficients (11). According to the results of Chapter VII in [14], these polynomials are orthogonal with respect to some measure $d\mu(p)$. (This measure is unique, up to a constant factor, since the operator P is self-adjoint; see [6].) A set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator P and $d\mu(p)$ determines a spectral measure of this operator. Moreover, the spectrum of P is simple.

Thus, to find the spectrum of the momentum operator P, we recall that the continuous q-Hermite polynomials $H_n(z) \equiv H_n(z|q)$ are orthogonal and the orthogonality relation is of the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H_m(\cos\theta)\, H_n(\cos\theta)\, w(\cos\theta)\, d\theta = (q;q)_n \delta_{mn},$$

where $w(\cos\theta) = (q;q)_\infty \left|(e^{2\mathrm{i}\theta};q)_\infty\right|^2$ (see formula (3.26.2) in [12]). This orthogonality relation can be written for polynomials in relation (11) as

$$\frac{1}{2\pi} \int_{-b}^{b} H_m(p/b|q^2)\, H_n(p/b|q^2) \tilde{w}(p)\, dp = c_n \delta_{mn}, \qquad (14)$$

where $c_n = (q^2;q^2)_n$, $b := (q^{-1/2} - q^{1/2})^{-1}$ and $\tilde{w}(p) = (q^2;q^2)_\infty \left|(e^{2\mathrm{i}\theta};q^2)_\infty\right|^2 / b \sin\theta$. This means that the spectrum of P coincides with the finite interval,

$$\mathrm{Spec}\, P = [-b, b], \qquad b = (q^{-1/2} - q^{1/2})^{-1}.$$

Thus, the spectrum is continuous and simple. Continuity of the spectrum means that the eigenfunctions $\psi_p(y)$ are not elements of the Hilbert space $\mathcal{H}_l$. These functions of y for $p \in [-b, b]$ form a continuous basis in $\mathcal{H}_l$ (similar to the basis $\{e^{\mathrm{i}px}\}$ in the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions on $\mathbb{R}$).

Note that the spectrum of P tends to the infinite interval $(-\infty, \infty)$ when $q \to 1$. When $q \to 0$, the spectrum tends to the zero point.

Eigenfunctions of P are determined up to constant factors. In order to normalize the eigenfunctions $\psi_p(y)$, we take into account the orthogonality relation (14) for the continuous q-Hermite polynomials. Since these polynomials are associated with the determinate moment problem (see, for example, [6] for the description of this correspondence), the set $P_n(p/b)$, $n = 0, 1, 2, \cdots$, is complete in the Hilbert space $L^2([-b, b], \tilde{w})$ with the scalar product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_{-b}^{b} f_1(p)\, \overline{f_2(p)}\, \tilde{w}(p)\, dp, \qquad (15)$$

where b and $\tilde{w}(p)$ are the same as in (14). This means that

$$\sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n} \tilde{w}(p)\, H_n\left(p/b|q^2\right)\, H_n\left(p'/b|q^2\right) = \delta(p - p').$$

Then, due to (12), we get

$$\langle \psi_p(y), \psi_{p'}(y) \rangle = \frac{\delta(p - p')}{\tilde{w}(p)}.$$

Therefore, the normalized functions are

$$\tilde{\psi}_p(y) = \tilde{w}(p)^{1/2} \psi_p(y), \qquad p \in [-b, b],$$

that is, $\langle \tilde{\psi}_p(y), \tilde{\psi}_{p'}(y) \rangle = \delta(p - p')$.

5. Spectrum and Eigenfunctions of the Position Operator

The position operator Q in the basis e_n^l, $n = 0, 1, 2, \cdots$, has the form

$$Q\, e_n^l = \frac{1}{2\mathrm{i}} \left(\frac{1-q^{2(n+1)}}{q^{-1/2}-q^{1/2}}\, e_{n+1}^l - \frac{1-q^{2n}}{q^{-1/2}-q^{1/2}}\, e_{n-1}^l \right).$$

By changing the basis $\{e_n^l\}$ to the basis $\{\tilde{e}_n^l\}$, where $\tilde{e}_n^l = \mathrm{i}^{-n} e_n^l$, we see that the position operator Q is given in the latter basis by the same formula as the momentum operator is given in the former basis $\{e_n^l\}$ (see Section 4). This means that the spectrum of the operator Q coincides with the spectrum of P, that is,

$$\mathrm{Spec}\; Q = [-b, b], \qquad b = (q^{-1/2} - q^{1/2})^{-1}.$$

Eigenfunctions of the position operator can be found (by using the basis $\{\tilde{e}_n^l\}$) in the same way as in the case of the momentum operator. For this reason, we exhibit here only the result.

If $\phi_x(y)$ is an eigenfunction of Q, corresponding to the eigenvalue x, $Q\phi_x(y) = x\phi_x(y)$, then

$$\phi_x(y) = \sum_{n=0}^{\infty} \tilde{h}_n(x)\, e_n^l(y),$$

where, as before, $e_n^l(y)$ are given by (3) and $\tilde{h}_n(x)$ are coefficients depending on the eigenvalues x.

Repeating the reasoning of the previous section, we derive a three-term recurrence relation for the polynomials $\tilde{h}_n(x)$ and conclude that

$$\tilde{h}_n(x) = \mathrm{i}^{-n} h_n(x) = \mathrm{i}^{-n} (q^2; q^2)_n^{-1/2} H_n(x/b|q^2), \tag{16}$$

where $b = (q^{-1/2} - q^{1/2})^{-1}$ and $H_n(z|q)$ is the continuous q-Hermite polynomial from Section 4. Thus, eigenfunctions of the position operator Q are of the form

$$\phi_x(y) = \sum_{n=0}^{\infty} \frac{\mathrm{i}^{-n}}{(q^2;q^2)_n^{1/2}} H_n(\cos\theta|q^2)\, e_n^l(y) =$$

$$= \sum_{n=0}^{\infty} \frac{\mathrm{i}^{-3n/2}}{(q;q)_n} H_n(x/b|q^2)\, y^n. \tag{17}$$

One can sum up expression (17) for the eigenfunctions $\phi_x(y)$ by the same method as in the case of functions (13). *Eigenfunctions of the position operator Q are of the form*

$$\phi_x(y) = \frac{((\mathrm{i}q)^{1/2}y;q)_\infty(-(\mathrm{i}q)^{1/2}y;q)_\infty}{(-\mathrm{i}^{1/2}e^{\mathrm{i}\theta}y;q)_\infty(-\mathrm{i}^{1/2}e^{-\mathrm{i}\theta}y;q)_\infty},$$

where $x = b\cos\theta$ with $b = (q^{-1/2} - q^{1/2})^{-1}$.

Eigenfunctions of the operator Q are determined up to constant factors. To normalize the eigenfunctions $\phi_x(y)$, we employ the orthogonality relation (14) for continuous q-Hermite polynomials. The set $H_n(x/b)$, $n = 0, 1, 2, \cdots$, is complete in the Hilbert space $L^2([-b, b], \tilde{w}(x))$, $b = (q^{-1/2} - q^{1/2})^{-1}$, with the scalar product

$$\langle f_1, f_2\rangle = \frac{1}{2\pi} \int_{-b}^{b} f_1(x)\, \overline{f_2(x)}\, \tilde{w}(x)\, dx,$$

where $\tilde{w}$ is the same as in (14). Consequently, the normalized functions are

$$\tilde{\phi}_x(y) = \tilde{w}(x)^{1/2}\phi_x(y), \qquad x \in [-b, b],$$

that is, $\langle\tilde{\phi}_x(y), \tilde{\phi}_{x'}(y)\rangle = \delta(x - x')$.

6. Momentum Realization of the Oscillator

In Section 3, we have constructed a realization of the oscillator on the space of functions of the supplementary variable y. It is natural to look for its realization on the space of functions of the coordinate x and on the space of functions of the momentum p.

Let $L^2([-b, b], \tilde{w})$, $b = (q^{-1/2} - q^{1/2})^{-1}$, be the space of square-integrable functions $f(p)$ (where p is the momentum of the oscillator) with respect to the scalar product (15). It is clearly seen from (14) that polynomials (11) constitute an orthonormal basis in $L^2([-b, b], \tilde{w})$.

First, we construct a one-to-one linear isometry Ω from the Hilbert space $\mathcal{H}_l$, considered in Section 2, onto the Hilbert space $L^2([-b, b], \tilde{w})$ given by the formula

$$\Omega:\ \mathcal{H}_l \ni e(y) \to f(p) = \langle e(y), \psi_p(y)\rangle_{\mathcal{H}_l} \in L^2([-b, b], \tilde{w}),$$

where $\psi_p(y)$ are eigenfunctions (13) of P. It follows from (12) that

$$\mathcal{H}_l \ni e_n^l(y) \to \langle e_n^l(y), \psi_p(y)\rangle_{\mathcal{H}_l} = h_n(p), \tag{18}$$

that is, Ω maps the orthonormal basis $\{e_n^l(y)\}$ of $\mathcal{H}_l$ onto the orthonormal basis $\{h_n(p)\}$ in $L^2([-b, b], \tilde{w})$. This means that Ω is, indeed, a one-to-one isometry.

The operator P acts on $L^2([-b, b], \tilde{w})$ as the multiplication operator,

$$P\, f(p) = p\, f(p).$$

Indeed, according to formula (18), if $\Omega e(y) = f(p) = \langle e(y), \psi_p(y)\rangle_{\mathcal{H}_l}$, then

$$Pe(y) \to Pf(p) = \langle Pe(y), \psi_p(y)\rangle_{\mathcal{H}_l} =$$

$$= \langle e(y), P\psi_p(y)\rangle_{\mathcal{H}_l} = \langle e(y), p\psi_p(y)\rangle_{\mathcal{H}_l} = p\, f(p).$$

We can find how P acts upon the basis elements $h_n(p)$, $n = 0, 1, 2, \ldots$, of the Hilbert space $L^2([-b, b], \tilde{w})$. According to the recurrence relation for polynomials (11) (which follows from the recurrence relation for the polynomials $H_n(z|q)$), we have, for $P\, h_n(p) = p\, h_n(p)$, the expression

$$Ph_n = \frac{1}{2}\left(\frac{1-q^{2(n+1)}}{q^{-1/2}-q^{1/2}}\, h_{n+1} + \frac{1-q^{2n}}{q^{-1/2}-q^{1/2}}\, h_{n-1}\right),$$

where $h_n \equiv h_n(p)$.

The Hamiltonian H acts upon the polynomials $h_n(p)$ in the Hilbert space $L^2([-b, b], \tilde{w})$ as

$$H\, h_n(p) = (n + 1/2)\, h_n(p).$$

Indeed, since $He^l_n(y) = (n+1/2)e^l_n(y)$, we have, according to (18),

$$Hh_n(p) = \langle He^l_n(y), \psi_p(y)\rangle_{\mathcal{H}_l} =$$

$$= (n + 1/2)\langle e^l_n(y), \psi_p(y)\rangle_{\mathcal{H}_l} = (n + 1/2)\, h_n(p).$$

Let us find how the position operator Q acts on the Hilbert space $L^2([-b, b], \tilde{w})$. To achieve this, we use the results of [15].

The polynomials $h_n(p)$ from (11) can be expressed in terms of the Askey–Wilson polynomials defined as

$$p_n\left(\tfrac{1}{2}(z + z^{-1}); a, b, c, d|q\right) \equiv p_n[z] =$$

$$= \frac{(ab, ac, ad; q)_n}{a^n}\, {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}\,\right|\, q, q\right),$$

where $(\alpha, \beta, \gamma; q)_n := (\alpha; q)_n(\beta; q)_n(\gamma; q)_n$ (see [7], Section 7.5). We have

$$h_n(p) = c_n p_n(\tfrac{1}{2}(z+z^{-1}); q^{1/2}, -q^{1/2}, 0, 0|\, q), \qquad (19)$$

where $c_n = (q^2; q^2)_n^{-1/2}$ and $z = e^{\mathrm{i}\theta}$. For convenience, we denote the polynomials $h_n(p)$ by $h_n[z]$, where z is the same as in (19). It follows from (19) and from formula (4.5) in [15] that the polynomials $h_n[z]$ satisfy the difference equation

$$D\, h_n[z] = \frac{2(q^{-n} - 1)}{1 - q^{-1}}\, h_n[z],$$

where the difference operator D acts as

$$D\, f(z) = \frac{1 + qz}{(1 - z^2)(1 - qz^2)} f(qz) -$$

$$-\left(\frac{1+qz}{(1-z^2)(1-qz^2)} + \frac{1+qz^{-1}}{(1-z^{-2})(1-qz^{-2})}\right) f(z) +$$

$$+\frac{1 + qz^{-1}}{(1 - z^{-2})(1 - qz^{-2})} f(q^{-1}z).$$

We also need an operator of the form

$$D' := \frac{1}{2}(1 - q^{-1})D + 1\,,$$

whose action on the polynomials $h_n[z]$ is

$$D'\, h_n[z] = q^{-n}\, h_n[z].$$

This means that the operator D' acts on the polynomials $h_n[z]$ as q^{-N}, where N is the number operator,

$$N\, h_n[z] = n\, h_n[z]\,.$$

In order to find a difference form for the position operator, we use the operator L which is defined as

$$L\, f(z) = \frac{(z^{-2} + q)}{z - z^{-1}} f(qz) - \frac{(z^2 + q)}{z - z^{-1}} f(q^{-1}z).$$

This is operator (4.7) in [15] for our case. Then, from formulas (4.11) and (4.12) in [15], we get

$$\left(L - q^{1-n}(z+z^{-1})\right) h_n[z] = -d_n\sqrt{1-q^{2(n+1)}}\, h_{n+1}[z], \qquad (20)$$

$$\left(L + q^{-n}(z+z^{-1})\right) h_n[z] = d_n\sqrt{1-q^{2n}}\, h_{n-1}[z], \qquad (21)$$

where $d_n = (1 + q)/q^n$. One may express the position operator $Q = q^{J_3/4} J_2\, q^{J_3/4}$ in terms of the difference operators (20) and (21) as

$$Q = \mathrm{i}\frac{q^{1/2}}{2(1-q^2)}\left[2L + (1-q)(z+z^{-1})q^{-N}\right] q^N,$$

which can be represented in the form

$$Q = \mathrm{i}\frac{q^{1/2}}{2(1 - q^2)}\left[2Lq^N + (1 - q)(z + z^{-1})\right]. \qquad (22)$$

Thus, formula (22) gives us the difference form of the position operator.

7. Coordinate Realization of the Oscillator

Let $\tilde{L}^2([-b,b],\tilde{w})$, $b=(q^{-1/2}-q^{1/2})^{-1}$, be the space of square-integrable functions $f(x)$ (where x is the coordinate of the oscillator) with respect to the same scalar product as in (15). It follows from (14) that the polynomials $\tilde{h}_n(x)$ from (16) constitute an orthonormal basis in $\tilde{L}^2([-b,b],\tilde{w})$.

We construct a one-to-one linear isometry $\tilde{\Omega}$ from the Hilbert space $\mathcal{H}_l$, considered in Section 2, onto the Hilbert space $\tilde{L}^2([-b,b],\tilde{w})$ given by the formula

$$\tilde{\Omega}:\ \mathcal{H}_l\ni e(y)\to f(x):=\langle e(y),\phi_x(y)\rangle_{\mathcal{H}_l}\in\tilde{L}^2([-b,b],\tilde{w}),$$

where $\phi_x(y)$ are eigenfunctions (17) of Q. It is evident from (17) that

$$\mathcal{H}_l\ni e^l_n(y)\to\langle e^l_n(y),\phi_x(y)\rangle_{\mathcal{H}_l}=\tilde{h}_n(x),$$

that is, $\tilde{\Omega}$ maps the orthonormal basis $\{e^l_n(y)\}$ from $\mathcal{H}_l$ onto the orthonormal basis $\{\tilde{h}_n(x)\}$ in $\tilde{L}^2([-b,b],\tilde{w})$. This means that $\tilde{\Omega}$ is, indeed, a one-to-one isometry.

The operator Q acts on $\tilde{L}^2([-b,b],\tilde{w})$ as the multiplication operator,

$$Q\,f(x)=x\,f(x).$$

We can find how Q acts upon the basis elements $\tilde{h}_n(x)$, $n=0,1,2,\cdots$, in the Hilbert space $\tilde{L}^2([-b,b],\tilde{w})$. According to the recurrence relation for polynomials (16), we have, for $Q\tilde{h}_n(x)=x\tilde{h}_n(x)$, the expression

$$Q\tilde{h}_n=\frac{1}{2\mathrm{i}}\left[\frac{1-q^{2(n+1)}}{q^{-1/2}-q^{1/2}}\,\tilde{h}_{n+1}-\frac{1-q^{2n}}{q^{-1/2}-q^{1/2}}\,\tilde{h}_{n-1}\right],$$

where $h_n\equiv h_n(x)$. Clearly, $H\,\tilde{h}_n(x)=(n+1/2)\,\tilde{h}_n(x)$.

One can also find a difference form for the momentum operator P in the coordinate space by repeating the reasoning of the previous section.

8. Evolution Operator in the Coordinate Space

According to (7), the time evolution operator $\exp(\mathrm{i}\tau H)$ acts upon the basis elements e^l_n, $n=0,1,2,\cdots$, of the Hilbert space $\mathcal{H}_l$ as

$$e^{\mathrm{i}\tau H}e^l_n=e^{-\mathrm{i}(l-1/2)\tau}\,e^{\mathrm{i}(l+n)\tau}e^l_n=e^{\mathrm{i}(n+1/2)\tau}\,e^l_n.$$

We wish to find how this operator acts in the coordinate space, that is, on the Hilbert space $\tilde{L}^2([-b,b],\tilde{w})$ from Section 7. If the isometry $\tilde{\Omega}$ maps a function $e(y)\in\mathcal{H}_l$ to a function $f(x)\in\tilde{L}^2([-b,b],\tilde{w})$, then $\exp(\mathrm{i}\tau H)\,e(y)\in\mathcal{H}_l$ corresponds to the function

$$e^{\mathrm{i}\tau H}f(x)=\langle e^{\mathrm{i}\tau H}e(y),\phi_x(y)\rangle_{\mathcal{H}_l}=\langle e(y),e^{-\mathrm{i}\tau H}\phi_x(y)\rangle_{\mathcal{H}_l}$$

$$=\sum_{n=0}^{\infty}\langle e(y),e^l_n\rangle_{\mathcal{H}_l}\langle e^l_n,e^{-\mathrm{i}\tau H}\phi_x(y)\rangle_{\mathcal{H}_l}=$$

$$=\sum_{n=0}^{\infty}\langle e(y),e^l_n\rangle_{\mathcal{H}_l}\langle e^{\mathrm{i}\tau H}e^l_n,\phi_x(y)\rangle_{\mathcal{H}_l}=$$

$$=\sum_{n=0}^{\infty}\frac{1}{2\pi}\int_{-b}^{b}\langle e(y),\tilde{\phi}_{x'}(y)\rangle_{\mathcal{H}_l}\langle\tilde{\phi}_{x'}(y),e^l_n\rangle_{\mathcal{H}_l}\,dx'\times$$

$$\times e^{\mathrm{i}\tau(n+1/2)}\langle e^l_n,\phi_x(y)\rangle_{\mathcal{H}_l}=$$

$$=\frac{1}{2\pi}\int_{-b}^{b}f(x')K^\tau(x,x')\tilde{w}(x')\,dx',$$

where the kernel $K^\tau(x,x')$ is given by

$$K^\tau(x,x')=\sum_{n=0}^{\infty}\langle\phi_{x'}(y),e^l_n\rangle_{\mathcal{H}_l}\,\langle e^l_n,\phi_x(y)\rangle_{\mathcal{H}_l}e^{\mathrm{i}\tau(n+1/2)}.$$

Taking into account the expression for $\tilde{h}_n(x)=\langle e^l_n,\phi_x(y)\rangle_{\mathcal{H}_l}$ with $x=b\cos\theta$, we obtain

$$K^\tau(x,x')=e^{\mathrm{i}\tau/2}\sum_{n=0}^{\infty}e^{\mathrm{i}n\tau}\,q^{-n}\,\frac{(-q;q)_n}{(q;q)_n}\times$$

$$\times{}_3\phi_2(q^{-n},q^{1/2}e^{\mathrm{i}\theta},q^{1/2}e^{-\mathrm{i}\theta};\ -q,0;q,q)\times$$

$$\times{}_3\phi_2(q^{-n},q^{1/2}e^{\mathrm{i}\theta'},q^{1/2}e^{-\mathrm{i}\theta'};\ -q,0;q,q).$$

Here, the explicit expression for the continuous q-Hermite polynomials from Section 4 was taken into account.

We can derive an explicit expression for the kernel $K^\tau(x,x')$. Due to formula (8.15) in [16], one calculates that $K^\tau(x,x')$ is of the form

$$K^\tau(x,x')=e^{\mathrm{i}\tau/2}\times$$

$$\frac{(ae^{\mathrm{i}\theta'}e^{\mathrm{i}\tau},ae^{-\mathrm{i}\theta'}e^{\mathrm{i}\tau},ae^{\mathrm{i}\theta}e^{\mathrm{i}\tau},ae^{-\mathrm{i}\theta}e^{\mathrm{i}\tau},-e^{\mathrm{i}\tau};q)_\infty}{(e^{\mathrm{i}(\theta+\theta')}e^{\mathrm{i}\tau},e^{\mathrm{i}(\theta'-\theta)}e^{\mathrm{i}\tau},e^{\mathrm{i}(\theta-\theta')}e^{\mathrm{i}\tau},e^{-\mathrm{i}(\theta+\theta')}e^{\mathrm{i}\tau},qe^{\mathrm{i}\tau};q)_\infty}$$

$$\times{}_8W_7(e^{\mathrm{i}\tau};\ ae^{\mathrm{i}\theta},ae^{-\mathrm{i}\theta},ae^{\mathrm{i}\theta'},ae^{-\mathrm{i}\theta'},-e^{\mathrm{i}\tau};\ q,\ -e^{\mathrm{i}\tau}),\quad(23)$$

where $(d_1,\cdots,d_r;q)_\infty\equiv(d_1;q)_\infty\cdots(d_r;q)_\infty$, $a=q^{1/2}$, and ${}_8W_7$ is the basic hypergeometric function (2.1.11) from [7]. Expressing the function ${}_8W_7$ in (23) in terms of the basic hypergeometric function ${}_8\phi_7$ (see [7], Section 2.1) and using relation (III.17) from Appendix III in [7], one can reduce ${}_8W_7$ in (23) to the basic hypergeometric function ${}_4\phi_3$:

$${}_8W_7(e^{\mathrm{i}\tau};\ ae^{\mathrm{i}\theta}, ae^{-\mathrm{i}\theta}, ae^{\mathrm{i}\theta'}, ae^{-\mathrm{i}\theta'}, -e^{\mathrm{i}\tau};\ q,\ -e^{\mathrm{i}\tau}) =$$

$$= \frac{(qe^{\mathrm{i}\tau}, e^{\mathrm{i}\tau}, -q^{1/2}e^{-\mathrm{i}\theta'}, -q^{1/2}e^{\mathrm{i}\theta'}; q)_\infty}{(q^{1/2}e^{-\mathrm{i}\theta'}e^{\mathrm{i}\tau}, q^{1/2}e^{\mathrm{i}\theta'}e^{\mathrm{i}\tau}, -q, -1; q)_\infty} \times$$

$$\times {}_4\phi_3\left(\begin{matrix} e^{\mathrm{i}\tau}, -e^{\mathrm{i}\tau}, q^{1/2}e^{\mathrm{i}\theta'}, q^{1/2}e^{-\mathrm{i}\theta'} \\ q^{1/2}e^{-\mathrm{i}\theta}e^{\mathrm{i}\tau}, q^{1/2}e^{\mathrm{i}\theta}e^{\mathrm{i}\tau}, -q \end{matrix} \middle|\, q, q\right).$$

As a result, we arrive at the following expression for the kernel $K^\tau(x, x')$:

$$K^\tau(x, x') =$$

$$= c\frac{(ae^{\mathrm{i}\theta}e^{\mathrm{i}\tau}, ae^{-\mathrm{i}\theta}e^{\mathrm{i}\tau}, e^{\mathrm{i}\tau}, -e^{\mathrm{i}\tau}, ae^{-\mathrm{i}\theta'}, -q^{1/2}e^{\mathrm{i}\theta'}; q)_\infty}{(e^{\mathrm{i}(\theta+\theta')}e^{\mathrm{i}\tau}, e^{\mathrm{i}(\theta'-\theta)}e^{\mathrm{i}\tau}, e^{\mathrm{i}(\theta-\theta')}e^{\mathrm{i}\tau}, e^{-\mathrm{i}(\theta+\theta')}e^{\mathrm{i}\tau}; q)_\infty} \times$$

$$\times e^{\mathrm{i}\tau/2}\, {}_4\phi_3\left(\begin{matrix} e^{\mathrm{i}\tau}, -e^{\mathrm{i}\tau}, q^{1/2}e^{\mathrm{i}\theta'}, q^{1/2}e^{-\mathrm{i}\theta'} \\ q^{1/2}e^{-\mathrm{i}\theta}e^{\mathrm{i}\tau}, q^{1/2}e^{\mathrm{i}\theta}e^{\mathrm{i}\tau}, -q \end{matrix} \middle|\, q, q\right), \qquad (24)$$

where $a = q^{1/2}$ and $c = (-1, -q; q)_\infty^{-1}$. Thus, *the evolution operator* $\exp(\mathrm{i}\tau H)$ *is a kernel operator given by the formula*

$$\exp(\mathrm{i}\tau H)\ f(x) = \frac{1}{2\pi}\int_{-b}^{b} K^\tau(x, x') f(x')\, \tilde{w}(x')\, dx',$$

where the kernel $K^\tau(x, x')$ *is given by* (24). Since $e^{\mathrm{i}\tau H}e^{\mathrm{i}\tau' H} = e^{\mathrm{i}(\tau+\tau')H}$, this kernel satisfies the relation

$$\frac{1}{2\pi}\int_{-b}^{b} K^\tau(x, x')\overline{K^\tau(x', x'')}\, \tilde{w}(x')\, dx' = K^{\tau+\tau'}(x, x'').$$

Observe that this relation leads to the corresponding integral relation for the basic hypergeometric function ${}_4\phi_3$ from relation (24).

Formula (24) gives a possibility to construct a transition from the coordinate space to the momentum space. Such a transition is fulfilled by an integral operator. The kernel of such an operator is a particular case of kernel (24) (see [10] for details).

9. Concluding Remarks

We have constructed new models of the quantum oscillator which are related to continuous q-Hermite polynomials (that is, the models can be realized on the bases of the coordinate and momentum Hilbert spaces expressed in terms of these q-orthogonal polynomials). Remark that the Macfarlane–Biedenharn q-oscillator is also related to these q-Hermite polynomials. This means that our models and the Macfarlane–Biedenharn q-oscillator in real are q-deformations of the standard harmonic quantum oscillator. A characteristic peculiarity of our models is that the spectrum of the position operator covers a finite interval of the real line. Note that there exist the systems in quantum physics with this property.

Our models may be useful for the description of quantum systems in a non-commutative space-time (for which a "motion group" is one of the quantum groups $SU_q(2)$ and $SL_q(2,\mathbb{R})$) and of quantum systems with the quantum algebra $\mathrm{su}_q(1,1)$ describing their dynamical symmetry. Principles of these applications are the same as in the case of the Biedenharn–Macfarlane q-oscillator.

This research was partially supported by Grant 14.01/016 of the State Foundation of Fundamental Research of Ukraine.

1. A.J. Macfarlane, J. Phys. A **22**, 4581 (1989).
2. L.C. Biedenharn, J. Phys. A **22**, L873 (1989).
3. D.V. Anchishkin, A.M. Gavrilik, and N.Z. Iorgov, Mod. Phys. Lett. A **15**, 1637 (2000).
4. L.V. Adamska and A.M. Gavrilik, J. Phys. A **37**, 4787 (2004).
5. E.V. Damaskinsky and P.P. Kulish, Zap. Nauch. Sem. LOMI **189**, 37 (1991).
6. A.U. Klimyk, J. Phys. A **38**, 4447 (2005).
7. G. Gasper and M. Rahman, *Basic Hypergeometric Functions* (Cambridge University Press, Cambridge, 2004).
8. A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations* (Springer, Berlin, 1997).
9. I.M. Burban and A.U. Klimyk, J. Phys. A **26**, 2139 (1993).
10. M.N. Atakishiyev, N.M. Atakishiyev, and A.U. Klimyk, J. Math. Phys. **47**, 093502 (2006).
11. N.M. Atakishiyev, A.U. Klimyk, and K.B. Wolf, J. Phys. A. **37**, 5569 (2004).
12. R. Koekoek and R.F. Swarttouw, *The Askey-Scheme of Hypergeometric Orthogonal Polynomials and Its q-Analogue* (Delft University of Technology, Report 98–17, 1998 [available from `ftp.tudelft.nl`]).
13. A.U. Klimyk and I.I. Kachurik, Commun. Math. Phys. **175**, 89 (1996).
14. Yu.M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators* (American Mathematical Society, Providence, R.I., 1969).
15. T.H. Koornwinder, arXiv: math.CA/0601303.
16. J. Van der Jeugt and R. Jagannathan, J. Math. Phys. **39**, 5062 (1998).

Received 23.04.07

НОВІ МОДЕЛІ КВАНТОВОГО ОСЦИЛЯТОРА

А.У. Клімик

Р е з ю м е

Побудовано нові моделі квантового осцилятора. Як і у випадку q-осцилятора Біденгарна–Макфарлейна, ці моделі зв'язані з q-многочленами Ерміта. Оператори положення та імпульсу в цих моделях є підходящими операторами представлень квантової алгебри $\mathrm{su}_q(1,1)$. Як і в стандартному квантовому гармонічному осциляторі, оператори положення та імпульсу мають неперервні прості спектри. Ці спектри покривають скінченний інтервал дійсної осі, що залежить від значення q. Власні функції цих операторів знайдені у явному вигляді. На противагу випадку q-осцилятора Біденгарна–Макфарлейна оператори положення та імпульсу Q і P в цих моделях задовольняють квантово-механічні співвідношення $[H,Q]=-\mathrm{i}P$ і $[H,P]=\mathrm{i}Q$.