GENERAL QUESTIONS OF THERMODYNAMICS, STATISTICAL PHYSICS, AND QUANTUM MECHANICS

ON THE MOTION OF VORTEX RING IN AN INCOMPRESSIBLE MEDIUM

O.K. CHEREMNYKH

Space Research Institute, Nat. Acad. Sci. of Ukraine and Nat. Space Agency of Ukraine
(40, Academician Glushkov Ave., Kyiv 03680, Ukraine; e-mail: ch_ ol@space.is.kiev.ua)

UDC 539

The stationary motion of an axisymmetric vortex ring in an incompressible medium, where the velocity $\vec{v}$ and the density ρ satisfy the equations div $\vec{v} = 0$ and $\vec{v}\nabla\rho = 0$, is considered. The latter equation allows the motion of a vortex ring with the density distributed in space to be analyzed. It has been shown that the density of the incompressible medium can be inhomogeneous only in the vortex motion region and is constant in the potential motion one. Taking this fact into account, the velocity of the ring and the shape of its atmosphere were found to depend not only on the geometrical dimensions of the vortex core and the amplitude of the external velocity vorticity, but also on the spatial distribution of the density in the vortex core.

1. Introduction

The paper reports some results of the theoretical research of the structure and the motion of axisymmetric vortex rings. The latter were known for a rather long time; they are easily produced and observed experimentally [1, 2]. At the end of the nineteenth century, this hydrodynamic phenomenon was intensively studied in connection with the attempts to build the vortex model of atoms [3]. Later on, the interest was caused by the necessity of studying the dynamics of characteristic mushroom-like clouds, which are formed after the explosions of large charges and whose structure is similar to that of the vortex ring [4, 5]. Recently, the vortex formations [6, 7] were regarded as potential generators of "magnetic clouds" or coronal mass ejections from the Sun [8].

The name "vortex ring" is usually associated with a vortical region, confined in space, which moves through a surrounding potential current, so that the vorticity $\vec{\omega}$ is distinct from zero only inside some boundary. Since the motion of the environment at infinity must be potential, the vorticity has to vanish beyond the region, which is bounded by a closed stream line, and the velocity components have to be continuous on this line. In the general case, there is a jump of the vorticity at the vortex region boundary, while the continuity of the pressure follows from the continuity of the velocity. As a result, the problem of matching the potential and the vortex flow arises [1, 2, 9, 10]. In such a formulation, the problem has been studied only for some elementary functions describing $\vec{\omega}$. As an example of the exact solution of the problem, the Hill spherical vortex [11] may be mentioned, which, however, is not observed in reality. The Maxwell vortex [1, 10] is more similar to what is observed; here, the vorticity region comprises a torus, whose transverse cross-section radius a is much less than the torus radius R.

A remarkable feature of all observable vortex rings is their capability to move almost stationary for a long time. The problem of describing such a motion drew attention of a good many theoreticians. Among a number of papers dealing with this subject, that of Lord Kelvin [12] ranks as the most outstanding one: in 1867, in his remark to Tait's translation of Helmholtz's article [13], he wrote down – without any proof – a valid result for the velocity V of motion of a vortex ring with a "uniform" vorticity in the core,

$$V = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a} - \frac{1}{4}\right],$$

where Γ is the circulation of the velocity in the external flow about the ring. The applicability of Lord Kelvin's result to a thin homogeneous ring was later confirmed by Hicks [10, 14], who has also derived the formula for the velocity of motion of a hollow vortex ring

$$V = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a} - \frac{1}{2}\right].$$

In 1970, three researchers independently published their solutions for the velocity of motion of a vortex ring with a small transverse cross-section and an arbitrary distribution of vorticity in the vortex core. These were Saffman [15], who, in order to find the solution, used the theorems on the energy and momentum of the vortex, as well as the transformation suggested by Lamb [10]; Fraenkel [16], who, by using the integral expression for the stream function, found an asymptotic solution for a nontwisted vortex ring; and Bliss [17], who obtained the solution, by using the method of matching the asymptotic expansions.

Fraenkel's and Bliss's solutions were derived in the framework of asymptotic methods and can be extended to include a higher order of the small parameter $\varepsilon = a/R$, the ratio between the core and ring radii. Fraenkel [16] showed that the error in the asymptotic formula for the velocity of motion of an arbitrary ring turned out by two orders higher in ε. Those two solutions give the equivalent expressions for the ring velocity,

$$V = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a} + A - \frac{1}{2}\right],$$

where the quantity A depends only on the vorticity distribution shape in the vortex ring core. For a uniform vorticity, $A = 1/4$, so that the final result for the ring velocity coincides with that obtained by Lord Kelvin, and the stream field in the vortex ring core can be determined in the framework of either Fraenkel's [16] or Bliss's [17] method.

An alternative method for the calculation of the motion velocity of thin vortex rings is based on a procedure proposed by Lamb [10]. Saffman [15] applied Lamb's method to describe vortex rings with an inhomogeneous distribution of vorticity and a nonzero twisting. In particular, he obtained the following formula for the ring velocity:

$$V = \frac{\Gamma}{4\pi R}\left[\ln\frac{8R}{a} - \frac{1}{2} + 2\pi^2\frac{a^2\overline{v_\omega^2}}{\Gamma^2} - 4\pi^2\frac{a^2\overline{v_\varphi^2}}{\Gamma^2}\right].$$

Here, the bar above a quantity denotes the averaging over the cross-section of the vortex thread core, and v_ω and v_φ are the poloidal and twist velocities, respectively. One can see that the twisting decelerates the motion of the ring, and, if the former is large enough, the ring motion direction can be changed.

Despite the availability of a plenty of articles devoted to vortex rings, the interesting issue – from both scientific and practical viewpoints – on their motion and structure has not been studied till now. It concerns the distribution of the density and the pressure in a moving vortex ring and in its atmosphere. It should be noted that all the results obtained for vortex rings by now are based on the assumption that the medium density is constant both in the ring and in its atmosphere. From the mathematical point of view, such an assumption substantially simplifies the vortex motion description, but, at the same time, considerably restricts the class of possible solutions. This work aims at studying the existence and the motion of vortex rings, the density and the pressure in which are spatially distributed.

2. Basic Equations

While considering a stationary motion in an incompressible medium, the viscosity of which can be neglected, the basic equations for the analysis are [1, 10]

$$\rho\,(\vec{v}\cdot\nabla)\,\vec{v} = -\nabla\,p, \tag{1}$$

$$\mathrm{div}\vec{v} = 0, \tag{2}$$

$$\vec{v}\cdot\nabla\rho = 0. \tag{3}$$

It follows from these equations, that, among three components of the velocity $\vec{v}$ that appear in the equations of motion, only two are independent. The third component can be obtained from Eq. (2). Equation (3) determines the density of the medium. The pressure in the medium is a passive function, determined by one of the components of Eq. (1). The equation, which couples the density and the pressure, can be omitted in this case, because those quantities are determined independently from Eqs. (1) and (3).

Let us rewrite Eqs. (1)—(3) in an arbitrary orthogonal coordinate system $\left(x^1, x^2, x^3\right)$, assuming that the flow is axisymmetric, and the condition for the axial symmetry of the motion looks like

$$\frac{\partial}{\partial x^3} = 0. \tag{4}$$

From the condition of medium incompressibility (2), it follows that the velocity field can be expressed in terms of the stream function $\psi\left(x^1, x^2\right)$ and the function $A\left(x^1, x^2\right)$ which describes the rate of twisting as

$$\vec{v} = \left[\nabla\,\psi\times\vec{e}^{\,3}\right] + \frac{A\,\left(x^1, x^2\right)}{\sqrt{g}}\vec{e}_3, \tag{5}$$

where $g = \det g_{ik}$ is the determinant of the metric tensor.

For the further analysis, it is convenient to express the Euler equation (1) as

$$\rho[\vec{\omega} \times \vec{v}] + \nabla p^* - \frac{\vec{v}^2}{2} \nabla \rho = 0, \tag{6}$$

where

$$\vec{\omega} \equiv \operatorname{rot} \vec{v}, \quad p^* \equiv p + \rho \vec{v}^2/2 \tag{7}$$

The vorticity $\vec{\omega}$ in Eq. (6) has the following components:

$$\omega^1 = \frac{1}{\sqrt{g}} \frac{\partial v_3}{\partial x^2}, \quad \omega^2 = -\frac{1}{\sqrt{g}} \frac{\partial v_3}{\partial x^1},$$

$$\omega^3 = \frac{1}{\sqrt{g}} \frac{\partial v_2}{\partial x^1} - \frac{1}{\sqrt{g}} \frac{\partial v_1}{\partial x^2}. \tag{8}$$

As a consequence of Eqs. (3) and (6), we obtain

$$\vec{v} \cdot \nabla p^* = 0, \tag{9}$$

It follows from Eqs. (3), (9), and the third covariant component of Eq. (6) that the quantities p^*, ρ, and v_3 have to be constant along the stream lines $\psi = \text{const}$, i.e. to depend only on ψ:

$$p^* = p^*(\psi), \quad \rho = \rho(\psi), \quad v_3 = v_3(\psi). \tag{10}$$

The first and second covariant projections of Eq. (6), taking Eqs. (8) and (10) into account, yield the same nonlinear equation

$$\rho v^3 \frac{dv_3}{d\psi} + \frac{\rho}{\sqrt{g}} \left(\frac{\partial v_1}{\partial x^2} - \frac{\partial v_2}{\partial x^1} \right) = \frac{dp^*}{d\psi} - \frac{\vec{v}^{\,2}}{2} \frac{d\rho}{d\psi}, \tag{11}$$

which, as is seen from Eq. (5), is the equation for finding the function ψ.

Equations (3), (7), and (11) comprise the basis for the further analysis of both the vortex and potential motions of the medium.

In this Section, we will also obtain two results which will be used below. In what follows, we will need the stream function $\bar{\psi}$ for a circular vortex thread with the circulation Γ. Let this tread lie, in cylindrical coordinates ($x^1 = R$, $x^2 = z$, $x^3 = \varphi$), in the plane $z = 0$, so that its center is located on the axis $R = 0$, and the radius of the ring is equal to R_0. In this case, the azimuthal vorticity ω_φ is distinct from zero,

$$\omega_\varphi(R, z) = \Gamma \delta(R - R_0)\, \delta(z) \tag{12}$$

and satisfies the equation (see the third of Eqs. (8))

$$\Delta^* \bar{\psi} = -R\omega_\varphi, \quad \Delta^* \bar{\psi} = \frac{\partial^2 \bar{\psi}}{\partial R^2} - \frac{1}{R} \frac{\partial \bar{\psi}}{\partial R^2} + \frac{\partial^2 \bar{\psi}}{\partial z^2}. \tag{13}$$

This equation can be integrated, and its solution, which vanishes at infinity, looks like [2]

$$\bar{\psi}(R, z) = \frac{1}{4\pi} \int R\omega_\varphi(R', z')\, R' dR' dz' \times$$

$$\times \int_0^{2\pi} \frac{\cos\theta d\theta}{\left[(z - z')^2 + R^2 + R'^2 - 2RR' \cos\theta \right]^{1/2}} =$$

$$= \frac{\Gamma R R_0}{4\pi} \int_0^{2\pi} \frac{\cos\theta d\theta}{[R^2 + z^2 + R_0^2 - 2RR_0 \cos\theta]^{\frac{1}{2}}}. \tag{14}$$

The integral obtained can be expressed in terms of complete elliptic integrals of the first and second kinds. Simple transformations of Eq. (14) (for more details, see, e.g., works [2, 10]) lead to

$$\bar{\psi} = \frac{\Gamma (RR_0)^{1/2}}{2\pi} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right], \tag{15}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and the second kind, respectively, and

$$k^2 = \frac{4RR_0}{\left[z^2 + (R + R_0)^2 \right]}.$$

As the second result, we now demonstrate that the density of the incompressible medium is constant in the region of its potential flow ($\operatorname{rot} \vec{v} = 0$). In this case, the pressure and the density are coupled, as follows from Eq. (6), by the equation

$$\nabla p^* = \frac{v^2}{2} \nabla \rho. \tag{16}$$

Since dependences (10) remain valid in the region of potential motion, Eq. (16) can be rewritten in the form

$$\frac{dp^*}{d\psi} = \frac{v^2}{2} \frac{d\rho}{d\psi}.$$

The latter equation can be presented qualitatively as

$$F(\psi) = g\left(\psi, x^i\right) f(\psi),$$

where F, f, and g are some arbitrary functions. Since the functions F and f depend only on ψ, there is a unique opportunity to satisfy this equation, namely, by putting

$$F(\psi) = 0, \; f(\psi) = 0,$$

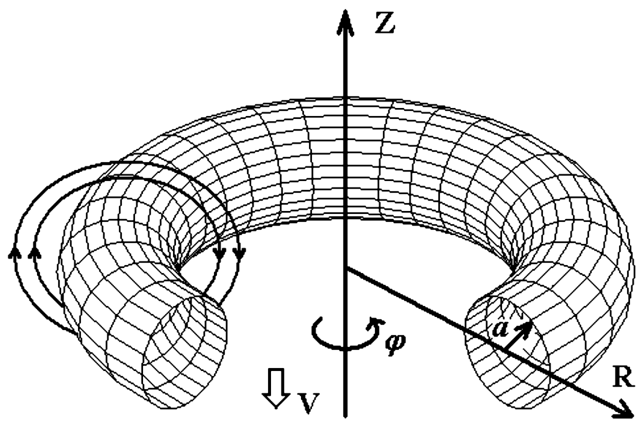

Fig. 1. Maxwell vortex

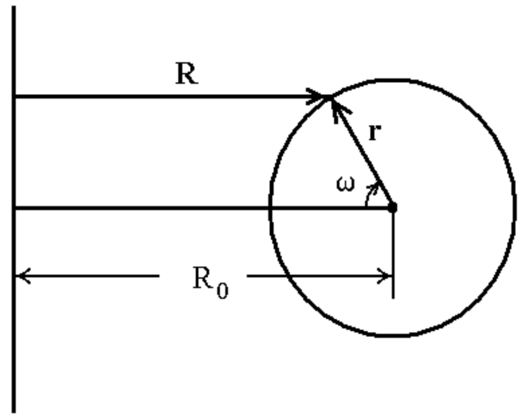

Fig. 2. Quasicylindrical coordinates

which, in physical variables, leads to the Cauchy—Lagrange integral [1, 2, 10] with a constant density of the medium

$$p+\rho\frac{v^2}{2}=\text{const},\ \rho=\text{const}. \tag{17}$$

Note that, in the region of the medium vortex flow, Eq. (6) can be expressed in the form

$$F(\psi)+G\left(\psi,x^i\right)=g\left(\psi,x^i\right)f(\psi),$$

which obviously has solutions with the nonzero F and f functions.

Thus, our problem concerning the motion of a vortex torus is reduced to the study of the spatial distribution of density in the vortex core and the spatial distributions of velocity and pressure both in the vortex core and in the potential flow region.

3. Interior Problem

Following Maxwell [1, 10], we assume that the vortex ring looks like a torus (see Fig. 1). Consider that the torus moves in an infinite surrounding incompressible medium with velocity V which is governed by the characteristic parameters of the torus (by its geometrical dimensions and vorticity). Let us change over to a quasicylindrical orthogonal coordinate system ($x^1=r$, $x^2=\omega$, $x^3=\varphi$) (see Fig. 2) which moves together with the torus and is connected with the cylindrical coordinates (R,z,φ) by the relations

$$R=R_0(1-kr\cos\omega),\quad z=r\sin\omega,\quad \varphi=\varphi,k=1/R_0,$$

$$g_{11}=1,\quad g_{22}=r^2,\quad g_{33}=R^2,\quad \sqrt{g}=rR. \tag{18}$$

In expressions (18), the torus radius, i.e. the distance reckoned from the coordinate origin to the geometrical center of the torus transverse cross-section, is denoted as R_0. In the limit $r/R_0\to 0$, the torus transforms into a vortex thread.

In the chosen coordinates, Eq. (11) reads

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial\omega^2}=kr\sin\omega\frac{\partial\psi}{\partial\omega}-kr^2\cos\omega\frac{\partial\psi}{\partial r}+$$

$$+\frac{r^2}{\rho}\left\{R^2\left(\frac{dp^*}{d\psi}-\frac{\vec{v}_\perp^2}{2}\frac{d\rho}{d\psi}\right)-\frac{d}{d\psi}\left[\rho\frac{\nu^2}{2}\right]\right\} \tag{19}$$

and describes the motion of the vortex core. Here, the following notations are used:

$$\nu=\frac{v_\varphi}{R}=\nu(\psi),\quad v_\perp^2=v_r^2+v_\omega^2,$$

$$v_r=\frac{1}{rR}\frac{\partial\psi}{\partial\omega},\qquad v_\omega=-\frac{1}{R}\frac{\partial\psi}{\partial r},\qquad v_\varphi=\frac{A(r)}{r}. \tag{20}$$

As the first approximation with respect to the small ratio r/R_0, the solution in the vortex is tried in the form

$$\psi=\psi_0(r)+f(r)\cos\omega,\quad \vec{v}=\vec{v}_0+\vec{v}_1\cos\omega,$$

$$p=p_0(r)+p_1(r)\cos\omega,\quad \rho=\rho_0(r)+\rho_1(r)\cos\omega. \tag{21}$$

Substituting Eq. (21) into Eq. (19), we obtain the zero-order approximation for the equation of dynamic equilibrium of the liquid in the vortex core:

$$\frac{\partial p_0}{\partial r}=\rho_0\frac{v_{\omega 0}^2}{r}, \tag{22}$$

where

$$v_{\omega 0}=-\frac{1}{R_0}\frac{\partial\psi_0}{\partial r}.$$

Equation (22) reflects the fact that a radial variation in the pressure generates a force necessary for preserving the motion of liquid elements along circular trajectories. It also demonstrates that the pressure at the vortex center is lower than that at some distance from the center. This result is in agreement with experimental data which testify that the vortices are characterized by the appearance of rarefactions near their centers [2].

From Eq. (19), we find, as the first approximation, that

$$\frac{\upsilon_{\omega 0}}{r}\frac{\partial}{\partial r}\left(\rho_0 r\frac{\partial f}{\partial r}\right)-\frac{f}{r}\frac{\partial}{\partial r}\left(\rho_0 r\frac{\partial \upsilon_{\omega 0}}{\partial r}\right)=$$

$$=3\rho_0\left(\upsilon_{\omega 0}\right)^2+\frac{r\partial}{\partial r}\left(\rho_0{\upsilon_{\omega 0}}^2\right)+\frac{r\partial}{\partial r}\left(\rho_0\upsilon_{\varphi 0}^2\right). \tag{23}$$

Noticing that the function $f=\upsilon_{\omega 0}(r)$ is a solution of the homogeneous equation (23), we try the solution of the inhomogeneous equation in the form

$$f=\upsilon_{\omega 0}g. \tag{24}$$

From Eqs. (23) and (24), it follows that the function g satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\rho_0\upsilon_{\omega 0}^2\frac{\partial g}{\partial r}\right]=3\rho_0\left(\upsilon_{\omega 0}\right)^2+$$

$$+r\frac{\partial}{\partial r}\left(\rho_0\upsilon_{\omega 0}^2\right)+r\frac{\partial}{\partial r}\left(\rho_0\upsilon_{\varphi 0}^2\right).$$

The solution of this equation looks like

$$g(r)=\frac{\left(r^2-a^2\right)}{2}+\int\limits_a^r\frac{dr'}{r'\rho_0\upsilon_{\omega 0}^2}\int\limits_0^{r'}r''\rho_0\upsilon_{\omega 0}^2dr''+$$

$$+\int\limits_a^r\frac{dr'}{r'\rho_0\upsilon_{\omega 0}^2}\int\limits_0^{r'}r''^2\frac{d}{dr''}\left(\rho_0\upsilon_{\varphi 0}^2\right)dr''. \tag{25}$$

Hence, in the approximation concerned ($r/R_0<1$), the stream function ψ is determined for *arbitrary spatial distributions* of the pressure and density in the vortex region.

With the help of the function ψ, we find the following quantities that describe a stationary flow of the liquid in the vortex ring:

$$\upsilon_r=\frac{1}{R_0}\left(1+kr\cos\omega\right)\frac{\partial\psi}{\partial\omega},$$

$$\upsilon_\omega=\upsilon_{\omega_0}^0(r)+\left[\upsilon_{\omega_0}^0(r)\,kr-\frac{1}{R_0}\frac{\partial}{\partial r}\left(\upsilon_{\omega_0}^0\cdot g\right)\right]\cos\omega,$$

$$\upsilon_\varphi=\frac{A(r)}{r},$$

$$p=p_0(r)-\left[\frac{\rho_0\upsilon_{\omega_0}^{02}}{R_0}\left(r+\frac{g}{r}-\frac{\partial g}{\partial r}\right)+\rho_0\upsilon_{\varphi_0}^{02}\frac{r}{R_0}\right]\cos\omega,$$

$$\rho=\rho_0(r)-\frac{g}{R_0}\frac{\partial\rho_0}{\partial r}\cos\omega. \tag{26}$$

Here, the function $g(r)$ is determined by expression (25).

To describe the internal structure of the vortex, it is convenient to change from the variable r, which describes a family of circles $r=\text{const}$, to the variable l, which describes a family of stream lines $\psi=\text{const}$. Expanding the stream function (21) into a series

$$\psi=\psi_0(l)+\left.\frac{\partial\psi_0}{\partial r}\right|_{r=l}(r-l)+f(l)\cos\omega$$

and setting $\psi=\psi_0(l)=\text{const}$, we obtain

$$r-l=-\frac{f(l)\cos\omega}{\left.\frac{\partial\psi_0}{\partial r}\right|_{r=l}}=\frac{g(l)}{R_0}\cos\omega=\xi\cos\omega. \tag{27}$$

Equation (27) coincides with the equation of a circle of radius l, the center of which is shifted by $\xi=g/R_0$ (in Fig. 2, the positive displacements are directed towards the symmetry axis, i.e. to the left of the point $R=R_0$). Hence, Eq. (27) describes a family of *nested circular stream lines*, and the displacement is absent at the end of the pinch, i.e. $\xi(l=a)=0$.

The straightforward calculations easily demonstrate that, in coordinates (l,ω,φ), the physical quantities in the toroidal pinch look like

$$\upsilon_l=0,\quad \upsilon_\omega=\upsilon_{\omega 0}(l)\left[1+\left(kl-\frac{\partial\xi}{\partial l}\right)\cos\omega\right],$$

$$\upsilon_\varphi=\upsilon_{\varphi 0}(l)\left(1+kl\cos\omega\right),\quad \rho=\rho(l),$$

$$p=p_0(l)+\left[\rho(l)\,\upsilon_{\omega 0}^2\left(\frac{\partial\xi}{\partial l}-kl\right)-\rho(l)\,\upsilon_{\varphi 0}^2kl\right]\cos\omega \tag{28}$$

and completely describe the internal configuration of the vortex torus. The displacement ξ in Eq. (28) is expressed as (see Eqs. (25) and (27))

$$\xi(l)=\frac{\left(l^2-a^2\right)}{2R_0}+\int\limits_a^l\frac{dr'}{r'R_0\rho\upsilon_{\omega 0}^2}\int\limits_0^{r'}r''\rho\upsilon_{\omega 0}^2dr''+$$

$$+\int\limits_a^l\frac{dr'}{r'R_0\rho\upsilon_{\omega 0}^2}\int\limits_0^{r'}r''^2\frac{d}{dr''}\left(\rho\upsilon_{\varphi 0}^2\right)dr''. \tag{29}$$

From the last equation, it is evident that $\xi(l) < 0$, i.e. the surfaces of constant velocity move to the right (outwards) of the geometrical center of the vortex.

In particular, if a "homogeneous" vortex without twisting is analyzed ($v_{\omega 0} = \Gamma l/(2\pi a^2)$, $v_{\varphi 0} = 0$, and $\rho = \mathrm{const}$), Eq. (29) transform into Fraenkel's result [16]

$$\xi(l) = \frac{5}{8}\frac{\left(l^2 - a^2\right)}{R_0}. \tag{30}$$

From Eq. (30), it follows that the surfaces of constant velocity (stream lines) possess the internal structure, which is exhibited in Fig. 3. In this case, the maximal displacement of the centers of the surfaces of constant velocity

$$\xi(0) = -\frac{5}{8}\frac{a^2}{R_0}$$

is expectedly shifted to the right from the center of the geometrical transverse cross-section of the torus.

4. Exterior Problem

In this section, we consider the potential flow of the medium around the vortex torus, which moves oppositely to the z-axis direction with velocity V. The examined problem is completely equivalent to the problem on the flow of a stream about an immovable torus, provided that the velocity of the stream equals V at infinity and the circulation velocity equals Γ at the torus surface. It is this flow in the medium that is observed in a coordinate system that moves together with the torus. Therefore, in order to describe the external potential flow, we also use the quasicylindrical coordinate system (18) linked with the torus, as it was done in the previous Section.

Asn it was shown in Section 2, the potential flow outside the vortex ring is described by the equations

$$\mathrm{rot}\vec{v} = 0 \tag{31}$$

and

$$p + \frac{\rho v^2}{2} = \mathrm{const},\ \rho = \mathrm{const}. \tag{32}$$

Equations (8) and (31) make it possible to obtain the following equation for the stream function:

$$r\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial \omega^2} = kr\sin\omega\frac{\partial \psi}{\partial \omega} - kr^2\cos\omega\frac{\partial \psi}{\partial r}. \tag{33}$$

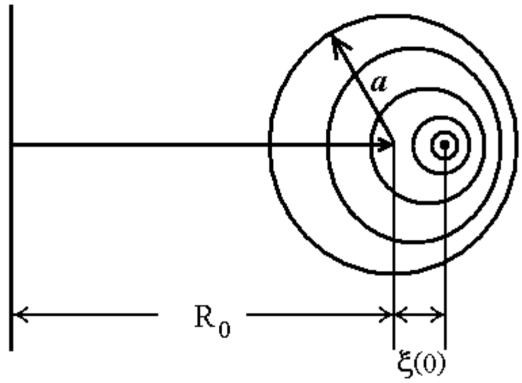

Fig. 3. Flow lines inside the vortex ring

Using the smallness of the quantity r/R_0, it is easy to verify that the general expression for the solution of Eq. (33) reads

$$\psi = A + \left(Br + \frac{C}{r}\right)\cos\omega + D\left(\ln r - \frac{r}{2R_0}\ln r\cos\omega\right) +$$
$$+\frac{1}{R_0}\left(Er + \frac{F}{r}\right)\cos\omega, \tag{34}$$

where A, B, C, D, E, and F are unknown arbitrary constants. Their values are found on the basis of the following considerations, often used in hydrodynamics. We try the required stream function as a sum of three functions: a homogeneous straight flow ψ_1, a vortical circular thread ψ_2, and the unknown flow function ψ_3:

$$\psi = \psi_1 + \psi_2 + \psi_3. \tag{35}$$

In the absence of the vortex ring, Eq. (33) would describe a stationary homogeneous liquid flow with velocity V along the z-axis direction. The function ψ_1, which describes such a flow, can be written down in cylindrical coordinates in form

$$\psi_1 = -\frac{1}{2}VR^2.$$

Taking relation (18) between the cylindrical and quasicylindrical coordinates into account, the function ψ_1 in the vicinity of the torus can be rewritten in quasicylindrical coordinates as follows:

$$\psi_1 = -\frac{1}{2}VR_0^2\left(1 - 2kr\cos\omega\right). \tag{36}$$

It is obvious from Eq. (34) that, to within the accuracy of the terms proportional to $(kr)^2$, function (36) is the solution of Eq. (33).

The function $\psi_2(r,\omega)$ for a circular vortex thread is given in Section 2 (see Eq. (15)). In the vicinity of the vortex ring, where $R \approx R_0$, $z \approx 0$, $k \approx \sqrt{1 - \varepsilon^2/4}$, and

$k' = \sqrt{1-k^2} \approx \varepsilon/2$ ($\varepsilon = r/R_0$), and the asymptotic expressions for elliptic integrals look like

$$K = \ln\frac{4}{k'} + \frac{1}{4}\left(\ln\frac{4}{k'} - 1\right)k'^2 + ...,$$

$$E = 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - 1\right)k'^2 + ...,$$

Equation (15) yields

$$\psi_2 = \frac{\Gamma R_0}{2\pi}\left[\ln\frac{8R_0}{r} - 2 - \frac{1}{2}\left(\ln\frac{8R_0}{r} - 1\right)\frac{r}{R_0}\cos\omega\right]. \quad (37)$$

It is evident that function (37), within the accuracy considered, is also the solution of Eq. (28).

At last, while solving the boundary-value problem concerning the flow about a thin vortex ring, the flow function is to be completed with a dipole-induced term ψ_3 proportional to cos ω and becoming zero at infinity:

$$\psi_3 = \frac{C}{R_0 r}\cos\omega. \quad (38)$$

Since this term is absent from Eqs. (36) and (37), the constant C is therefore to be determined from the matching condition at the torus surface.

From Eqs. (34)—(38), we obtain that the solution of Eq. (33) looks like

$$\psi = -\frac{1}{2}VR_0^2\left(1 - \frac{2r}{R_0}\cos\omega\right) + \frac{\Gamma R_0}{2\pi}\left[\ln\frac{8R_0}{r} - 2 - \right.$$

$$\left. -\frac{1}{2}\left(\ln\frac{8R_0}{r} - 1\right)\frac{r}{R_0}\cos\omega\right] + \frac{C}{R_0 r}\cos\omega. \quad (39)$$

With the help of Eq. (39), we find that

$$v_r = -\left[V + \frac{C}{R_0^2 r^2} - \frac{\Gamma}{4\pi R_0}\left(\ln\frac{8R_0}{r} - 1\right)\right]\sin\omega,$$

$$v_\omega = \frac{\Gamma}{2\pi r}\left[1 + \frac{r}{2R_0}\left(1 - \ln\frac{r}{a}\right)\cos\omega + \right.$$

$$\left. + \frac{2\pi r C}{\Gamma R_0^2}\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\cos\omega\right]. \quad (40)$$

From the condition that the flow function is constant at the torus surface, i.e. $\psi(a) = \text{const}$, we find the expression for the vortex motion velocity:

$$V = \frac{\Gamma}{4\pi R_0}\left(\ln\frac{8R_0}{a} - 1\right) - \frac{C}{R_0^2 a^2}. \quad (41)$$

As stems from Eqs. (40) and (41), the radial component of the velocity becomes zero at the torus surface.

The condition that the poloidal components (26) and (40) of the velocity are continuous at the torus surface gives rise to the equalities

$$\Gamma = 2\pi a v_{\omega 0}(a),$$

$$C = \frac{\Gamma a^2 R_0}{4\pi}\left(\frac{1}{2} - \frac{1}{a}\left.\frac{\partial g}{\partial r}\right|_{r=a}\right). \quad (42)$$

Substituting the expression for g into Eq. (42), we obtain

$$C = -\frac{\Gamma a^2 R_0}{4\pi}\left[\frac{1}{2} + \Delta\right], \quad (43)$$

where

$$\Delta = \frac{1}{\rho(a)\, v_{\omega 0}^2(a)\, a^2} \times$$

$$\times \left[\int_0^a r' \rho v_{\omega 0}^2 dr' + \int_0^a r'^2 \frac{d}{dr'}\left(\rho v_{\varphi 0}^2\right) dr'\right]. \quad (44)$$

Making use of Eqs. (41) and (43), we find the final expression for the velocity of the vortex ring V,

$$V = \frac{\Gamma}{4\pi R_0}\left[\ln\frac{8R_0}{a} - \frac{1}{2} + \Delta\right], \quad (45)$$

which can be also expressed in the form

$$V = v_{\omega 0}(a)\frac{a}{2R_0}\left[\ln\frac{8R_0}{a} - \frac{1}{2} + \Delta\right].$$

As follows from the last expression, the ring velocity tends to zero in the limiting case $R_0 \gg a$.

In the case of a "hollow vortex" ($v_{\omega 0} = v_{\varphi 0} = 0$, $\omega_{\varphi 0} = 0$, and $\Delta = 0$), Eq. (45) gives the result of Hicks [10, 14]

$$V = \frac{\Gamma}{4\pi R_0}\left[\ln\frac{8R_0}{a} - \frac{1}{2}\right].$$

But if the vortex is "homogeneous" ($v_{\omega 0} = \Gamma r/(2\pi a^2)$, $v_{\varphi 0} = 0$, $\omega_{\varphi 0} = \Gamma/(\pi a^2)$, $\rho = \text{const}$, and $\Delta = 1/4$), we get Lord Kelvin's result [12]

$$V = \frac{\Gamma}{4\pi R_0}\left[\ln\frac{8R_0}{a} - \frac{1}{4}\right].$$

If $\rho = \text{const}$ and $v_{\varphi 0} = 0$, Eq. (45) leads to Fraenkel's result [16], while, if $\rho = \text{const}$, to Saffman's one [15].

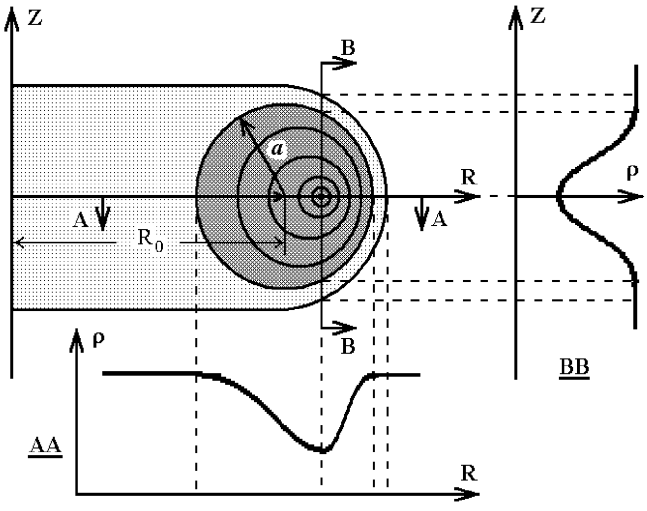

Fig. 4. Density distribution inside and near the vortex ring

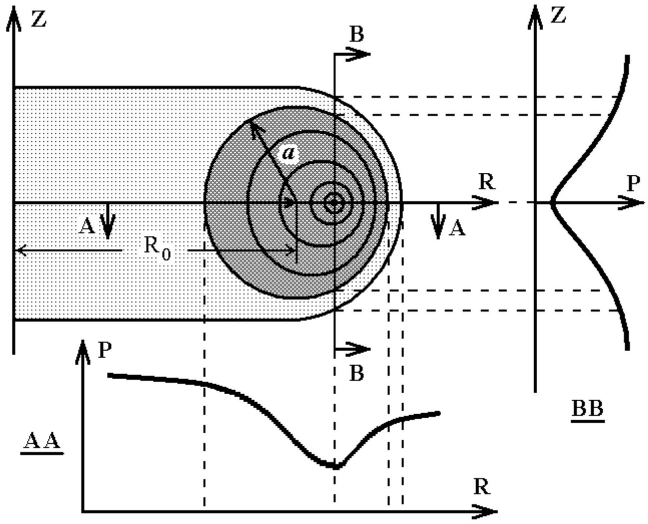

Fig. 5. Pressure distribution inside and near the vortex ring

Hence, solution (45) is a generalization of Saffman's [15] and Fraenkel's [16] solutions to the case of arbitrary dependences $\rho = \rho(r)$, $p_0 = p_0(r)$, $v^0_{\omega 0} = v^0_{\omega 0}(r)$, and $v^0_{\varphi 0} = v^0_{\varphi 0}(r)$.

Combining Eqs. (40), (42), and (45), we obtain that the field of velocities around the torus is described by the expressions

$$v_r = \frac{\Gamma}{4\pi R_0}\left\{\ln\frac{a}{r} - \left(\frac{1}{2} + \Delta\right)\left(1 - \frac{a^2}{r^2}\right)\right\}\sin\omega,$$

$$v_\omega = \frac{\Gamma}{2\pi r}\left\{1 + \frac{r}{2R_0}\left[1 - \ln\frac{r}{a} - \right.\right.$$

$$\left.\left. - \left(\frac{1}{2} + \Delta\right)\left(1 + \frac{a^2}{r^2}\right)\right]\cos\omega\right\},$$

$$v_\varphi = 0. \tag{46}$$

From Eq. (32), it follows that the density of and the pressure in the external flow of the medium can be expressed in the form

$$\rho = \rho(a) = \text{const},$$

$$p = p_0(r) + p_1(r)\cos\omega = p_0(r) -$$

$$-\frac{\rho(a)\, v^2_{\omega 0}(r)}{2}\frac{r}{R_0}\left[1 - \ln\frac{r}{a} - \left(\frac{1}{2} + \Delta\right)\left(1 + \frac{a^2}{r^2}\right)\right]\cos\omega, \tag{47}$$

where

$$p_0(r) = p_0(a) + \frac{\rho(a)\, v^2_{\omega 0}(a)}{2}\left(1 - \frac{a^2}{r^2}\right),$$

$$v_{\omega 0}(r) = v_{\omega 0}(a)\,\frac{a}{r}.$$

Taking into account that (see Eqs. (29) and (44))

$$\left.\frac{\partial \xi}{\partial l}\right|_{l=a} = ka + \Delta$$

and

$$v_{\varphi 0}(l = a) = 0,$$

we obtain from Eqs. (28), (46), and (47) that all physical quantities are continuous at $r = a$, so that the vortex under consideration is "smooth" [1]. The distributions of the density and the pressure in and near the vortex ring are shown in Figs. 4 and 5, respectively.

5. Vortex Atmosphere

The region of potential flow of the liquid, which is entrained by the vortex ring and is called the "vortex atmosphere", presents some interest for the vortex ring dynamics [1, 2]. In the reference frame which moves together with the torus, the non-whirling portion of the liquid entrained by the vortex ring is concentrated in a volume which is bounded by the stream surface that contains the external stagnant points located outside the ring's "body". It is known [1, 2] that, depending on the ring characteristics, there exists one or two such points. In the latter case, the points are located on the ring axis, where the liquid flow velocity exceeds that of the ring

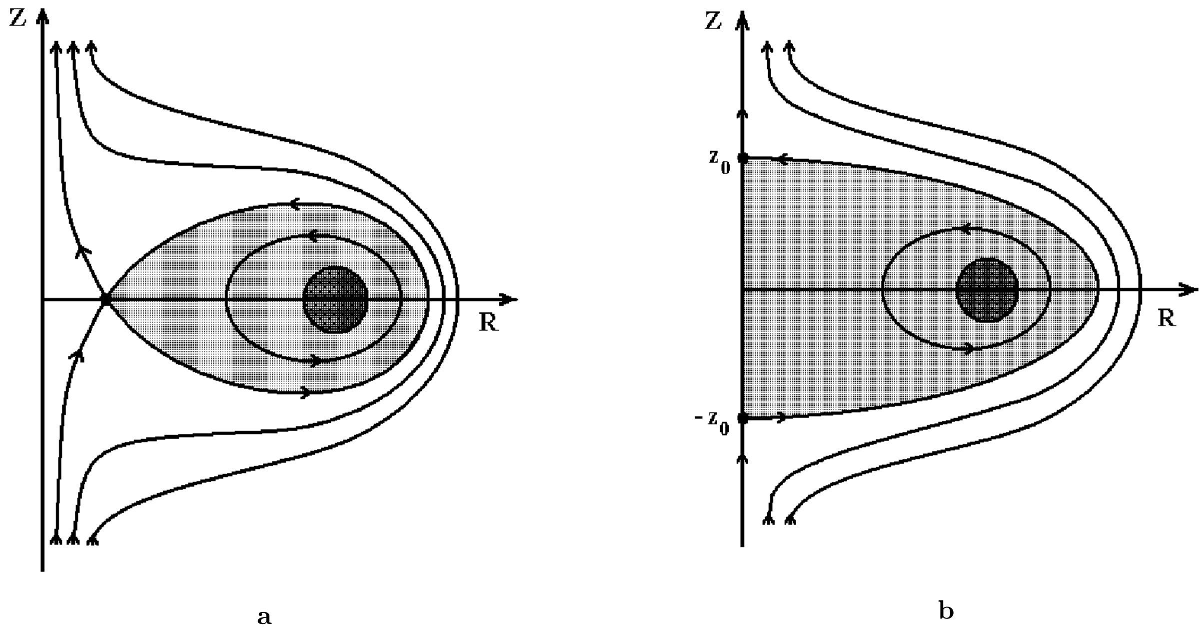

Fig. 6. Sketches of stream lines in the cases of thin (a) and thick (b) vortex rings

motion, and the transverse cross-section of the captured region of the non-whirling liquid has the shape of a closed oval. In the former case, the single stagnant point is located between the ring's "body" and its axis, and the region of the non-whirling liquid entrained by the ring has the shape of a ring. Qualitatively, the atmosphere of the vortex is depicted in Fig. 6. The vortex ring region in the figure is more shadowed than the vortex atmosphere. It is known [1] that vortices are not realized in the form of toroidal rings; therefore, only the vortex ring exhibited in Fig. 6,b is of practical interest.

The reconstruction of the flow structure occurs at a certain radius of the transverse cross-section of the ring, which can be estimated by substituting a circular vortex thread for the vortex ring and applying formula (15). Using the relations

$$\frac{dK}{dk} = \frac{E}{k(1-k^2)} - \frac{K}{k}, \quad \frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}$$

between the elliptic integrals, we obtain from Eq. (15) that

$$v_r = \frac{\Gamma}{2\pi R} \frac{z}{\left[(R+R_0)^2 + z^2\right]^{1/2}} \times$$

$$\times \left[-K(k) + \frac{R^2 + R_0^2 + z^2}{(R-R_0)^2 + z^2} E(k)\right],$$

$$v_z = \frac{\Gamma}{2\pi R} \frac{z}{\left[(R+R_0)^2 + z^2\right]^{1/2}} \times$$

$$\times \left[K(k) + \frac{R_0^2 - R^2 - z^2}{(R-R_0)^2 + z^2} E(k)\right], \quad v_\varphi = 0. \tag{48}$$

On the symmetry axis, the flow velocities are

$$v_r = 0, v_z = \Gamma R_0^2 / 2\left(z^2 + R_0^2\right)^{\frac{3}{2}}. \tag{49}$$

Then, the coordinates $\pm z_0$ of the stagnant points in the coordinate system, which moves together with the ring, are determined from the equation

$$\frac{\Gamma R_0^2}{2\left(z_0^2 + R_0^2\right)^{\frac{3}{2}}} = \frac{\Gamma}{4\pi R_0}\left(\ln\frac{8R_0}{a} - \frac{1}{2} + \Delta\right) \tag{50}$$

and depend on the distributions of density and pressure in the vortex core.

For a homogeneous vortex, Eq. (50) is solvable in the intervals of the parameters $1 < R_0/a \le 86$ and $-14 < z/a \le 14$; the corresponding solutions are shown in Fig. 7. Provided such values of the ratio R_0/a, the medium at the center of the ring and the ring itself move together. The figure illustrates that the vortex atmosphere is confined in space, being close by its shape to an ellipse. In the case $R_0/a > 86$, the vortex atmosphere has the shape of a torus and is exhibited in Fig. 6,a.

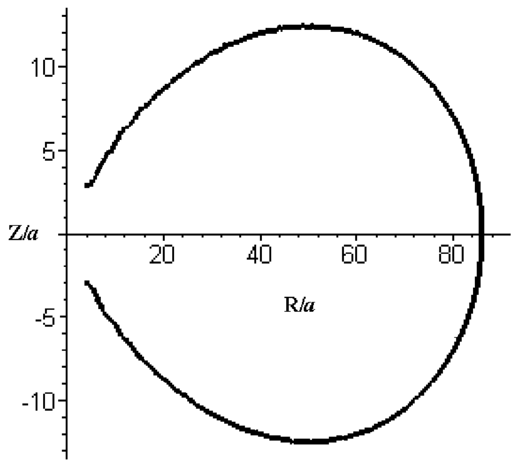

Fig. 7. Positions of stagnant points for a thick vortex as a function of the ratio R/a value

6. Conclusions

In this work, the following results have been obtained:
— The velocity of a vortex ring was shown to depend on the distribution of density in the vortex core.
— Expression (45) for the velocity of the vortex ring in an incompressible medium was obtained. This expression generalizes the solutions which have been obtained earlier [15–17] to the case of distributed density and pressure.
— Expressions for the velocity, density, and pressure were derived, for both the flow of the medium inside the ring (expressions (28) and (29)) and the external flow near the ring (expressions (46) and (47)).
— Equation (50), which describes the vortex atmosphere, was obtained and used to analyze its shape. The atmosphere shape was demonstrated to depend on the density distribution in the vortex core.

1. M.A. Lavrent'ev and B.V. Shabat, *Problems in Hydrodynamics and Their Mathematical Models* (Nauka, Moscow, 1973) [in Russian].
2. G.K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge Univ. Press, Cambridge, 1967).
3. W. Thomson (Lord Kelvin), Philos. Mag. **34**, 15 (1867).
4. A.T. Onufriev, Prikladn. Matem. Teor. Fiz. N 2, 3 (1967).
5. E. Teller and A.L. Latter, *Our Nuclear Future: Facts, Dangers, and Opportunities* (Criterion Books, New York, 1958).
6. E.I. Mogilevskii, *Fractals on the Sun* (Fizmatgiz, Moscow, 2001) [in Russian].
7. Yu.P. Ladikov-Roev, A.A. Linnik, N.N. Salnikov, and O.K. Cheremnykh, Kosmich. Nauka Tekhnol. **10**, 131 (2004).
8. M.-B. Kallenrode, *Space Physics: An Introduction to Plasmas and Particles in the Heliosphere and Magnetospheres* (Springer, Berlin, 2004).
9. P.G. Saffman, *Vortex Dynamics* (Cambridge Univ. Press, Cambridge, 1992).
10. H. Lamb, *Hydrodynamics* (Cambridge Univ. Press, Cambridge, 1932).
11. M.J.M. Hill, Phil. Trans. R. Soc. A **185**, 213 (1894).
12. W. Thomson (Lord Kelvin), Philos. Mag. **33**, 511 (1867).
13. H. Helmholtz, Philos. Mag. **33**, 485 (1867).
14. W.M. Hicks, Phil. Trans. Roy. Soc. London A **176**, 725 (1885).
15. P.G. Saffman, Stud. Appl. Math. **49**, 371 (1970).
16. L.E. Fraenkel, Proc. Roy. Soc. A **316**, 29 (1970).
17. D.B. Bliss, *MS thesis* (Massachusetts Inst. Technol, Cambridge, 1970).

Received 13.04.06.
Translated from Russian by O.I. Voitenko

ПРО РУХ ВИХРОВИХ КІЛЕЦЬ У НЕСТИСЛИВОМУ СЕРЕДОВИЩІ

О.К. Черемних

Р е з ю м е

Розглянуто стаціонарний рух осесиметричного вихрового кільця у нестисливому середовищі, в якому швидкість $\vec{v}$ і густина ρ задовольняють рівняння div$\vec{v} = 0$, $\vec{v} \cdot \nabla\rho = 0$. Друге рівняння дозволяє розглядати рух вихрового кільця з розподіленою в просторі густиною. Показано, що густина нестисливого середовища може бути неоднорідною тільки в області вихрового руху і є постійною величиною в області потенціального руху. З урахуванням цієї обставини встановлено, що швидкість кільця і форма його атмосфери визначаються не тільки геометричними розмірами вихрового ядра і величиною циркуляції швидкості але й просторовим розподілом густини у ядрі вихора.