S-MATRIX APPROACH CALCULATION OF THE POLARIZATION CONTRIBUTION TO THE NUCLEOSYNTHESIS CROSS-SECTION

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In the framework of the S-matrix approach, the influence of the electric dipole polarizability of colliding particles upon the low-energy nucleosynthesis cross-section is estimated. It is shown that the relative contribution of the polarization effects to the reaction cross-section does not exceed the quantity of the order of 0.1%.

1. Introduction

When considering the low-energy scattering of like-charged structural particles, it is important to take correctly into account the effects of the electric dipole polarizability of the colliding particles. The corresponding polarization potential falls off at large distances as r^{-4}, which results in a substantial distortion of the scattering wave function at large distances and also in the divergence [1,2] of the parameters of the standard effective range theory [3,4]. To describe the low-energy scattering of charged structural particles, it is necessary to make use of the modified effective range theory [1, 5—7] which defines the nuclear phase shift with respect to the phase shift from the Coulomb and polarization long-range potentials. The scattering length and the effective range obtained in such an approach [5—8] are very close to the relevant quantities of the nuclear-Coulomb problem.

The account of the polarization effects in low-energy nucleosynthesis reactions needs a particularly accurate approach. In works [7,9—11], by means of the calculation of the corresponding matrix elements, the possibility to increase the astrophysical nucleosynthesis reaction cross-sections due to the long-range dipole polarization attraction was examined. It was established that the relative contribution of the polarization corrections to the nucleosynthesis reaction cross-section is characterized by a magnitude of the polarization potential at the boundary of the nuclear force range and does not exceed a value of the order of 10^{-3}. The result is in excellent agreement with the calculations [12,13] and with the estimation in the framework of the quasiclassical approximation approach [8].

On the other hand, the estimation [14] of the upper bound to the polarization effects in the reactions of the deuteron-nucleus synthesis gave the result $\sim 10^{-8}$ that is 5 orders less than our conclusion [7,9—11].

We will show that the S-matrix approach with a due account of all relevant contributions from the polarization potential also leads to the result consistent with the conclusions in [7—13].

2. Reaction cross-section

In the framework of the S-matrix formalism, the integral cross-section (summed over all final states) of the reaction involving two colliding particles can be expressed in terms of the diagonal matrix element of the scattering S-matrix which also determines the elastic scattering cross-section.

For the S-wave collision dominating at low energies, the corresponding formula takes the form [15]

$$\sigma_r = \frac{\pi}{k^2}(1 - |S|^2). \qquad (1)$$

The diagonal matrix element in Eq. (1),

$$S = e^{2i\delta}, \qquad (2)$$

is expressed in terms of the scattering phase shift $\delta \equiv \delta(k)$ which depends on the relative motion energy

$$E = \hbar^2 k^2/(2\mu), \qquad (3)$$

where k is the wave number, and μ is the reduced mass of the two particles.

When the phase shift is real, only the elastic scattering takes place. For the complex phase shift

$$\delta = \delta' + i\delta'', \qquad (4)$$

we have $|S| < 1$, and the inelastic processes are present in the system. In this case, the reaction cross-section (1) is expressed in terms of the imaginary part of the phase shift:

$$\sigma_r = \frac{\pi}{k^2}(1 - e^{-4\delta''}). \tag{5}$$

For small values of the phase shift, Eq. (5) can be approximated with a good accuracy by the expression

$$\sigma_r \simeq \frac{4\pi}{k^2}\delta'', \quad |\delta''| \ll 1. \tag{6}$$

Let us find the expression for the modulus of the diagonal S-matrix element (2) in terms of the real and imaginary parts of a tangent of the complex scattering phase shift (4). Using the relation

$$e^{2i\delta} = \frac{1 + i\mathrm{tg}\delta}{1 - i\mathrm{tg}\delta}$$

and (4), we get

$$\left|e^{2i\delta}\right|^2 = \left(\frac{1-\tau''}{1+\tau''}\right)^2 \left[1 + \left(\frac{\tau'}{1-\tau''}\right)^2\right] \times$$

$$\times \left[1 + \left(\frac{\tau'}{1+\tau''}\right)^2\right]^{-1}, \tag{7}$$

where we used the definition

$$\tau \equiv \tau(k) \equiv \mathrm{tg}\delta(k), \quad \tau = \tau' + i\tau''.$$

The substitution of relation (7) into Eq. (1) gives a more complex expression for the cross-section as compared with Eq. (5). But, at small values of the tangent of the phase shift, the right-hand side of Eq. (7) is simplified to the form

$$\left|e^{2i\delta}\right|^2 \simeq 1 - 4\tau'' + O(|\tau|^2), \quad |\tau| \ll 1, \tag{8}$$

which allows us to express cross-section (1) in terms of the imaginary part of the tangent of the phase shift τ'':

$$\sigma_r \simeq \frac{4\pi}{k^2}\tau'' + O(|\tau|^2), \quad |\tau| \ll 1. \tag{9}$$

Since, at small values of the phase shift, the approximation

$$\mathrm{tg}\delta \simeq \delta, \quad |\delta| \ll 1,$$

holds, the obtained relation (9) coincides, in this case, with (6) expressing the cross-section in terms of the imaginary part of the phase shift.

3. Nuclear-Coulomb Problem

Let us consider the S-wave collision of two like-charged structural particlesZ_1e and Z_2e with the relative motion energy (3) and an effective interaction in the form of a sum of the nuclear V_N and Coulomb V_{C} potentials,

$$V_{N\mathrm{C}} = V_N + V_{\mathrm{C}}.$$

In this case, the nuclear phase shift $\delta_{N,\mathrm{C}}(k)$ modified (renormalized) by the Coulomb field is normally used [3,4] in order to describe the effects of interactions in the system. This phase shift is defined as the difference between the total phase shift $\delta_{N\mathrm{C}}(k)$due to the scattering by the nuclear+Coulomb potential and the true Coulomb phase shift $\delta_{\mathrm{C}}(k)$:

$$\delta_{N,\mathrm{C}}(k) = \delta_{N\mathrm{C}}(k) - \delta_{\mathrm{C}}(k). \tag{10}$$

The low-energy behavior of the phase shift $\delta_{N,\mathrm{C}}(k)$ is defined by the well-known expansion in the effective range theory [3,4] which looks as

$$-\frac{1}{a_{N,\mathrm{C}}(k)} + \frac{2}{a_{\mathrm{B}}}h(\eta) = -\frac{1}{A_{N,\mathrm{C}}} + \frac{r_{N,\mathrm{C}}}{2}k^2 + O(k^4). \tag{11}$$

The function

$$a_{N,\mathrm{C}}(k) \equiv -\frac{\mathrm{tg}\delta_{N,\mathrm{C}}(k)}{C^2 k} \tag{12}$$

in (11) is the generalized (energy-dependent) Fermi scattering length [16] which transforms at the threshold energies into the constant, the scattering length $A_{N,\mathrm{C}}$. The denominator of relation (12) contains the Coulomb barrier penetration factor

$$C^2 \equiv C^2(\eta) = 2\pi\eta(e^{2\pi\eta} - 1)^{-1} \tag{13}$$

which depends on the Sommerfeld parameter

$$\eta = \frac{1}{ka_{\mathrm{B}}}. \tag{14}$$

Here, the quantity

$$a_{\mathrm{B}} = \frac{\hbar^2}{\mu e^2 Z_1 Z_2} \tag{15}$$

is the Bohr radius of the system. At low energies, $k \ll 1$ ($\eta \gg 1$ due to (14)), and the Coulomb penetration factor (13) decreases exponentially as

$$C^2(\eta) \simeq 2\pi\eta e^{-2\pi\eta}, \quad \eta \gg 1. \tag{16}$$

The function $h(\eta)$ on the left-hand side of Eq. (11) is defined in terms of the digamma function $\psi(z)$ [17],

$$h(\eta) = \mathrm{Re}\psi(i\eta) - \ln\eta,$$

and decreases at small energies ($k \ll 1$, $\eta = (ka_{\rm B})^{-1} \gg 1$) as the inverse power of η:

$$h(\eta) \simeq \frac{1}{12\eta^2}\left(1+\frac{1}{10\eta^4}\right), \quad \eta \gg 1. \tag{17}$$

For the short-range nuclear potentials, the left-hand side of Eq. (11) is a meromorphic function of the energy and can be expanded in terms of the energy with the parameters $A_{N,{\rm C}}$ (the modified nuclear scattering length) and $r_{N,{\rm C}}$ (the modified effective range).

Employing Eqs. (11) and (17), it is easy to establish the low-energy behavior of the modified generalized nuclear scattering length,

$$a_{N,{\rm C}}(k) = A_{N,{\rm C}}\Big[1 - A_{N,{\rm C}}\left(\frac{a_{\rm B}}{6} - \frac{r_{N,{\rm C}}}{2}\right)k^2 + O(k^4)\Big], \tag{18}$$

that gives at the threshold energy

$$a_{N,{\rm C}}(k=0) = A_{N,{\rm C}}.$$

According to Eq. (12), the tangent of the modified nuclear phase shift can be expressed in terms of the corresponding generalized scattering length as

$$\text{tg}\delta_{N,{\rm C}}(k) = -C^2 k a_{N,{\rm C}}(k). \tag{19}$$

As a result of relations (16) and (18), the tangent of the modified nuclear phase shift at low energies is a very small exponentially decreasing quantity:

$$\text{tg}\delta_{N,{\rm C}}(k) \simeq -\frac{2\pi}{a_{\rm B}}e^{-2\pi\eta}A_{N,{\rm C}}, \quad \eta \gg 1. \tag{20}$$

4. Allowance for the Polarization Interaction

Let us now consider the S-wave collision of the two charged particles with the effective interaction

$$V_{NP{\rm C}} = V_N + V_{\rm C} + V_P,$$

which includes the polarization potential

$$V_P(r) = -\frac{\beta^2}{r^4}\Theta(r-R), \quad \beta^2 = \frac{\alpha_E}{a_{\rm B}} \tag{21}$$

in addition to the nuclear V_N and Coulomb $V_{\rm C}$ potentials. Potential (21) describes the additional long-range attraction in the system caused by the electric dipole polarizability of the structural particles. In Eq. (21), $\Theta(x)$ is the step function, and R is the radius which separates the nuclear $V_N(r)\Theta(R-r)$ and dipole polarization $V_P(r)$ parts in the effective two-particle interaction. The strength of potential (21) is defined by the ratio of the effective polarizability of the system,

$$\alpha_E \equiv \alpha(1)\frac{Z_2}{Z_1} + \alpha(2)\frac{Z_1}{Z_2},$$

to its Bohr radius $a_{\rm B}$ (15), $\alpha(i)$, $i=1,2$, are the electric dipole polarizabilities of the particles. For the nuclear systems, the estimation

$$\beta^2 \lesssim 10^{-2} \quad \text{Fm}^2$$

holds [1]. Therefore, the polarization potential V_P (21) by its intensity is a small correction to the repulsive Coulomb potential

$$V_C(r) = \frac{2}{a_{\rm B} r}.$$

It has been shown in [1] that the modified polarization phase shift $\delta_{P,{\rm C}}$ decreases at low energies by the power law:

$$\text{tg}\delta_{P,{\rm C}}(k) \sim k^5, \quad k \ll 1. \tag{22}$$

Let us represent the Coulomb-modified nuclear-polarization phase shift $\delta_{NP,{\rm C}}$ as a sum of the modified nuclear phase shift $\delta_{N,{\rm C}}$ and the polarization phase shift $\delta_{P,N{\rm C}}$ reckoned from the total phase shift of the nuclear-Coulomb problem, $\delta_{N{\rm C}}$ (10):

$$\delta_{NP,{\rm C}} = \delta_{N,{\rm C}} + \delta_{P,N{\rm C}}. \tag{23}$$

Employing Eq. (23) and calculating the phase shift $\delta_{P,N{\rm C}}$ in the Born approximation for the potential V_P[1], we get an explicit expression for the tangent of the modified nuclear-polarization phase shift:

$$\tau^{\rm B}_{NP,{\rm C}} = \frac{\tau^{\rm B}_{P,{\rm C}} + \left[1+\varepsilon(k)+\tau^2_{N,{\rm C}}\right]\tau_{N,{\rm C}}}{1 - \left[\tau^B_{P,{\rm C}} - (1-\varepsilon(k))\tau_{N,{\rm C}}\right]\tau_{N,{\rm C}}}. \tag{24}$$

In Eq. (24), $\tau^{\rm B}_{NP,{\rm C}}$ and $\tau^{\rm B}_{P,{\rm C}}$ are the Born approximations for the corresponding functions, and the quantity

$$\varepsilon(k) = 2b(k) - d(k)a_{N,{\rm C}}(k) \tag{25}$$

is expressed in terms of the modified generalized nuclear scattering length $a_{N,{\rm C}}(k)$ and the matrix elements of the polarization potential V_P between the regular

[1]According to [18, 19], the phase shift $\delta_{P,{\rm C}}$ in the energy range $1 \div 10^3$ keV is reproduced by the Born approximation for V_P with the accuracy $\sim 0.01\%$.

$F(k,r)$ and irregular $G(k,r)$ solutions of the Schrödinger equations for the Coulomb scattering as

$$b(k) = -\frac{1}{k}\int_R^\infty F V_P G dr, \quad d(k) = -\int_R^\infty V_P (CG)^2 dr. \tag{26}$$

Deriving Eq. (24), we used the following Born approximation for the tangent of the phase shift $\delta_{P,NC}$:

$$(\mathrm{tg}\delta_{P,NC}(k))^{\mathrm{B}} \simeq -\frac{\cos^2\delta_{N,\mathrm{C}}(k)}{k}\int_R^\infty V_P(r) u_{N\mathrm{C}}^2(k,r)dr.$$

Here, the regular solution of the Schrödinger equation for the nuclear-Coulomb problem $u_{N\mathrm{C}}(k,r)$ in the integration region is expressed directly in terms of the Coulomb functions $F(k,r)$ and $G(k,r)$:

$$u_{N,\mathrm{C}}(k,r) \underset{r\geq R}{=} F(k,r) + \mathrm{tg}\delta_{N,\mathrm{C}} G(k,r).$$

It has been shown in [9—11] that the function $b(k)$ is bounded uniformly in k as

$$|b(k)| \lesssim \sqrt{\frac{\pi}{2}} M, \tag{27}$$

where the parameter M takes on small values

$$M \equiv R^2 |V_P(R)| = \frac{\beta^2}{R^2} \sim 10^{-3}. \tag{28}$$

In addition, the function $d(k)$ (26) at low energies ($E \lesssim$ 100 keV) is also bounded by a small value,

$$d(k) \lesssim \frac{11}{a_{\mathrm{B}}} M. \tag{29}$$

Using Eq. (25) and taking bounds (27)—(29) and the low-energy behavior of the modified generalized nuclear scattering length $a_{N,\mathrm{C}}(k)$ (18) into account, we obtain a bound to the function $\varepsilon(k)$ in the form

$$|\varepsilon(k)| \lesssim O(10^{-3}). \tag{30}$$

Relation (24) allows us to find the energy dependence of the modified nuclear-polarization phase shift of the S-wave scattering of charged particles. The right-hand side of Eq. (24) contains separately the tangents of the modified nuclear and polarization phase shifts $\tau_{N,\mathrm{C}}$ and $\tau_{P,\mathrm{C}}$. At low energies, they decrease according to Eqs. (20) and (22) and are very small values. Therefore, we can restrict ourselves by keeping only the terms linear in $\tau_{N,\mathrm{C}}$ and $\tau_{P,\mathrm{C}}^{\mathrm{B}}$ in Eq. (24), which leads to

$$\tau_{NP,\mathrm{C}}^{\mathrm{B}} \simeq \tau_{P,\mathrm{C}}^{\mathrm{B}} + [1+\varepsilon(k)]\tau_{N,\mathrm{C}}. \tag{31}$$

Considering Eq. (31) and taking constraint (30) to $\varepsilon(k)$ into account, we can conclude that the tangent of the modified nuclear-polarization phase shift $\tau_{NP,\mathrm{C}}$ is also a small value. In this case, the use of approximation (8) is justified and, because of Eq. (9), the effect of the dipole polarization upon the low-energy reaction cross-section can be evaluated by a deviation of the ratio of the imaginary parts of the tangents of the corresponding phase shifts (4)

$$\frac{\sigma_{NP,\mathrm{C}}^r}{\sigma_{N,\mathrm{C}}^r} \simeq \frac{\tau_{NP,\mathrm{C}}''}{\tau_{N,\mathrm{C}}''} \tag{32}$$

from 1. Representing the complex quantities $a_{N,\mathrm{C}}$, $\tau_{N,\mathrm{C}}$, and $\tau_{NP,\mathrm{C}}^{\mathrm{B}}$ in Eqs. (31) and (25) as

$$z = z' + iz'',$$

we find the following expression for the imaginary part of the function $\tau_{NP,\mathrm{C}}^{\mathrm{B}}$:

$$\tau_{NP,\mathrm{C}}''^{B} \simeq \{1 + 2[b(k) - d(k)a_{N,\mathrm{C}}'(k)]\}\tau_{N,\mathrm{C}}''. \tag{33}$$

Substituting expression (33) in the right-hand side of Eq. (32), we obtain the required estimation for the polarization effects:

$$\frac{\sigma_{NP,\mathrm{C}}^r}{\sigma_{N,\mathrm{C}}^r} \simeq 1 + 2[b(k) - d(k)a_{N,\mathrm{C}}'(k)]. \tag{34}$$

The second term in Eq. (34) is just the quantity that determines a relative contribution of the dipole polarization of the particles to the reaction cross-section. With regard for Eq. (18) and the bounds to the functions $b(k)$ (27), (28) and $d(k)$ (29), (28), this contribution does not exceed a value of the order of 10^{-3}:

$$\frac{\sigma_{NP,\mathrm{C}}^r}{\sigma_{N,\mathrm{C}}^r} = 1 + O(10^{-3}). \tag{35}$$

The obtained result (35) is in agreement with the conclusions of works [7, 9—11] and is also supported by the numerical calculations of the $\mathrm{pp} \to \mathrm{de}^+\nu$ [12] and $^7\mathrm{Be}(\mathrm{p},\gamma)^8\mathrm{B}$ [13] reactions, as well as by the quasiclassical estimation in [8]. Alternatively in [14], the upper bound to the polarization contribution to the reaction cross-section $\sim 10^{-8}$, which is 5 orders smaller as compared with Eq. (35), has been obtained. Our analysis reveals that such large underestimation of the polarization effects in [14] is a consequence of using the insufficiently correct approximation for the modified nuclear-polarization phase shift $\delta_{NP,\mathrm{C}}$ (Eq. (4) in [14]). It differs substantially from our expression for $\tau_{NP,\mathrm{C}}^{\mathrm{B}}$ (24) mainly by the absence of the term $\varepsilon(k)\tau_{N,\mathrm{C}}$ in a numerator, which gives finally just the main polarization contribution to the reaction cross-section.

5. Conclusion

We have shown that the application of the S-matrix formalism to the evaluation of the effect of the electric dipole polarizability of colliding particles on the low-energy nucleosynthesis reaction cross-section leads to result (35) that agrees with those in works [7—13] based on the other approaches. It is established that the large underestimation of the polarization contribution in [14] is a consequence of using the incorrect approximation for the modified nuclear-polarization phase shift $\delta_{NP,\mathrm{C}}$.

In general, the polarization effects in the low-energy nucleosynthesis play a very insignificant role and do not exceed a value of the order of 0.1%. The physical reason for this is that the reactions concerned are realized in a small region of the configuration space, where the polarization potential (21) represents a small correction to the nuclear and Coulomb interactions. The value of the polarization potential at the boundary of the nuclear force range just determines the parameter $M \sim 10^{-3}$ (28) which characterizes the magnitude of the polarization contributions to the low-energy nucleosynthesis.

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РОЗРАХУНОК ПОЛЯРИЗАЦІЙНОГО ВНЕСКУ В ПЕРЕРІЗ РЕАКЦІЇ СИНТЕЗУ ЯДЕР В РАМКАХ S-МАТРИЧНОГО ФОРМАЛІЗМУ

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Р е з ю м е

В рамках S-матричного формалізму проведено оцінку впливу електричної дипольної поляризовності частинок, що стикаються, на переріз реакцій синтезу ядер за низьких енергій. Показано, що відносний внесок поляризаційних ефектів в переріз реакцій за порядком величини не перевищує 0,1%.