ALGEBRAIC DERIVATION OF THE SPECTRUM OF THE DIRAC HAMILTONIAN FOR AN ARBITRARY COMBINATION OF THE LORENTZ-SCALAR AND LORENTZ-VECTOR COULOMB POTENTIALS[1]

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The spectrum of the Dirac Equation is obtained algebraically for an arbitrary combination of the Lorentz-scalar and Lorentz-vector Coulomb potentials using the Witten's superalgebra approach. The result coincides with that known from the explicit solution of the Dirac equation.

Recently [1], we have derived the dynamical symmetry operator for the Dirac Hamiltonian in the case of an arbitrary combination of the Lorentz-scalar and Lorentz-vector Coulomb potentials. The operator of this kind was known earlier [2] , but our derivation seems to be very simple and more transparent. Our consideration concerns the general central potentials $V(r)$ and $S(r)$ in the Dirac equation. By requiring the anticommutativity of the above-mentioned operator with the Dirac operator

$$K = \beta \left(\vec{\Sigma} \cdot \vec{l} + 1\right) \tag{1}$$

and its commutativity with the Dirac Hamiltonian

$$H = \vec{\alpha} \cdot \vec{p} + \beta m + V(r) + \beta S(r), \tag{2}$$

we have obtained the following expression for it:

$$X = \gamma^5 (\vec{\alpha} \cdot \hat{\vec{r}})(m a_V + H a_S) + \imath \gamma^5 K (H - \beta m). \tag{3}$$

Here, $a_{V,S}$ are the strengths of the corresponding Coulomb potentials.

It is worth to underline that the commutativity (i.e. the symmetry) with the Dirac Hamiltonian (2) is valid only for Coulomb potentials, as a result of the degeneracy relative to two signs of the eigenvalue of the K operator, $\pm\kappa$.

In the present paper, we want to obtain the spectrum of the Dirac Hamiltonian pure algebraically, without any referring to the equations of motion. Our method is based on the Witten's superalgebra which appears immediately as soon as the anticommutator with the K operator is constructed. It is only sufficient to introduce the SUSY generators as follows:

$$Q_1 = X, \quad Q_2 = \imath \frac{XK}{|\kappa|}. \tag{4}$$

Then the anticommutativity, $\{X, K\} = 0$, yields

$$\{Q_1, Q_2\} = 0, \quad Q_1^2 = Q_2^2 \equiv h. \tag{5}$$

Therefore, we are faced with the Witten's algebra $S(2)$, where $h = X^2$ is the Witten's Hamiltonian.

Now we explore this algebra in a manner as in work [3]. Let us define a SUSY ground state $|0\rangle$:

$$h|0\rangle = X^2|0\rangle = 0 \quad \longrightarrow \quad X|0\rangle = 0. \tag{6}$$

Because X^2 is the square of the Hermitian operator, it has a positive definite spectrum, and it is convenient to put this operator to zero in the ground state. By this requirement, we will obtain the Hamiltonian in this ground state and, correspondingly, the ground state energy. Then, by using the well-known ladder procedure, we can construct the energies of all excited levels. We

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believe that this method requires a stronger justification, but nevertheless we are sure that it is true.

Let us equate $X = 0$. It follows from Eq. (3) that

$$H = m[(\vec{\alpha}\cdot\hat{\vec{r}})a_S + \imath K]^{-1}[\imath K\beta - a_V(\vec{\alpha}\cdot\hat{\vec{r}})] = \frac{m}{\kappa^2 + a_S^2}N, \quad (7)$$

where

$$N \equiv [(\vec{\alpha}\cdot\hat{\vec{r}})a_S - \imath K][\imath K\beta - a_V(\vec{\alpha}\cdot\hat{\vec{r}})] =$$

$$= -a_S a_V + K[K\beta + \imath a_V(\vec{\alpha}\cdot\hat{\vec{r}})] - \imath a_S K\beta(\vec{\alpha}\cdot\hat{\vec{r}}). \quad (8)$$

Now we will try to diagonalize this operator using a transformation similar to the Foldy–Wouthuysen one [4]. Because the second and third terms do not commute with each other, we need several (at least two) such transformations.

We choose the first transformation in the form

$$\exp(\imath S_1) = \exp\left(-\frac{1}{2}\beta(\vec{\alpha}\cdot\hat{\vec{r}})w_1\right). \quad (9)$$

It is evident that

$$\exp(\imath S_1)(\vec{\alpha}\cdot\hat{\vec{r}})\exp(-\imath S_1) = \exp(2\imath S_1)(\vec{\alpha}\cdot\hat{\vec{r}}),$$

$$\exp(\imath S_1)\beta\exp(-\imath S_1) = \exp(2\imath S_1)\beta. \quad (10)$$

Moreover,

$$\exp(\imath S_1)K\exp(-\imath S_1) = K,$$

$$\exp(\imath S_1)\beta K\exp(-\imath S_1) = \exp(2\imath S_1)\beta K, \quad (11)$$

and

$$\exp(\imath S_1)\beta(\vec{\alpha}\cdot\hat{\vec{r}})\exp(-\imath S_1) = \beta(\vec{\alpha}\cdot\hat{\vec{r}}). \quad (12)$$

Therefore, the first transformation acts as

$$N' \equiv \exp(\imath S_1)N\exp(-\imath S_1) = -a_S a_V +$$

$$+K\exp(2\imath S_1)[K\beta + \imath a_V(\vec{\alpha}\cdot\hat{\vec{r}})] - \imath a_S K\beta(\vec{\alpha}\cdot\hat{\vec{r}}). \quad (13)$$

Using the relation $\exp(2\imath S_1) = \cosh w_1 + \imath\beta(\vec{\alpha}\cdot\hat{\vec{r}})\times \times\sinh w_1$, we have

$$\exp(2\imath S_1)[K\beta + \imath a_V(\vec{\alpha}\cdot\hat{\vec{r}})] = \beta[K\cosh w_1 +$$

$$+a_V\sinh w_1] + K(\vec{\alpha}\cdot\hat{\vec{r}})[\imath a_V\cosh w_1 + \imath K\sinh w_1]. \quad (14)$$

Now, in order to get rid of the non-diagonal $(\vec{\alpha}\cdot\hat{\vec{r}})$ terms, we must choose

$$\tanh w_1 = -\frac{a_V}{K}. \quad (15)$$

Using simple trigonometric relations, we arrive at

$$\exp(2\imath S_1)[K + \imath a_V(\vec{\alpha}\cdot\hat{\vec{r}})] = K^{-1}\beta\sqrt{\kappa^2 - a_V^2}. \quad (16)$$

Let us perform the second Foldy–Wouthuysen transformation

$$N'' = \exp(\imath S_2)N'\exp(-\imath S_2), \quad (17)$$

where

$$S_2 = -\frac{1}{2}(\vec{\alpha}\cdot\hat{\vec{r}})w_2.$$

Now we have

$$\exp(\imath S_2)K\beta\exp(-\imath S_2) = \exp(2\imath S_2)K\beta, \quad (18)$$

$$\exp(\imath S_2)K\beta(\vec{\alpha}\cdot\hat{\vec{r}})\exp(-\imath S_2) = \exp(2\imath S_2)K\beta(\vec{\alpha}\cdot\hat{\vec{r}}),$$

$$\exp(2\imath S_2) = \cos w_2 - \imath(\vec{\alpha}\cdot\hat{\vec{r}})\sin w_2.$$

Therefore

$$N'' = -a_S a_V + K\sqrt{\kappa^2 - a_V^2}\exp(2\imath S_2)\beta -$$

$$-\imath a_S\exp(2\imath S_2)K\beta(\vec{\alpha}\cdot\hat{\vec{r}})$$

$$= -a_S a_V + K\sqrt{\kappa^2 - a_V^2}\beta\cos w_2 +$$

$$+\imath K\sqrt{\kappa^2 - a_V^2}\beta(\vec{\alpha}\cdot\hat{\vec{r}})\sin w_2 -$$

$$-\imath a_S K\beta(\vec{\alpha}\cdot\hat{\vec{r}})\cos w_2 + a_S K\beta\sin w_2 \quad (19)$$

Requiring absence of $(\vec{\alpha}\cdot\hat{\vec{r}})$ terms we have

$$\tan w_2 = \frac{a_S}{\sqrt{\kappa^2 - a_V^2}} \quad (20)$$

Therefore,

$$N'' = -a_S a_V + K\beta\sqrt{\kappa^2 - a_V^2 + a_S^2}, \quad (21)$$

and, finally, we get

$$H = \frac{m}{\kappa^2 + a_S^2}\{-a_S a_V + K\beta\sqrt{\kappa^2 - a_V^2 + a_S^2}\}. \quad (22)$$

For eigenvalues in the ground state, we have

$$E_0 = \frac{m}{\kappa^2 + a_S^2}\left[-a_S a_V \pm \kappa\sqrt{\kappa^2 - a_V^2 + a_S^2}\right]. \quad (23)$$

Now let us remember the explicit solution of the Dirac equation in this case [5],

$$\frac{E}{m} = \frac{-a_S a_V}{a_V^2 + (n - |\kappa| + \gamma)^2} \pm$$

$$\pm\sqrt{\left(\frac{a_S a_V}{a_V^2+(n-|\kappa|+\gamma)^2}\right)^2+\frac{(n-|\kappa|+\gamma)^2-a_S^2}{a_V^2+(n-|\kappa|+\gamma)^2}}, \quad (24)$$

where

$$\gamma^2=\kappa^2-a_V^2+a_S^2. \quad (25)$$

In the ground state, $n=1$, $j=1/2\longrightarrow|\kappa|=j+1/2=1$. Thus, we get

$$E_0=m\left[\frac{-a_S a_V}{a_V^2+\gamma^2}\pm\sqrt{\left(\frac{a_S a_V}{a_V^2+\gamma^2}\right)^2+\frac{\gamma^2-a_S^2}{a_V^2+\gamma^2}}\right]. \quad (26)$$

By the obvious manipulations, this formula is reduced to our above-derived expression (23). Therefore, by only algebraic methods, we have obtained the correct expression for the ground state energy. In order to obtain the total spectrum, it is sufficient now to use the Witten's algebra. Following work [3], the ordinary ladder procedure consists in the change (in our case)

$$\gamma\longrightarrow\gamma+n-|\kappa|.$$

In view of this change, our lowest-energy formula (23) yields the correct expression for the total energy spectrum, Eq. (24).

Thus, by using the pure algebraic manipulations, we have obtained the spectrum of a generalized Coulomb problem of the Dirac equation for an arbitrary combination of the Lorentz-scalar and Lorentz-vector potentials. This fact demonstrates a power of symmetry considerations, while the method developed here requires a stronger justification, which will be made in future investigations.

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АЛГЕБРИЧНЕ ВИВЕДЕННЯ СПЕКТРА ГАМІЛЬТОНІАНА ДІРАКА ДЛЯ ДОВІЛЬНОЇ КОМБІНАЦІЇ ЛОРЕНТЦ-СКАЛЯРНОГО ТА ЛОРЕНТЦ-ВЕКТОРНОГО КУЛОНІВСЬКИХ ПОТЕНЦІАЛІВ

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Р е з ю м е

Отримано спектр енергії рівняння Дірака для довільної комбінації лорентц-скалярного та лорентц-векторного кулонівських потенціалів за допомогою супералгебри Віттена. Результат збігається з відомим точним розв'язком рівняння Дірака.