CONICAL REFRACTION OF RELATIVISTIC PARTICLES WITH SPIN 1/2 AND 1

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The quasiclassical equations of motion for relativistic particles with spins 1/2 and 1 which possess anomalous magnetic moments have been considered. The motion of particles in a constant uniform electromagnetic field has been analyzed. It has been demonstrated that, under definite conditions, phenomena similar to the optical conical refraction can be observed.

1. Introduction

The motion of particles with a large spin in external fields has been considered in a lot of works (see, e.g., works [1–3]). It was proved that the application of relativistically invariant equations for the description of such particles gives rise to the violation of causality. In work [2], a possibility for a particle with spin 1/2 to achieve the speed of light was indicated as well.

There is a standard way — by using the Dirac equation — to describe a particle with spin 1/2. A good number of approaches has been developed for particles with higher spin, but equations, which are eligible for a free particle, can become invalid even if one would take into account, at a minimal level, the electromagnetic interaction (for example, the Bargman—Wigner equation). There is also a substantial ambiguity with respect to the account of the anomalous magnetic moment. For example, there are different ways to introduce the corresponding terms in the Proca equation used by the majority of authors; accordingly, different conclusions can be arrived at.

In this work, in the framework of the Kemmer—Duffin formalism, we analyze the behavior of a neutral particle with spin 1 and an anomalous magnetic moment in an electromagnetic field. The particle's motion not only reveals noncausality but discovers another property, a possibility of conical refraction.

Conical refraction in physical optics means a specific refraction of light in biaxial crystals, when the light beam splits into many beams propagating along a conical surface. This phenomenon was predicted theoretically in 1832 by W.R. Hamilton, who used the Huygens—Fresnel principle while examining the propagation of light in a biaxial crystal along the optical axis. In 1833, conical refraction was discovered experimentally by H. Lloyd.

In work [4], the quasiclassical motion of a neutral particle with spin 1/2 and an anomalous magnetic moment has been considered. It was proved that the conical refraction can take place for such a particle too, and this refraction can be observed in very high-intensity electric fields.

2. Conical Refraction of Spin-1/2 Particles

Consider the motion of a neutral particle with spin 1/2 and an anomalous magnetic moment μ in a constant uniform electromagnetic filed $F_{\mu\nu}$ [4]. The dispersion equation looks like

$$\left|i\gamma_\nu p_\nu - \frac{\mu}{2}\sigma_{\rho\nu}F_{\rho\nu} + mI\right| = 0, \quad (1)$$

where $\sigma_{\rho\nu} = \frac{\gamma_\rho\gamma_\nu - \gamma_\nu\gamma_\rho}{4i}$, and the matrices and the metrics are the same as, e.g., in work [5] (hereafter, $c = \hbar = 1$). The determinant is equal to

$$\left(\varepsilon^2 - \vec{E}^2 - \vec{H}^2 - \vec{p}^2 - m^2\right)^2 - 4\Bigg(\left[\vec{E}\vec{p}\right]^2 + \left[\vec{H}\vec{p}\right]^2 +$$

$$+ \left[\vec{E}\vec{H}\right]^2 + m^2\vec{H}^2 - 2\varepsilon\vec{p}\left[\vec{E}\vec{H}\right]\Bigg) = 0. \quad (2)$$

Equation (2) makes it possible to calculate the group velocity

$$\vec{V} = \frac{\partial\varepsilon}{\partial\vec{p}} = \left\{\left(\varepsilon^2 - \vec{E}^2 - \vec{H}^2 - \vec{p}^2 - m^2\right)\vec{p} - \right.$$

$$\left. -2\left[\vec{E}\left[\vec{E}\vec{p}\right]\right] - 2\left[\vec{H}\left[\vec{H}\vec{p}\right]\right] - 2\varepsilon\left[\vec{E}\vec{H}\right]\right\}\times$$

$$\times\left\{\varepsilon\left(\varepsilon^2 - \vec{E}^2 - \vec{H}^2 - \vec{p}^2 - m^2\right) + 2\vec{p}\left[\vec{E}\vec{H}\right]\right\}^{-1}. \quad (3)$$

If the invariant $\left(\vec{E}\vec{H}\right)$ differs from zero, one can always pass to a reference system, where $\vec{E}\|\vec{H}$. In such a system, formula (3), taking Eq. (2) into account, transforms into

$$\vec{V} = \frac{\vec{p}}{\varepsilon} \pm \frac{\left[\vec{E}\left[\vec{E}\vec{p}\right]\right] + \left[\vec{H}\left[\vec{H}\vec{p}\right]\right]}{\varepsilon\sqrt{\left[\vec{E}\vec{p}\right]^2 + \left[\vec{H}\vec{p}\right]^2 + m^2\vec{H}^2}}, \quad (4)$$

where different signs correspond to different polarizations of the particle. It is evident from Eq. (2) that ε and $\vec{p}$ can be equal to zero simultaneously only if $\left(\vec{E}\vec{H}\right) = 0$ and $\vec{E}^2 - \vec{H}^2 + m^2 = 0$. In this case, the particle with one of the polarizations moves with the speed of light along the magnetic field in the reference system, where $\vec{E} = 0$ [2]. One can also prove that the inequality $\varepsilon \neq 0$ always holds true, provided that $\vec{p} \neq 0$ (the causality is not violated in the case of spin-1/2 particle).

The denominator in formula (4) can equal zero in another case, namely, if $\vec{H} = 0$. Then, from Eq. (4), we obtain

$$\vec{V} = \frac{\vec{p}}{\varepsilon} \pm \frac{\left[\vec{E}\left[\vec{E}\vec{p}\right]\right]}{\varepsilon\left|\left[\vec{E}\vec{p}\right]\right|}, \quad (5)$$

where $\varepsilon = \sqrt{\vec{E}^2 + \vec{p}^2 + m^2 + 2\left|\left[\vec{E}\vec{p}\right]\right|}$. If $\vec{p}\|\vec{E}$, there appears an uncertainty of the type 0/0 in Eq. (5). If the vector $\vec{p}$ has an infinitesimally small perpendicular component $\vec{p}_\perp$, Eq. (5) transforms into

$$\vec{V} = \frac{\vec{p}}{\varepsilon} \mp \frac{\left|\vec{E}\right|\vec{p}_\perp}{\varepsilon\left|\vec{p}_\perp\right|}. \quad (6)$$

Equation (6) demonstrates that the infinitesimally small component $\vec{p}_\perp$ affects the particle's velocity. Therefore, the beam of particles propagating in one direction splits, and the particles start to move along a conical surface.

3. Impossibility of Causality Violation for Moving Particles with Spin 1/2

Consider the Dirac equation

$$\left(\gamma^\mu\frac{\partial}{\partial x_\mu} + Q\right)\Psi = 0, \quad (7)$$

where Q is a 4×4-matrix, which does not depend on coordinates and corresponds to the condition that the wave function $\overline{\Psi} = \Psi^*\gamma^4$ would satisfy the equation

$$\overline{\Psi}\left(\gamma^\mu\frac{\partial}{\partial x_\mu} + Q\right) = 0, \quad (8)$$

where the derivative $\frac{\partial}{\partial x_\mu}$ operates to the left. Equations (7) and (8) result in

$$\frac{\partial}{\partial x_\mu} i\left(\overline{\Psi}\gamma^\mu\Psi\right) = \frac{\partial J_\mu}{\partial x_\mu} = 0,$$

i.e. the continuity equation for the probability 4-current J_μ holds true.

In the case of a plane wave $\Psi = u\left(\vec{p}\right)e^{i\vec{p}\vec{x}-\varepsilon t}$, we obtain the equations

$$\left(-\varepsilon\gamma^4 + i\vec{p}\vec{\gamma} + Q\right)u\left(\vec{p}\right) = 0,$$

$$\overline{u}\left(\vec{p}\right)\left(-\varepsilon\gamma^4 + i\vec{p}\vec{\gamma} + Q\right) = 0. \quad (9)$$

Differentiating the first of them with respect to $\vec{p}$, we obtain

$$\left(-\vec{v}\gamma^4 + i\vec{\gamma}\right)u\left(\vec{p}\right) -$$

$$-\left(-\varepsilon\gamma^4 + i\vec{p}\vec{\gamma} + Q\right)\frac{\partial u\left(\vec{p}\right)}{\partial\vec{p}} = 0. \quad (10)$$

Multiplying Eq. (10) by $\bar{u}(\vec{p})$, we obtain

$$-\vec{v}\left(\overline{u}\gamma^4 u\right) + i\left(\overline{u}\vec{\gamma}u\right) = 0$$

or

$$\vec{v} = \frac{i\left(\overline{u}\vec{\gamma}u\right)}{\left(\overline{u}\gamma^4 u\right)} = \frac{\vec{j}}{\rho}.$$

The velocity $\vec{v}$ becomes infinitely large in none of the reference systems, because ρ is a positive definite quantity, so that the particle's velocity is always less than the speed of light.

Notice that the Dirac equation is the only example among the finite-dimensional relativistic wave equations, in which the probability density has a positive value, i.e., in all other cases, the violation of causality is possible.

4. Conical Refraction of Spin-1 Particles

The Kemmer—Duffin equation for a neutral spin-one particle with an anomalous magnetic moment looks like

$$\left[\left(\gamma_\nu^{(1)} + \gamma_\nu^{(2)}\right)\frac{\partial}{\partial x_\nu} + \right.$$

$$\left. + \frac{\mu}{2}\left(\sigma_{\rho\nu}^{(1)} + \sigma_{\rho\nu}^{(2)}\right)F_{\rho\nu} + 2m\right]\Phi = 0, \qquad (11)$$

where the superscripts indicate the (first or second) index of the wave function $\Phi_{\alpha\beta} = \Phi_{\beta\alpha}$, over which the summation is carried out when multiplying by the Dirac matrices.

In order to pass to the quasiclassical description, a solution is sought in the form

$$\Phi = e^{ipx}u. \qquad (12)$$

Then, from Eq. (11), we obtain

$$\left[i\left(\gamma_\nu^{(1)} + \gamma_\nu^{(2)}\right)p_\nu + \frac{\mu}{2}\left(\sigma_{\rho\nu}^{(1)} + \sigma_{\rho\nu}^{(2)}\right)F_{\rho\nu} + 2m\right]u =$$

$$= \left[Q^{(1)} + Q^{(2)}\right]u = 0, \qquad (13)$$

where

$$Q = i\gamma_\nu p_\nu + \frac{\mu}{2}\sigma_{\rho\nu}F_{\rho\nu} + mI.$$

To obtain the characteristic equation, we use the following relation:

$$\left|Q^{(1)} + Q^{(2)}\right| = \frac{8}{9}\left|Q\right|\left\{\left(\mathrm{Sp}\left(Q\right)\right)^6 - 2\left(\mathrm{Sp}\left(Q^3\right)\right)^2 - \right.$$

$$-3\left(\mathrm{Sp}\left(Q\right)\right)^4\mathrm{Sp}\left(Q^2\right) + \left(\mathrm{Sp}\left(Q\right)\right)^3\mathrm{Sp}\left(Q^3\right) +$$

$$\left. +3\mathrm{Sp}\left(Q\right)\mathrm{Sp}\left(Q^2\right)\mathrm{Sp}\left(Q^3\right) - 18\left|Q\right|\left(\mathrm{Sp}\left(Q\right)\right)^2\right\}, \qquad (14)$$

where Q is an arbitrary 4×4-matrix. The validity of formula (14) can be proved rather easily for a diagonal matrix Q. In the general case, the matrix Q can be diagonalized by applying the corresponding similarity transformation. Taking into account that the determinant and the trace of the matrix are invariant with respect to such transformations, we arrive at a conclusion that formula (14) is eligible in our case. This formula turns out very useful, because it makes possible to split our characteristic equation into two ones. The first equation reads

$$|Q| = 0. \qquad (15)$$

Another factor in Eq. (14) gives rise to the equation

$$\left(m^2 - \vec{H}^2\right)\varepsilon^2 + 2\vec{p}\left[\vec{E}\vec{H}\right]\varepsilon + \left(\vec{E}\vec{H}\right)^2 + \left(\vec{H}\vec{p}\right)^2 -$$

$$-m^2\left(m^2 + \vec{p}^2 + \vec{E}^2 - \vec{H}^2\right) - \left[\vec{E}\vec{p}\right]^2 = 0. \qquad (16)$$

Equation (15) is equivalent to Eq. (2) and was discussed above. It describes the projections ± 1 of the spin; this means that, for those projection values, the result completely coincides with that obtained for a spin-$1/2$ particle. Therefore, the motion of particles with spin projections ± 1 does not differ from that of particles with spin $1/2$ and spin projections $\pm 1/2$, so that the phenomenon of conical refraction also takes place for such particles.

If the spin projection of a particle is equal to zero, the violation of causality becomes possible. For example, from Eq. (16), we obtain the following formula for the particle's velocity in the case of the particle's motion in a magnetic field:

$$\vec{V} = \frac{m^2\vec{p} - (\vec{H}\vec{p})\vec{H}}{\sqrt{(m^2 - H^2)(m^2(m^2 + p^2 - H^2) - (\vec{H}\vec{p})^2)}}. \qquad (17)$$

If $H^2 = m^2$ and the particle does not move along the direction of the magnetic field, its velocity becomes infinite, i.e. the causality is violated.

Note that the substitution of Eq. (11) by the equation

$$[(\gamma_\nu^{(1)} + \gamma_\nu^{(2)})\frac{\partial}{\partial x_\nu} + \frac{\mu}{2}F_{\mu\nu}(\sigma_{\mu\nu}^{(1)} + \sigma_{\mu\nu}^{(2)}) +$$

$$+\frac{\eta}{2}F_{\mu\nu}(\sigma^{(1)}_{\mu\nu}+\sigma^{(2)}_{\mu\nu})(\gamma^{(1)}_5\gamma^{(2)}_5)+2m]\Psi=0 \quad (18)$$

does not change the anomalous magnetic moment of the particle, but does change the characteristic equation. In a plenty of works, including work [2], the condition $\eta=\mu$ was adopted. It explains the difference between our results and those obtained in work [2]. But we believe that the relation $\eta=0$ is more natural.

APPENDIX
Calculation of the Determinants of Matrices Associated with the Algebra of Dirac Matrices

The determinant of an arbitrary matrix can be expressed in terms of the traces (spurs) of the powers of this matrix. For example, for a 4×4-matrix, the corresponding formula looks like

$$|Q|\equiv\frac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma}Q^\mu_\alpha Q^\nu_\beta Q^\rho_\gamma Q^\sigma_\delta=$$

$$=\frac{1}{24}(\mathrm{Sp}\ Q)^4-\frac{1}{4}(\mathrm{Sp}\ Q)^2\mathrm{Sp}\ (Q^2)+\frac{1}{8}(\mathrm{Sp}\ (Q^2))^2+$$

$$+\frac{1}{3}\mathrm{Sp}\ Q\ \mathrm{Sp}\ (Q^3)-\frac{1}{4}\mathrm{Sp}\ (Q^4). \quad \text{(D1)}$$

Formula (D1) is the result of calculating the determinant and the known formula (see work [6]) which expresses the product of two completely antisymmetric unit tensors $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma}$ as the determinant of a matrix that includes Kronecker symbols. Formula (D1) is very useful in the case where the spin equals 1/2, because the procedures of multiplication of the Dirac matrices and the calculation of the traces of their products are well elaborated. Unfortunately, for spin 1, when a necessity to calculate the determinants of 10×10-matrices emerges, a direct way to calculate the determinants in a formula similar to Eq. (D1) is very difficult. Nevertheless, there is an alternative, more attractive opportunity. Note that formula (D1) is invariant with respect to the similarity transformation of the matrix Q: $Q\to Q'=\Lambda Q\Lambda^{-1}$. Let the matrix Q can be reduced by carrying out the similarity transformation to a diagonal form with the diagonal elements x_1, x_2, x_3, and x_4. (If it is not so, the matrix Q can be deformed by summing up with another infinitesimally small matrix ε, $Q\to Q+\varepsilon$, so that the new matrix can be reduced to a diagonal one. In the final formulas obtained, ε should be put equal to zero.[1]) For diagonal matrices, formula (D1) becomes an identity law, known as one of Waring's formulas,

$$x_1x_2x_3x_4=\frac{1}{24}(x_1+x_2+x_3+x_4)^4-$$

$$-\frac{1}{4}(x_1+x_2+x_3+x_4)^2(x_1^2+x_2^2+x_3^2+x_4^2)+$$

$$+\frac{1}{8}(x_1^2+x_2^2+x_3^2+x_4^2)^2+\frac{1}{3}(x_1+x_2+x_3+x_4)\times$$

$$\times(x_1^3+x_2^3+x_3^3+x_4^3)-\frac{1}{4}(x_1^4+x_2^4+x_3^4+x_4^4). \quad \text{(D2)}$$

It is easy to validate the following identity which corresponds to formula (16) for the calculation of the determinant:

$$2^4x_1x_2x_3x_4(x_1+x_2)(x_1+x_3)(x_1+x_4)\times$$

$$\times(x_2+x_3)(x_2+x_4)(x_3+x_4)=\frac{8}{9}x_1x_2x_3x_4\times$$

$$\times[(x_1+x_2+x_3+x_4)^6-2(x_1^3+x_2^3+x_3^3+x_4^3)^2-$$

$$-3(x_1+x_2+x_3+x_4)^4(x_1^2+x_2^2+x_3^2+x_4^2)+$$

$$+(x_1+x_2+x_3+x_4)^3(x_1^3+x_2^3+x_3^3+x_4^3)+$$

$$+3(x_1+x_2+x_3+x_4)(x_1^2+x_2^2+x_3^2+x_4^2)\times$$

$$\times(x_1^3+x_2^3+x_3^3+x_4^3)-18x_1x_2x_3x_4(x_1^2+x_2^2+x_3^2+x_4^2)]. \quad \text{(D3)}$$

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КОНІЧНА РЕФРАКЦІЯ РЕЛЯТИВІСТСЬКИХ ЧАСТИНОК ЗІ СПІНАМИ 1/2 ТА 1

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Р е з ю м е

Розглянуто квазікласичні рівняння руху релятивістських частинок зі спінами 1/2 і 1 та аномальними магнітними моментами. Проаналізовано рух частинок у постійному та однорідному електромагнітному полі. Виявлено, що за певних умов виникають явища, аналогічні конічній рефракції в оптиці.

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