THE ENERGY-MOMENTUM TENSOR FOR 1/2-SPIN PARTICLES WITH ELECTRIC AND MAGNETIC POLARIZABILITIES

S.A. LUKASHEVICH, N.V. MAKSIMENKO

F. Skorina Gomel State University
(103, Sovetskaya Str., Gomel 246699, Belarus; e-mail: lukashevich@gsu.unibel.by)

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Using the formalism of the relativistic electrodynamics of continuous media and the main principles of relativistic quantum field theory, the covariant Lagrangian of the interaction of an electromagnetic field with polarizable 1/2-spin particles has been obtained. This Lagrangian allows us to determine the canonical and metric energy-momentum tensors as well as the low-energy Compton scattering amplitude.

1. Introduction

At present time, the description of the Compton scattering by hadrons is realized with the use of a nonrelativistic Hamiltonian function [1, 2]. However, for the extraction of the more essential experimental and theoretical information about hadron polarizabilities not only from the Compton scattering process but from other virtual and real two-photon processes, it is necessary to use the Lagrangian of the interaction of an electromagnetic field with a polarizable particle in the covariant form.

The covariant Lagrangian of the interaction of the electromagnetic field with polarizable particles was constructed in [3, 4], where the phenomenological formfactor-based approach was used. In the present paper, we apply the approach developed in [4] to the construction of the similar Lagrangian in a more consistent and complete way, which allows us to determine also the energy-momentum tensor of the interaction of the electromagnetic field with polarizable particles and to take the spin polarizabilities of hadrons introduced in [4, 5] into account.

2. Lagrangian for the Interaction of the Electromagnetic Field in the Electrodynamics of Continuous Media

We present the Lagrangian responsible for the interaction of the electromagnetic field with a structural particle as

$$\mathcal{L}_I = -{j_\mu}^{(M)}(x)A^\mu(x), \tag{1}$$

where ${j_\mu}^{(M)}$ is the current density due to a motion of constituents, and A^μ is the four-dimensional potential of the electromagnetic field.

The current density ${j_\mu}^{(M)}(x)$ has to satisfy the conservation law

$$\partial^\mu {j_\mu}^{(M)} = 0,$$

which allows us to present it in the form

$${j_\mu}^{(M)} = -\partial^\rho G^I_{\rho\mu},$$

where $G^I_{\rho\mu}$ is an antisymmetric tensor.

Let us introduce the vectors which are the relativistic generalization of the electric

$$D_\nu = G^I_{\nu\rho}U^\rho \tag{2}$$

and magnetic

$$M^\sigma = \varepsilon^{\sigma\rho\kappa\mu}G^I_{\rho\kappa}U_\mu \tag{3}$$

dipole moments of the system. In expressions (2) and (3), the four-dimensional velocity of a structural particle U has the components

$$U^\mu \left\{\gamma, \mathbf{v}\gamma\right\},$$

where $\gamma = \dfrac{1}{\sqrt{1-\mathbf{v}^2}}$, and $\mathbf{v}$ is the medium velocity.

In the rest system of the structural particle, the vectors D_ν and M^σ can be presented as a multiple expansion relative to the center of gravity [9]:

$$D_k = \sum_{L=1}^{\infty}(-1)^{L-1}\, Q^{(L)}_{kl\ldots n}\, \partial_{l\ldots}\, \partial_n\, \delta(\overrightarrow{x}),$$

$$M_k = \sum_{L=1}^{\infty}(-1)^{L-1}\, M^{(L)}_{kl\ldots n}\, \partial_{l\ldots}\, \partial_n\, \delta(\overrightarrow{x}),$$

where $k, l = 1, 2, 3$; $Q^{(L)}$ and $M^{(L)}$ are multiple tensors which are determined from a displacement of constituents with regard to the center of gravity.

It follows from (2) and (3) that $U^\nu D_\nu = 0, U^\nu M_\nu = 0$.

Using the vectors U^ν, D^ν, and M_ν, one can construct the antisymmetric tensor $G^I_{\mu\nu}$. Taking into account the space inversion, this tensor looks like

$$G^I_{\mu\nu} = U_\mu D_\nu - U_\nu D_\mu + \varepsilon_{\mu\nu\rho\sigma} U^\rho M^\sigma. \tag{4}$$

On the other hand, the effective Lagrangian (1) can be presented as

$$\mathcal{L}_I = -j^{(M)}_\mu A^\mu = (\partial^\rho G^I_{\rho\mu}) A^\mu.$$

To avoid the ambiguity of the Lagrangian, we set

$$\mathcal{L}_I = -\frac{1}{2} G^I_{\rho\mu} F^{\rho\mu} \tag{5}$$

or, by taking (4) into account,

$$\mathcal{L}_I = -\frac{1}{2}(D_\mu e^\mu - M_\mu h^\mu), \tag{6}$$

where e^μ, h^μ can be expressed via the electromagnetic field tensor as $e^\mu = F^{\mu\nu} U_\nu$, $h^\mu = \widetilde{F}^{\mu\nu} U_\nu$.

In the case where

$$D_\mu = 4\pi \alpha_{\mu\nu} e^\nu, \tag{7}$$

$$M_\mu = 4\pi \beta_{\mu\nu} h^\nu, \tag{8}$$

and tensors $\alpha^{\mu\nu}$ and $\beta^{\mu\nu}$ are represented in the diagonal form, by using the metric tensor $g^{\mu\nu}$, as $\alpha^{\mu\nu} = g^{\mu\nu}\alpha$, $\beta^{\mu\nu} = g^{\mu\nu}\beta$, Lagrangian (6) can be split into two parts,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \tag{9}$$

where $\mathcal{L}_0 = -\frac{1}{2}(e^2 - h^2) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F^2$ and

$$\mathcal{L}_I = -2\pi \left[(\alpha - \beta)\, e^2 + \frac{\beta}{2} F^2 \right]. \tag{10}$$

Inserting Lagrangian (9) into the Euler–Lagrange equation of motion

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \tag{11}$$

we can find

$$\partial_\mu F^{\mu\nu} = j^{(M)\nu} = -\partial_\mu G^{I\mu\nu}, \tag{12}$$

where

$$G^{I\mu\nu} = 4\pi \left[(\alpha - \beta)(e^\mu U^\nu - e^\nu U^\mu) + \beta F^{\mu\nu} \right]. \tag{13}$$

From Eqs. (12) for the medium at rest, the definitions of charge and current densities for bound charges are as follows:

$$\rho^{(M)} = -4\pi\alpha(\nabla \mathbf{E}) = -4\pi\alpha \cdot \mathrm{div}\, \mathbf{E}, \tag{14}$$

$$\mathbf{j}^{(M)} = 4\pi(\alpha \cdot \partial_t \mathbf{E} - \beta \cdot \mathrm{rot}\, \mathbf{H}), \tag{15}$$

while the Maxwell's equations for the medium have the form

$$\mathrm{rot}\, \mathbf{E} = -\partial_t \mathbf{H}, \quad \mathrm{div}\, \mathbf{H} = 0,$$

$$\mathrm{rot}\, \mathbf{B} = \partial_t \mathbf{D}, \quad \mathrm{div}\, \mathbf{D} = 0, \tag{16}$$

where $\mathbf{D} = \widehat{\varepsilon} \mathbf{E}$ and $\mathbf{B} = \widehat{\mu} \mathbf{H}$ are the vectors of electric and magnetic inductions, respectively.

Equations (14) and (15) for the medium at rest can be presented in the covariant form by the introduction of the polarizability tensor of the medium [10]

$$\widehat{M} = \begin{pmatrix} 0 & -\mathbf{P} \\ \mathbf{P} & \mathbf{M}^\times \end{pmatrix}, \tag{17}$$

where $\mathbf{P}$ and $\mathbf{M}$ are the vectors of electric and magnetic polarizabilities of the medium. By definition, they are the dipole moments of volume scale.

As a result, the charge and current densities are transformed into

$$\mathbf{j}^{(M)} = \frac{\partial \mathbf{P}}{\partial t} - \mathrm{rot}\, \mathbf{M}, \quad \rho^{(M)} = -\mathrm{div}\, \mathbf{P}, \tag{18}$$

while the equation of motion reads as

$$\partial_\mu (F^{\mu\nu} + G^{I\mu\nu}) = j^\nu \tag{19}$$

for the moving medium and

$$\partial_\mu (F^{\mu\nu} + M^{\mu\nu}) = j^\nu, \tag{20}$$

for the medium at rest. Here, $M^{\mu\nu}$ is the polarizability tensor of the medium (17), and j^ν is the current density of free charges.

Let us introduce the tensor [9, 11, 12]

$$G^{\mu\nu} = d^\mu U^\nu - d^\nu U^\mu + \varepsilon^{\mu\nu\rho\sigma} U_\rho b_\sigma, \tag{21}$$

with $d^\mu = \varepsilon^{\mu\sigma} e_\sigma$, $b_\rho = \mu_{\rho\sigma} h^\sigma$. Then Lagrangian (2) for the moving medium can be presented as

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} G_{\mu\nu} = -\frac{1}{2}(e \widehat{\varepsilon} e - h \widehat{\mu} h), \tag{22}$$

where $\widehat{\varepsilon} = I + 4\pi\widehat{\alpha}$, $\widehat{\mu} = I + 4\pi\widehat{\beta}$.

Lagrangian (2) helps us to determine the canonical energy-momentum tensor. Indeed, the Noether's theorem yields [8]

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} (\partial^\nu A^\rho) - g^{\mu\nu} \mathcal{L}. \tag{23}$$

Inserting (22) into (23), one can find that

$$T^{\mu\nu} = -G^{\mu\rho}(\partial^\nu A_\rho) - g^{\mu\nu}\frac{1}{4}(F_{\rho\sigma}G^{\rho\sigma}). \qquad (24)$$

Now we determine the metric energy-momentum tensor as

$$\widetilde{T}^{\mu\nu} = -G^{\mu\rho}(\partial^\nu A_\rho) - g^{\mu\nu}L + \partial_\rho(A^\nu G^{\mu\rho}). \qquad (25)$$

After some reconstruction of (25) with the use of the equation of motion (19), we find

$$\widetilde{T}^{\mu\nu} = F_\rho{}^{\nu}G^{\mu\rho} + \frac{1}{4}g^{\mu\nu}(F_{\rho\sigma}G^{\rho\sigma}). \qquad (26)$$

From relation (26), we can obtain that the energy density for the medium at rest is

$$\widetilde{T}^{00} = \omega = \frac{1}{2}(\varepsilon\mathbf{E}^2 + \mu\mathbf{H}^2). \qquad (27)$$

Using the correspondence principle [7, 8], we now consider the quantum-mechanical description of the interaction of the electromagnetic field with polarizable particles. The electric and magnetic polarizabilities extracted from Lagrangian (22) can be presented as

$$\mathcal{L}_I = -2\pi(\alpha F_{\mu\rho}F_\sigma{}^{\mu} - \beta\widetilde{F}_{\mu\rho}\widetilde{F}_\sigma{}^{\mu})U^\rho U^\sigma, \qquad (28)$$

where $\widetilde{F}_{\mu\rho} = \frac{1}{2}\varepsilon_{\mu\rho\sigma\kappa}F^{\sigma\kappa}$, $\varepsilon_{0123} = -1$.

3. Lagrangian of the Interaction of the Electromagnetic Field with a Polarizable Spin-1/2 Particle

To pass from Lagrangian (28) to the Lagrangian describing the interaction of the electromagnetic field with a polarizable 1/2-spin particle, we perform the following transition based on the correspondence principle between classical and quantum mechanics. First of all, instead of P_μ, we introduce a momentum operator acting on the wave function of the particle, $-i\partial_\mu\psi$. Taking into account that $\overline{\psi}\gamma_\mu\psi$ is the current density of particles, making symmetrization of operators, and providing the hermiticity, as well as the relativistic and gauge invariance, we can present the Lagrangian describing the interaction of the electromagnetic field with a polarizable 1/2-spin particle as

$$\mathcal{L}_I = \frac{2\pi}{m}\left[\alpha F_{\mu\rho}F_\sigma{}^{\mu} - \beta\widetilde{F}_{\mu\rho}\widetilde{F}_\sigma{}^{\mu}\right]\widetilde{\Theta}^{\rho\sigma}, \qquad (29)$$

where $\Theta^{\rho\sigma}$ is the energy-momentum tensor of the spinor field.

The polarizabilities α and β introduced in expression (29) are used in hadronic physics and are connected with the polarizabilities of expression (28):

$$\alpha' = \rho\alpha, \quad \beta' = \rho\beta,$$

where ρ is the density of particles.

To verify that Lagrangian (29) is correct, it is sufficient to define a Hamiltonian for the interaction of the electromagnetic field with a polarizable particle and the Compton scattering amplitude by this particle.

To define the movement of a charged polarizable spinor particle in the electromagnetic field, we write the total Lagrangian in the following form:

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\overset{\wedge}{\overleftrightarrow{\partial}}\psi - m\overline{\psi}\psi - e\overline{\psi}\widehat{A}\psi -$$
$$-\frac{1}{4}F^2 - \frac{1}{4}F_{\mu\nu}G^{(S)I\mu\nu}. \qquad (30)$$

Then tensor (13) is determined as

$$G^{(S)I\mu\nu} = -\frac{4\pi}{m}\Big\{(\alpha-\beta)\left[F^{\mu\sigma}\widetilde{\Theta}_\sigma{}^{\nu} - F^{\nu\sigma}\widetilde{\Theta}_\sigma{}^{\mu}\right] +$$
$$+\beta\widetilde{\Theta}_\rho{}^{\rho}F^{\mu\nu}\Big\}. \qquad (31)$$

In view of Lagrangian (30) and the antisymmetric tensor (31), the metric momentum-energy tensor can be defined as

$$\widetilde{T}^{\mu\nu} = \widetilde{\Theta}^{\mu\nu} + F_\rho{}^{\nu}F^{\mu\rho} + \frac{1}{4}g^{\mu\nu}F^2 -$$
$$-\frac{e}{2}\overline{\psi}(\gamma^\mu A^\nu + \gamma^\nu A^\mu)\psi + \widetilde{T}_I^{\mu\nu}, \qquad (32)$$

where

$$\widetilde{T}_I^{\mu\nu} = F_\rho{}^{\nu}G_I^{(S)\mu\rho} + \frac{1}{4}g^{\mu\nu}(F_{\rho\sigma}G^{(S)I\rho\sigma}). \qquad (33)$$

One can see from expression (33) that, for the particle at rest, its interaction energy due to its polarizability will have the form [2]

$$H_I = -2\pi(\alpha\mathbf{E}^2 + \beta\mathbf{H}^2). \qquad (34)$$

Lagrangian (30) yields the following equation of motion:

$$\partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi + j^{(M)\nu}, \qquad (35)$$

where $j^{(M)\nu} = -\partial_\mu G^{(S)I\mu\nu}$.

Taking into account (34), the scattering amplitude of the second order in the photon energy gives the following contribution to the electric and magnetic polarizabilities:

$$T_{fi}^{pol} = \frac{8\pi m\omega\omega'}{N(t)}\left[\mathbf{e}'^{*}\cdot\mathbf{e}\alpha_0 + \mathbf{s}'^{*}\cdot\mathbf{s}\beta_0\right]. \qquad (36)$$

Here, $\mathbf{s} = \mathbf{n}\times\mathbf{e}$; $\mathbf{s}'^{*} = \mathbf{n}'\times\mathbf{e}'^{*}$, $\mathbf{e}$ and $\mathbf{e}'$ are the polarization vectors, ω_1 and ω_2 are the energies of the incident and scattered photons, $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$, and $\mathbf{n}' = \mathbf{k}'/|\mathbf{k}'|$.

4. Conclusion

On the basis of the relativistic electrodynamics of continuous media and the main principles of relativistic quantum field theory, we have constructed the covariant Lagrangian of the interaction of the electromagnetic field with polarizable 1/2-spin particles. This Lagrangian allowed us to determine the canonical and metric energy-momentum tensors as well as the low-energy Compton scattering amplitude.

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ТЕНЗОР ЕНЕРГІЇ-ІМПУЛЬСУ ДЛЯ ЧАСТИНОК ЗІ СПІНОМ 1/2 З УРАХУВАННЯМ ЇХ ЕЛЕКТРИЧНОЇ ТА МАГНІТНОЇ ПОЛЯРИЗОВНОСТЕЙ

С.А. Лукашевич, Н.В. Максименко

Р е з ю м е

З використанням формалізму релятивістської електродинаміки суцільних середовищ та основних положень релятивістської квантової теорії поля отримано коваріантний лагранжіан взаємодії електромагнітного поля з частинкою зі спіном 1/2, що поляризується. За допомогою отриманого лагранжіана визначено канонічний і метричний тензори, а також амплітуду низькоенергетичного комптонівського розсіяння.