CALCULATION OF THE IMAGINARY PART OF THE SPECTRUM OF ELEMENTARY EXCITATIONS OF A STRONGLY CORRELATED BOSE SYSTEM

A. CHUMACHENKO

Taras Shevchenko Kyiv National University
(2, Build. 1, Academician Glushkov Prosp., Kyiv 03127, Ukraine)

UDC 539

On the base of the microscopic approach, the damping of the spectrum of elementary excitations for a strongly correlated Bose-system with condensation at nonzero temperatures is calculated, by using the Popov's method of hydrodynamic variables.

1. Introduction

The analysis of modern experimental and theoretical data indicates that the study of the unique phenomenon of superfluidity of ^{4}He is far from the end. There is a set of contradictions between the theory and experiments related to the hydrodynamics of a superfluid Bose-liquid of ^{4}He and to the shape of the quasiparticle spectrum; these contradictions have no satisfactory explanation in a frame of the microscopic theory.

The exact first-principles calculation of the elementary excitation spectrum and its damping in a superfluid Bose-liquid with strong interparticle interaction is an extremely complicated problem of many-body quantum theory. The first success in the construction of the microscopic theory of Bose-systems was achieved by Bogolyubov [1]. Using the fact that the Bose condensation can be presented like the accumulation of a macroscopic amount of particles on the lowest energy level with zero momenta, Bogolyubov suggested to consider the corresponding field operator $\hat{a}_0$ as a -number, $\hat{a}_0 \to \langle \hat{a}_0 \rangle = \sqrt{n_0}$, here n_0 – condensate density. Using this representation, Bogolyubov was able to diagonalize the many-particle Hamiltonian of the weakly interacting Bose-gas and obtained the elementary excitation spectrum for this system:

$$E(\mathbf{q}) = \left[\frac{\mathbf{q}^2}{2m} \left(\frac{\mathbf{q}^2}{2m} + 2n_0 V(\mathbf{q}) \right) \right]^{1/2}. \tag{1}$$

For small momenta ($\mathbf{q} \to 0$), this spectrum has an acoustic behavior: $E = c|\mathbf{q}|$ with $c = \sqrt{n_0 V(0)/m}$. It is stable, when the Fourier transform of the interaction potential $V(\mathbf{q})$ satisfies the conditions $\mathbf{q}^2 > 4mn_0 V(\mathbf{q})$, here, m – the mass of a particle.

Later on, using the methods of quantum field theory, Belyaev [2, 3] has also obtained the Bogolyubo's result. In the higher orders of perturbation theory, Belyaev has described the decay of the quasiparticle spectrum at zero temperature. At the same time, the calculation of the long wavelength phonon part of the spectrum $E(\mathbf{q}) \simeq c|\mathbf{q}|$ (here, c – velocity of the first (hydrodynamic) sound in a liquid ^{4}He) on the base of the microscopic perturbation field theory, was faced with fundamental difficulties. This is related to the fact that, in the non-renormalized perturbation theory at small values of the four-momenta $q = (\omega, \mathbf{q}) \to 0$, the so-called infrared divergences and nonanalyticities appear.

An efficient method to avoid the problem of infrared divergences was developed by Popov [4]. Separating the Fourier transform of the field variables Ψ with respect to the momenta q_0 into the "slow" Ψ_s and "fast" Ψ_f ones and integrating over the fast variables, Popov obtained the effective renormalized functional of a quantum mechanical action S_{eff}. The field theory, which is based on this functional and formulated in terms of the "hydrodynamic" variables of the phase $\varphi(q)$ and the amplitude $\pi(q)$, does not contain the infrared divergences, so it can be used for the calculation of low-energy properties of the system. In the Bose-gas approximation, Popov obtained the expression for the damping of the elmentary excitation spectrum, which coincides with the Belyaev's result [3] at $T = 0$.

Unfortunately, the models mentioned above describe such a strongly interacting Bose system as ^{4}He not very well. Therefore, despite a good agreement with experimental data in the rotom minimum region attained with the use of the Monte-Carlo method with the so-called "shadow wave function" and the method of correlation basic function with modern interparticle potentials, the calculation of the elementary excitation spectrum in the superfluid helium remains a very difficult urgent problem even now. One of the reasons for that is that the physical reason for the roton minimum in the quasiparticle spectrum and the damping of this spectrum in a Bose-liquid remains unclear.

The goal of this paper is the calculation of the imaginary part (damping) of the spectrum for the strongly interacting Bose system using the method of effective potential developed in [4]. Representing the thermodynamical coefficients of a quadratic form of the action S_{eff} by the measurable physical parameters of the system like the speed of sound c, the system density n, and the condensate compressibility $dn_0/d\mu$ (μ – chemical potential of the system of Bose particles), like it was done in work [5], we find new hydrodynamic Green's functions for the strongly interacting Bose system. Using these Green's functions in the second order of perturbation theory, we will calculate the imaginary part of the spectrum of elementary excitations which define its damping.

2. Calculation of the Imaginary Part of the Elementary Excitation Spectrum

In the momentum representation, the Popov's effective action S_{eff} has the following form [4]:

$$S_{\mathrm{eff}}[\pi,\varphi]=\frac{1}{2}\sum_q\Big\{-\left(\frac{p_\mu}{m}\mathbf{q}^2+p_{\mu\mu}\omega_s^2\right)\varphi(q)\varphi(-q)-$$

$$-2p_{\mu n_0}\omega_s\varphi(q)\pi(-q)+$$

$$+\left(p_{n_0n_0}-\frac{\mathbf{q}^2}{4mn_0}\right)\pi(q)\pi(-q)\Big\}+$$

$$+\frac{1}{\sqrt{\beta\omega_s}}\sum_{q_1+q_2+q_3=0}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)}{2m}\varphi(q_1)\varphi(q_2)\pi(q_3), \tag{2}$$

where $q=(\omega_s/c,\mathbf{q})$ – 4-vector of the momentum and energy, c – the speed of sound (will be defined below); $\omega_s=2\pi s/\beta$ – Matsubara frequency, s – integer even numbers, and $\beta=(k_{\mathrm{B}}T)^{-1}$ – dimensionless inverse temperature, (k_{B} – Boltzmann constant). The coefficients $p_\mu, p_{\mu\mu}$, $p_{\mu n_0}$ and $p_{n_0n_0}$ are the thermodynamical derivatives of the pressure with respect to the chemical potential μ and the condensate density n_0. In work [4], it was shown that the term $\mathbf{q}^2/4mn_0$ in expression (2) describes a deviation of the elementary excitation spectrum from a linear one. At small momenta, this term can be neglected. The cubic term in action (2) leads to the appearance of the imaginary part in the spectrum.

Considering the quadratic part of expression (2) as the unperturbed action and the cubic term as a perturbation, it is possible to find the free hydrodynamic Green's functions:

$$g_{\varphi\varphi}=\langle\varphi(q)\varphi(-q)\rangle_0,\ g_{\varphi\pi}=\langle\varphi(q)\pi(-q)\rangle_0,$$

$$g_{\pi\varphi}=\langle\pi(q)\varphi(-q)\rangle_0,\ g_{\pi\pi}=\langle\pi(q)\pi(-q)\rangle_0; \tag{3}$$

here, $\langle...\rangle_0$ symbolizes the average with respect to the quadratic part of the action. The specific expression for this function is defined by the inverse matrix of the coefficients in the quadratic part of action (2):

$$G_0(q)=\begin{pmatrix} g_{\varphi\varphi} & g_{\varphi\pi}\\ g_{\pi\varphi} & g_{\pi\pi}\end{pmatrix}=$$

$$=-\begin{pmatrix} -\left(\frac{\mathrm{d}n_0}{\mathrm{d}\mu}\right)^2\omega_s^2 & \frac{\mathrm{d}n_0}{\mathrm{d}\mu}\omega_s\\ -\frac{\mathrm{d}n_0}{\mathrm{d}\mu}\omega_s & 1\end{pmatrix}\frac{m}{n}\frac{c^2}{\omega_s^2+c^2\mathbf{q}^2}+$$

$$+\begin{pmatrix} \frac{1}{p_{n_0n_0}} & 0\\ 0 & \frac{1}{8nn_0}\frac{1}{p_{n_0n_0}}\end{pmatrix}. \tag{4}$$

Here, we used the expressions obtained in [4, 6]

$$\frac{1}{m}\frac{p_{n_0n_0}p_\mu}{p_{n_0n_0}p_{\mu\mu}-p_{\mu n_0}^2}=\frac{n}{m}\frac{d\rho}{d\mu}=\frac{dp}{d\rho}=c^2,\quad p_\mu=n \tag{5}$$

and

$$\frac{p_{\mu n_0}}{p_{n_0n_0}}=-\frac{\mathrm{d}n_0}{\mathrm{d}\mu}. \tag{6}$$

Expressions (5) and (6) represent the thermodynamical derivatives of the pressure p with respect to the physical quantities. In relation (5), c – the macroscopic speed of sound (first sound) which can be defined from the experiment, and n – the system density. Equality (6) defines the inverse condensate density which is also the experimentally measurable quantity.

In the Popov's work [4], the hydrodynamic Green's functions are obtained in the gas approximation, when the thermodynamical derivatives of the pressure can be presented in the form

$$p_\mu=n\approx n_0,\quad p_{\mu\mu}=0,\quad p_{\mu n_0}=1,\ p_{n_0n_0}=-\frac{mc^2}{n_0}. \tag{7}$$

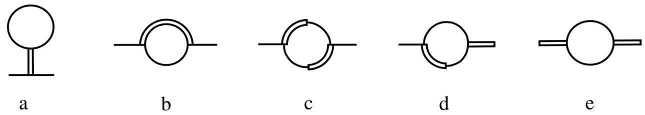

Fig. 1. Diagrams of the second-order corrections for the self-energy parts $\Sigma_{\varphi\varphi}$, $\Sigma_{\varphi\pi}$, and $\Sigma_{\pi\pi}$. Diagrams a, b, and c correspond to the component $\Sigma_{\varphi\varphi}$ of the self-energy matrix, diagrams d and e are the contributions to the $\Sigma_{\varphi\pi}$ and $\Sigma_{\pi\pi}$, respectively

In this paper, we didn't make any assumptions about the smallness of the Bose-system density. The Green's functions (4) and the damping of the spectrum calculated below correspond to the system with strong interparticle interaction.

As well known [7], the elementary excitation spectrum can be obtained from the equation

$$\det G^{-1}(q) = \det(G_0^{-1}(q) - \Sigma(q)) = 0, \tag{8}$$

where $\Sigma(q)$ – the self-energy matrix. Because we have $\Sigma(q) = 0$ in the lowest order of perturbation theory, the elementary excitation spectrum is defined from the condition that the matrix determinant G_0^{-1} is equal to zero. In the higher orders of perturbation theory, $\Sigma(q)$ is not equal to zero. This leads to the appearance of the imaginary part $\Delta(q)$ which describes the damping of the elementary excitation spectrum. The contribution to the self-energy in the second order of perturbation theory is given by the diagrams depicted in Fig. 1. The expression for the imaginary part of the spectrum $\Delta(q)$ can be found after the substitution of the matrix (4) in Eq. (8). After the analytic continuation $i\omega \to E$, Eq. (8) in the small-momentum approximation $(\mathbf{q} \to 0)$ can be written as

$$E^2\left[1 + \Sigma_{\pi\pi}(E)\ mc^2\frac{p_{\mu\mu}}{p_{n_0n_0}p_\mu}\right] -$$

$$-2iE\ \Sigma_{\varphi\pi}(E)\frac{mc^2}{n}\frac{p_{\mu n_0}}{p_{n_0n_0}} -$$

$$-c^2\mathbf{q}^2\left[1 - \Sigma_{\pi\pi}(E)\frac{1}{p_{n_0n_0}} + \Sigma_{\varphi\varphi}(E)\frac{mc^2}{p_\mu\mathbf{q}^2}\right] = 0. \tag{9}$$

This equation cannot be solved exactly, therefore we search for an approximate solution making assumption that components of the self-energy matrix Σ_{ij} are small quantities for the small values of the energy and momenta. This approximation will be justified later on. Let us expand $\Sigma_{ij}(E)$ around the unperturbed solution $E = \varepsilon(\mathbf{q}) = c|\mathbf{q}|$. This allows us to find the imaginary part of the solution of Eq. (9), by using expressions (5) and (6), in the following form:

$$\Delta(\mathbf{q} \to 0) = \mathrm{Im}\ E_{1,2}(\mathbf{q} \to 0) =$$

$$= -\frac{mc^2}{n}\frac{dn_0}{d\mu}\mathrm{Re}\ \Sigma_{\varphi\pi}(\varepsilon(\mathbf{q}))\mp$$

$$\mp c|\mathbf{q}|\left[\frac{mc^2}{2n}\left(\frac{dn_0}{d\mu}\right)^2 \mathrm{Im}\ \Sigma_{\pi\pi}(\varepsilon(\mathbf{q})) +\right.$$

$$\left.+\frac{m}{2n}\frac{1}{\mathbf{q}^2}\mathrm{Im}\ \Sigma_{\varphi\varphi}(\varepsilon(\mathbf{q}))\right]. \tag{10}$$

This expression gives the damping of the Bose-system elementary excitation spectrum via the components of the self-energy matrix. Exact expressions for the self-energy matrix components can be found by calculating the corresponding diagrams depicted in Fig. 1.

The diagram a gives no contribution to the self-energy as far as the intermediate propagator $g_{\pi\pi}$ which connects the external lines to the loop has zero energy and momentum. There are other possibilities to construct the diagrams topologically identical to a, but all of them will give no contribution to the self-energy because the elementary vertices equal zero in these diagrams. All other diagrams give the finite contribution to the corresponding self-energy components.

Analytic expressions for the components $\Sigma_{\varphi\pi}(q)$, $\Sigma_{\varphi\varphi}(q)$, and $\Sigma_{\pi\pi}(q)$ can be obtained from the corresponding diagrams in Fig. 1 using the Feynman rules. For the small values of the energy and momenta in Eq. (4), we can neglect the terms proportional to $1/p_{n_0n_0}$. Calculating Σ_{ij}, we will integrate over the internal momenta and will take a sum over the internal frequencies. As an example, let us calculate an expression for the diagram b which gives a contribution to the component $\Sigma_{\varphi\varphi}$ of the self-energy matrix:

$$\Sigma^b_{\varphi\varphi}(\mathbf{q}, i\omega) = \frac{T}{(2\pi)^3m^2}\int d\mathbf{q}_2\, d\mathbf{q}_3\, (\mathbf{q}\cdot\mathbf{q}_3)^2 \times$$

$$\times \sum_{\omega_2,\omega_3}\eta(q, q_2, q_3)\ g_{\pi\pi}(\mathbf{q}_2, i\omega_2)g_{\varphi\varphi}(\mathbf{q}_3, i\omega_3), \tag{11}$$

with

$$\eta(q, q_2, q_3) \equiv \frac{1}{2}\left(\delta(\mathbf{q} + \mathbf{q}_2 - \mathbf{q}_3)\delta_{\omega+\omega_2,+\omega_3} +\right.$$

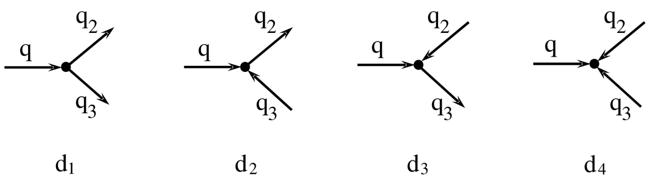

Fig. 2. All possible ways to construct the elementary vertex part

$$+\delta(\mathbf{q}-\mathbf{q}_2+\mathbf{q}_3)\delta_{\omega-\omega_2,-\omega_3} +\delta(\mathbf{q}-\mathbf{q}_2-\mathbf{q}_3)\delta_{\omega-\omega_2,\omega_3}) \, .$$

The factor η can be explained due to the fact that, for the calculation of the contribution for a specific diagram, it is necessary to take a sum over all directions of the internal lines in this diagram. The factor $1/2$ takes care of the identical diagrams which appear, when the directions of all four-momenta in the diagram are changed. All possible variants of the directions of four-momenta q_2 and q_3 in the diagram can be found if one consider the elementary vertex part. From Fig. 2, one can see that there exist only four vertex parts with different numbers of incoming and outgoing lines. Each vertex part obeys the energy and momentum conservation laws.

The vertex part d_4 which has all lines pointed at (from) the vertex corresponds to the conservation law in the form $\delta(\mathbf{q}+\mathbf{q}_2+\mathbf{q}_3)\delta_{\omega+\omega_2,-\omega_3}$ and, obviously, equals zero. Therefore, the diagrams which contain the vertex functions of this type also do not contribute to the self-energy. So, the contribution of each diagram will be the sum of the contributions of three topologically equal diagrams which are constructed with the use of the vertices from Fig. 2.

Let us calculate an expression for the component $\Sigma_{\varphi\varphi}$ of the self-energy matrix. The contribution to this component is given by two terms (let's define them as $\Sigma^b_{\varphi\varphi}$ and $\Sigma^c_{\varphi\varphi}$) which correspond to the diagrams b and c in Fig. 1. Let us calculate exactly the expression for $\Sigma^b_{\varphi\varphi}$ by performing the summation over the internal Matsubara frequencies. Assuming that the internal frequency $\epsilon_s = 2\pi s/\beta$ is positive, the sum S over the frequencies,

$$S = \frac{1}{\beta}\sum_s f(i\epsilon_s), \tag{12}$$

can be found by integration over the contour depicted in Fig. 3

$$I = \frac{1}{2\pi i}\int_c dz\; n_{\mathrm{B}}(z) f(z), \tag{13}$$

where $n_{\mathrm{B}}(z) = (e^{\beta z}-1)^{-1}$ – the Bose distribution function.

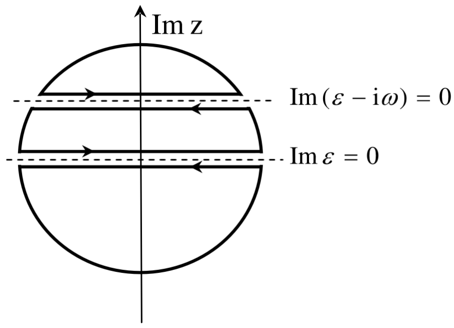

Fig. 3. Integration contour with the cuts on the complex plane

First of all, we will find an expression for the first component of η for the diagram with vertex part d_1:

$$\Sigma^b_{\varphi\varphi}(q) = \frac{1}{2(2\pi)^3 m^2}\int d\mathbf{q}_2\, d\mathbf{q}_3\, (\mathbf{q}\cdot\mathbf{q}_3)^2 \times$$

$$\times\delta(\mathbf{q}-\mathbf{q}_2-\mathbf{q}_3) I^b(i\omega,\mathbf{q}_2,\mathbf{q}_3). \tag{14}$$

Here,

$$I^b(i\omega,\mathbf{q}_2,\mathbf{q}_3) = \frac{m^2c^4}{n^2}\left(\frac{dn_0}{d\mu}\right)^2 \frac{1}{2\pi i}\int_c dz\; n_{\mathrm{B}}(z)\times$$

$$\times\frac{z^2}{z^2-\varepsilon^2(\mathbf{q}_2)}\frac{1}{(i\omega-z)^2-\varepsilon^2(\mathbf{q}_3)}. \tag{15}$$

In order to calculate integral (15), we assume that Green's functions are "full," i.e., saying differently, include high-order corrections to $\Sigma(q)$ in the denominator. The components of the self-energy matrix $\Sigma(q)$ have poles along the lines parallel to the real axes. Therefore, we have to use a different integration contour than that in the case of free propagators. In our case, the Green's functions have singularities on the lines shown in Fig. 3.

As far as the frequencies ω are even, the contour in Fig. 3 includes all the poles situated on the imaginary axes excluding two poles which coincide with cuts. The contribution of these two poles should be added to integral (15). But, during the integration over the small semicircles around these poles, their contributions are compensated [7]. The integrals over the big circles in the limit when the borders of the contour go to infinity are equal to zero. Therefore, we left with only four integrals along the cuts:

$$I^b = I_1 + I_2 + I_3 + I_4,$$

$$I_1 = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} dx\ g^R_{\pi\pi}(\mathbf{q}_2, x+i\omega) g^R_{\varphi\varphi}(\mathbf{q}_3, x)\ n_\mathrm{B}(x+i\omega),$$

$$I_2 = -\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} dx\ g^R_{\pi\pi}(\mathbf{q}_2, x+i\omega) g^A_{\varphi\varphi}(\mathbf{q}_3, x)\ n_\mathrm{B}(x+i\omega),$$

$$I_3 = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} dx\ g^R_{\pi\pi}(\mathbf{q}_2, x) g^A_{\varphi\varphi}(\mathbf{q}_3, x-i\omega)\ n_\mathrm{B}(x),$$

$$I_4 = -\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} dx\ g^A_{\pi\pi}(\mathbf{q}_2, x) g^A_{\varphi\varphi}(\mathbf{q}_3, x-i\omega)\ n_\mathrm{B}(x), \tag{16}$$

where g^R and g^A – advanced and retarded Green's functions. In view of the analytic properties of these functions and the parity of the external frequency ω, we obtain the expressions

$$g^R(\mathbf{q}, x) - g^A(\mathbf{q}, x) = 2i\ \mathrm{Im}\ g^R(\mathbf{q}, x),$$

$$n_\mathrm{B}(x+i\omega) = n_\mathrm{B}(x), \tag{17}$$

which allows us to write the sum of integrals (16) in the form

$$I^b = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} dx \left[\mathrm{Im}\ g^R_{\varphi\varphi}(\mathbf{q}_3, x) g^R_{\pi\pi}(\mathbf{q}_2, x+i\omega) + \right.$$

$$\left. + \mathrm{Im}\ g^R_{\pi\pi}(\mathbf{q}_2, x) g^A_{\varphi\varphi}(\mathbf{q}_3, x-i\omega)\right] n_\mathrm{B}(x). \tag{18}$$

In the last expression, we make an analytic continuation of the external frequency $i\omega \to E$ and consider the retarded component $(I^b)^R$ of the integral. Its real and imaginary parts are presented by the following expressions:

$$\mathrm{Re}(I^b)^R = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} dx \left[\mathrm{Im} g^R_{\varphi\varphi}(\mathbf{q}-\mathbf{k}, x) \mathrm{Re}\ g^R_{\pi\pi}(\mathbf{q}, x+E) + \right.$$

$$\left. + \mathrm{Im}\ g^R_{\pi\pi}(\mathbf{q}, x)\ \mathrm{Re}\ g^R_{\varphi\varphi}(\mathbf{q}-\mathbf{k}, x-E)\right] n_\mathrm{B}(x), \tag{19}$$

$$\mathrm{Im}\ (I^b)^R = -\frac{1}{\pi}\int\limits_{-\infty}^{+\infty} dx\ \mathrm{Im}\ g^R_{\pi\pi}(\mathbf{q}, x) \times$$

$$\times \mathrm{Im}\ g^R_{\varphi\varphi}(\mathbf{q}-\mathbf{k}, x-E)\left(n_\mathrm{B}(x) - n_\mathrm{B}(x-E)\right). \tag{20}$$

Expression (19) defines the correction to the real part of the spectrum which will not be discussed here. Let us consider the imaginary part of the integral $(I^b)^R$ which is given by Eq. (20) in more details. Using the well-known expression for the bosonic Green's functions [7, 8],

$$\mathrm{Im}\ D_R(\mathbf{q}, E) = -\frac{\pi}{2\varepsilon(\mathbf{q})}\left[\delta(E - \varepsilon(\mathbf{q})) - \delta(E + \varepsilon(\mathbf{q}))\right], \tag{21}$$

we found an expression for the imaginary parts of the hydrodynamic Green's functions $g_{\varphi\varphi}$ and $g_{\pi\pi}$:

$$\mathrm{Im}\ g^R_{\pi\pi}(\mathbf{q}_2, x) = \frac{\pi x^2}{2\varepsilon(\mathbf{q}_2)}\left[\delta(x - \varepsilon(\mathbf{q}_2)) - \delta(x + \varepsilon(\mathbf{q}_2))\right], \tag{22}$$

$$\mathrm{Im}\ g^R_{\varphi\varphi}(\mathbf{q}_3, x-\omega) = \frac{\pi}{2\varepsilon(\mathbf{q}_3)} \times$$

$$\times \left[\delta(x - E - \varepsilon(\mathbf{q}_3)) - \delta(x - E + \varepsilon(\mathbf{q}_3))\right]. \tag{23}$$

Substituting (23) and (23) into Eq. (20) and then into Eq. (14), we obtain the contribution to the component $(\Sigma^b_{\varphi\varphi})^R$ of the self-energy from the diagram constructed on the basis of the vertex part d_1:

$$\mathrm{Im}\ (\Sigma^b_{\varphi\varphi}(E, \mathbf{q}))^R = -\frac{1}{64}\frac{c^4}{n^2\pi^2}\left(\frac{dn_0}{d\mu}\right)^2 \int d\mathbf{q}_2\, d\mathbf{q}_3 \times$$

$$\times (\mathbf{q}\cdot\mathbf{q}_3)^2\, \delta(\mathbf{q} - \mathbf{q}_2 - \mathbf{q}_3)\frac{\varepsilon_2}{\varepsilon_3} \times$$

$$\times \left[(\delta(E + \varepsilon_2 - \varepsilon_3) - \delta(E + \varepsilon_2 + \varepsilon_3)) \times \right.$$

$$\times \left(\frac{1}{e^{-\beta\varepsilon_2} - 1} - \frac{1}{e^{-\beta(\varepsilon_2 + E)} - 1}\right) +$$

$$+ (\delta(E - \varepsilon_2 + \varepsilon_3) - \delta(E - \varepsilon_2 - \varepsilon_3)) \times$$

$$\left. \times \left(\frac{1}{e^{\beta\varepsilon_2} - 1} - \frac{1}{e^{\beta(\varepsilon_2 - E)} - 1}\right)\right]. \tag{24}$$

The contribution of another two diagrams from the vertices d_2 and d_3 is identical to that of (24). Taking this fact into account, we find a complete expression for $(\Sigma^b_{\varphi\varphi})^R$. Separating the common multiplier and multiplying the last equation by the dimensionless combination $\varepsilon_3^2/c^2\mathbf{q}_3^2$, we find

$$\mathrm{Im}\,(\Sigma^b_{\varphi\varphi}(\varepsilon(\mathbf{q}),\mathbf{q}))^R = -\frac{3}{64}\frac{c^4(1-e^{-\beta\varepsilon})}{n^2\pi^2}\left(\frac{dn_0}{d\mu}\right)^2\times$$

$$\times\int d\mathbf{q}_2\,d\mathbf{q}_3\frac{(\mathbf{q}\cdot\mathbf{q}_3)^2}{\mathbf{q}_3^2}\varepsilon_2\varepsilon_3 n_2 n_3\times$$

$$\times\left[2\delta(1+2-3)e^{\beta\varepsilon_3}+\delta(1-2-3)e^{\beta\varepsilon}\right], \tag{25}$$

with

$$\delta(1\pm 2-3)=\delta(\mathbf{q}\pm\mathbf{q}_2-\mathbf{q}_3)\delta(\varepsilon\pm\varepsilon_2-\varepsilon_3),$$

$$n_i=(e^{\beta\varepsilon_i}-1)^{-1}.$$

According to the argumentation presented after Eq. (9), the right-hand side of equality (25) is the first term of the expansion of (24) around the unperturbed solution $E=\varepsilon(\mathbf{q})=c|\mathbf{q}|$. Such an expansion is justified, as we can see from (25) that $\mathrm{Im}\,(\Sigma^b_{\varphi\varphi}(\mathbf{q}))^R\to 0$ if $\mathbf{q}\to 0$. Substituting expression (25) into formula (10), we find the contribution from the component $\Sigma^b_{\varphi\varphi}(\mathbf{q})$ of the self-energy matrix to the imaginary part of the spectrum:

$$\Delta^b_{\varphi\varphi}(\mathbf{q}) = -3\frac{mc^2(1-e^{-\beta\varepsilon})}{128\pi^2n^3}\left(\frac{dn_0}{d\mu}\right)^2\int d\mathbf{q}_2\,d\mathbf{q}_3\times$$

$$\times\frac{(\mathbf{q}\cdot\mathbf{q}_3)^2}{\mathbf{q}^2\mathbf{q}_3^2}\varepsilon\varepsilon_2\varepsilon_3 n_2 n_3\times$$

$$\times\left[2\delta(1+2-3)e^{\beta\varepsilon_3}+\delta(1-2-3)e^{\beta\varepsilon}\right]. \tag{26}$$

In the same way, all the contributions of another diagrams depicted in Fig. 1 can be calculated. The result of calculations of the diagram c which corresponds to the component $\Sigma^c_{\varphi\varphi}(\mathbf{q})$ of the self-energy matrix can be presented in the following form:

$$\Delta^c_{\varphi\varphi}(\mathbf{q}) = -\frac{mc^2(1-e^{-\beta\varepsilon})}{128\pi^2n^3}\left(\frac{dn_0}{d\mu}\right)^2\int d\mathbf{q}_2\,d\mathbf{q}_3\times$$

$$\times\frac{(\mathbf{q}\cdot\mathbf{q}_3)(\mathbf{q}\cdot\mathbf{q}_2)}{\mathbf{q}^2|\mathbf{q}_2||\mathbf{q}_3|}\varepsilon\varepsilon_2\varepsilon_3 n_2 n_3\times$$

$$\times\left[2\delta(1+2-3)e^{\beta\varepsilon_3}+\delta(1-2-3)e^{\beta\varepsilon}\right]. \tag{27}$$

In this expression, we already took the contributions of the all types of the diagrams constructed using the vertices d_1, d_2, and d_3 into account. The contributions from the diagrams d_2 and d_3 are exactly the same by the modulus but have different signs. Therefore, they cancel each other.

The contributions to the component $\Sigma_{\pi\pi}$ from the all topologically equivalent diagrams constructed with the use of the vertices in Fig. 2 are also identical. The contribution from this self-energy part to the imaginary part of the spectrum has the following form:

$$\Delta_{\pi\pi}(\mathbf{q}) = -3\frac{mc^2(1-e^{-\beta\varepsilon})}{128\pi^2n^3}\left(\frac{dn_0}{d\mu}\right)^2\int d\mathbf{q}_2\,d\mathbf{q}_3\times$$

$$\times\frac{(\mathbf{q}_2\cdot\mathbf{q}_3)^2}{\mathbf{q}_2^2\mathbf{q}_3^2}\varepsilon\varepsilon_2\varepsilon_3 n_2 n_3\left[2\delta(1+2-3)+\delta(1-2-3)e^{\beta\varepsilon}\right]. \tag{28}$$

The calculation of the contribution from the components $\Sigma_{\varphi\pi}(q)=-\Sigma_{\pi\varphi}(q)$ (see the diagram d in Fig. 1) requires a more detailed explanation. From Eq. (10), one can see that the contribution to the imaginary part of the spectrum given by the real part of $\Sigma_{\varphi\pi}(q)$ looks as

$$\mathrm{Re}\,\Sigma_{\varphi\pi}(i\omega,\mathbf{q}) = \frac{1}{2(2\pi)^3m^2}\int d\mathbf{q}_2\,d\mathbf{q}_3\,(\mathbf{q}\cdot\mathbf{q}_2)\,(\mathbf{q}_2\cdot\mathbf{q}_3)\times$$

$$\times\delta(\mathbf{q}-\mathbf{q}_2-\mathbf{q}_3)\ \mathrm{Re}\,I(i\omega,\mathbf{q}_2,\mathbf{q}_3). \tag{29}$$

Therefore, it is necessary to calculate the real part of the integral

$$I(i\omega,\mathbf{q}_2,\mathbf{q}_3) = \frac{1}{2\pi}\frac{m^2c^4}{n^2}\frac{dn_0}{d\mu}\int_c dz\ n_{\mathrm{B}}(z)\times$$

$$\times\frac{z}{z^2-\varepsilon_2^2}\frac{1}{(i\omega-z)^2-\varepsilon_3^2}. \tag{30}$$

Integrating over the contour given in Fig. 3, we find

$$I(i\omega,\mathbf{q}_2,\mathbf{q}_3) = \frac{1}{2\pi}\int_c dx\ \frac{m^2c^4}{n^2}\frac{dn_0}{d\mu}\ n_{\mathrm{B}}(x)\times$$

$$\times\left\{\frac{x}{(x-i\omega)^2-\varepsilon_3}\left[\frac{1}{(x+i\delta)^2-\varepsilon_2^2}-\frac{1}{(x-i\delta)^2-\varepsilon_2^2}\right]+\right.$$

$$\left.+\frac{x+i\omega}{(x+i\omega)^2-\varepsilon_3}\left[\frac{1}{(x+i\delta)^2-\varepsilon_3^2}-\frac{1}{(x-i\delta)^2-\varepsilon_3^2}\right]\right\}. \quad (31)$$

In the last expression, the retarded and advanced functions are written in their explicit forms. Using Eqs. (17) and (21), the multipliers in the square brackets can be presented as

$$\left[\frac{1}{(x+i\delta)^2-\varepsilon_i^2}-\frac{1}{(x-i\delta)^2-\varepsilon_i^2}\right]=$$

$$=-\frac{i\pi}{\varepsilon_i}\left[\delta(x-\varepsilon_i)-\delta(x+\varepsilon_i)\right]. \quad (32)$$

After the analytic continuation $i\omega\to E$, changing the variables $x+\omega=x'$ in the second term, and introducing the redefinition $x'=x$, we get the real part of expression (31) as

$$\mathrm{Re}\, I(E,\mathbf{q}_2,\mathbf{q}_3)=\frac{\pi}{4}\frac{m^2c^4}{n^2}\frac{dn_0}{d\mu}\frac{1}{\varepsilon_2\varepsilon_3}\int\limits_c x\,dx\times$$

$$\times\left[\delta(x-E-\varepsilon_3)-\delta(x-E+\varepsilon_3)\right]\times$$

$$\times\left[\delta(x-\varepsilon_2)-\delta(x+\varepsilon_2)\right]\left\{n_{\mathrm{B}}(x)-n_{\mathrm{B}}(x-E)\right\}. \quad (33)$$

Using Eqs. (29) and (10), we find the corresponding contribution to the imaginary part of the spectrum from the diagrams constructed with the help of the vertex parts d_1:

$$\Delta_{\varphi\pi}(\mathbf{q})=2\frac{mc^2(1-e^{-\beta\varepsilon})}{128\pi^2n^3}\left(\frac{dn_0}{d\mu}\right)^2\int d\mathbf{q}_2\,d\mathbf{q}_3\times$$

$$\times\frac{(\mathbf{q}\cdot\mathbf{q}_3)\,(\mathbf{q}_2\cdot\mathbf{q}_3)}{|\mathbf{q}||\mathbf{q}_2|\mathbf{q}_3^2}\varepsilon\varepsilon_2\varepsilon_3 n_2n_3\times$$

$$\times\left[2\delta(1+2-3)e^{\beta\varepsilon_3}+\delta(1-2-3)e^{\beta\varepsilon}\right]. \quad (34)$$

As far as the contributions from the vertex parts d_2 and d_3 have different signs, the last expression corresponds to the contribution from the component $\Sigma_{\varphi\pi}(q)$ to the imaginary part of the spectrum.

According to Eq. (10), the imaginary part of the elementary excitation spectrum is defined as the sum of the contributions of all components of the self-energy matrix which are given by expressions (26) – (28) and (34). The signs of the components $\Sigma_{\pi\pi}$ and $\Sigma_{\varphi\varphi}=\Sigma^b_{\varphi\varphi}+\Sigma^c_{\varphi\varphi}$ in this sum are determined from the condition of positiveness of the expression for the damping of the elementary excitation spectrum

$$\Delta(\mathbf{q})=\frac{9mc^2(1-e^{-\beta\varepsilon})}{128\pi^2n^3}\left(\frac{dn_0}{d\mu}\right)^2\int d\mathbf{q}_2\,d\mathbf{q}_3\;\varepsilon\varepsilon_2\varepsilon_3\times$$

$$\times n_2n_3\left[2\delta(1+2-3)e^{\beta\varepsilon_3}+\delta(1-2-3)e^{\beta\varepsilon}\right]. \quad (35)$$

The last expression is obtained from the condition that factors of the form $\frac{(\mathbf{q}_i\cdot\mathbf{q}_j)}{|\mathbf{q}_i||\mathbf{q}_j|}$ under the integrals in expressions (26) – (28) and (34) are proportional to $\cos\theta$, where θ – the angle between the vectors $\mathbf{q}_i$ and $\mathbf{q}_j$; for the small momenta, this angle is also small, so we can put $\cos\theta\approx 1$.

Expression (35) in the gas approximation (see expression (7)) coincides with the result obtained by Popov and corresponds to the Belyaev's result [3] at zero temperature ($\beta\to\infty$),

$$\Delta(\mathbf{q})=\frac{3\mathbf{q}^5}{640\pi nm}. \quad (36)$$

3. Conclusions

The perturbation theory constructed on the basis of the Popov's hydrodynamic action allows us to avoid the problem of infrared divergences. The singularities of the transverse $g_{\pi\pi}$ and intermediate (longitudinal-transverse) $g_{\phi\pi}$ Green's functions in the higher orders of perturbation theory are compensated by the vertex parts $\sim q^2$ which are given by the cubic term in the expression for the quantum-mechanical action. This behavior of the correlation functions can be understood from the physical point of view, by using the picture of the symmetry breaking discussed by Patashinsky and Pokrovsky [9] for a magnetic system with analogous symmetry.

The representation of free propagators in terms of the inverse condensate compressibility $dn_0/d\mu$, the hydrodynamic speed of sound c, and the density n allows us to find the Green's functions for the system with strong interparticle interaction. Based on this approach in the second order of perturbation theory, the imaginary part Δ of the elementary excitation spectrum was calculated at nonzero temperatures. As far as the obtained expression was calculated for the system with strong interaction between the particles, it can be applied to systems like ^{4}He. In the low-density approximation, the obtained result coincides with the Popov's expression [4]. For small $\mathbf{q}$, the expression for the damping of the spectrum

$$\Delta(\mathbf{q}) = \frac{3\pi^3 T^4 mq}{40n^3}\left(\frac{dn_0}{d\mu}\right)^2$$

is linear in q. The linear dependence becomes improper in the kinetic region $ql \lesssim 1$, $E\tau \lesssim 1$, where l – mean free path, and τ – time of relaxation. The correct answer for this region can be obtained after the summation of the diagrams which can be reduced to the solution of equations of the kinetic type.

All the results are obtained for small values of the momentum and the energy, which corresponds to the phonon part of the spectrum. The calculation of the Green's functions for the higher momenta for the maxon-roton part of the spectrum is rather difficult in this approach, because the effective action is written in the terms of "hydrodynamic" (wave) variables, and it will be the aim of the following research.

The author thanks Dr. C. Vilchynskyy and Dr. M. Weyrauch for the useful discussions and expresses his special gratitude to Dr. M. Weyrauch for the help and to the Physik.-Technische Bundesanstalt (Braunschweig, Germany) for the hospitality and the financial support.

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Received 05.09.06

ДО РОЗРАХУНКУ УЯВНОЇ ЧАСТИНИ
СПЕКТРА ЕЛЕМЕНТАРНИХ ЗБУДЖЕНЬ
СИЛЬНОВЗАЄМОДІЮЧОЇ БОЗЕ-СИСТЕМИ

А. Чумаченко

Р е з ю м е

На основі мікроскопічного підходу з використанням методу гідродинамічних змінних Попова розраховано уявну частину спектра елементарних збуджень сильнозв'язаної бозе-системи з конденсацією при малих температурах.