# KINETIC EQUATION FOR NON-MARKOVIAN GAUSSIAN PROCESSES IN LINEAR MULTIDIMENSIONAL SYSTEMS

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We develop a consistent operator formalism for the derivation of a generalized Fokker–Planck equation (GFPE) for linear multidimensional stochastic systems driven by coloured Gaussian noise. We study the relaxation in the phase space to the Maxwellian distribution for two types of correlation functions (CFs) of the external Gaussian noise, namely for the exponential and power-law CFs, and have found that, with the proper normalization, the latter relaxation process is slower than that for the exponential CF with short (in comparison with the inverse friction constant) correlation time, but faster than for the exponential CF with long correlation time. We study also in detail the next stage of relaxation, that is the diffusion in a real space. Here, we pay a special attention to the phenomenon of superdiffusion driven by a power-law CF on the infinite axis and the superdiffusion with adsorption on the semiaxis.

## 1. Introduction

The term "coloured Gaussian noise" (a noise with frequency-dependent power spectrum) has arisen in the description of the system of macroscopic particles being under the thermal influence of the surrounding medium (the Brownian motion [1]). At the same time, for many stochastic systems, a characteristic time of thermal fluctuations is much smaller than that of a Brownian particle, and it is a good approximation to consider that the external random forces possess a zero memory, that is, they are delta-correlated. Such a noise with frequency-independent power spectrum is called a white Gaussian noise. There is a plenty of scientific works devoted to studying the systems with the white Gaussian noise, see, e.g. [2–5] and the articles cited there. The theory of Markovian Brownian motion (also known as an ordinary Brownian motion) is successfully used to describe experiments and observations.

However, in most of the natural systems, the white Gaussian noise remains quite an idealistic model giving no satisfactory results. The processes in such systems are either Markovian and non-Gaussian, or Gaussian and non-Markovian, or non-Gaussian and non-Markovian. Recently, the first of them, that is, the Markovian Lévy processes in external fields were studied [6–9] by means of the technique of Langevin and fractional kinetics. The other case, namely, that of non-Markovian Gaussian processes is discussed in the present paper.

When the external random force is a white Gaussian noise, it is quite easy to obtain a differential equation for the probability density function from the Langevin equation. The pioneer work on this problem was done by Lord Rayleigh [10] within the approach of a zero external potential and an overdamped discrete motion of a heavy Brownian particle. A more consistent method was developed by Fokker, Smoluchowski, and Planck (a detailed historical sketch can be found in [11]).

However, in the case of coloured noise, the derivation of such an equation is connected with much troubles. The most common example of the derivation of a one-dimensional Fokker–Planck equation (FPE) for coloured noise can be found in [11]; for a particular case of a linear oscillator, it was obtained and studied in [12].

The theory of such generalized Brownian motion can find its applications in many problems of modern physics and astronomy. Indeed, the fluctuations of the magnetic field in Earth's magnetospheric tail turbulent plasma turn out to have colour: in the range of frequencies $\omega \leq 10^{-2}$ Hz, they have the properties of a flicker-noise (their power spectrum is proportional to $1/\omega$); when $\omega$ is about $10^{-1}$ Hz, they are a brown noise with the tendency of "blackening" at lower frequencies, see, e.g., work [13] and references cited there.

Moreover, a similar situation is known from experiments with laboratory plasmas: it was found, that the power spectra of a saturation current, electrostatic potential fluctuations, and the turbulence-induced flux measured in various plasma devices [14] have power-law dependences. At high frequencies, an asymptotic power fall-off of the fluctuation spectra with characteristic decay indices close to 2 was denoted; at intermediate frequencies, the decay indices were about 1, by gaining a weak frequency dependence at the lowest frequencies.

Another important application comes from the single-molecule dynamics. In work [15], it is shown that the experimental data on fluctuations of the distance between two components of a fluorescein-tyrosine complex can be described within the framework of the theory of coloured noise with the correlation function $K(t) \sim t^{-0.51\pm 0.07}$ in a harmonic potential well.

Below, we present a consistent method of derivation of a multidimensional generalized Fokker–Planck equation for linear stochastic systems driven by a coloured Gaussian noise.

## 2. Basis of the Method

As the basis of the suggested method for derivation of a generalized FPE, we use the approach for obtaining an ordinary FPE for linear systems with a delta-correlated noise described in [16]. We start from the Langevin equations written in the multidimensional form,

$$\dot{\xi}_i = -a_{ik}\xi_k + Y_i\left(t\right), \tag{1}$$

where $\xi_i$ is the generalized coordinate, $a_{ik}$ is the coefficient matrix, $Y_i$ is the external noise; and a dot stands for the time derivative. Let the initial conditions be $\xi_i\left(t=0\right) = \xi_i\left(0\right)$. Then, the formal solution of Eq. (1) is

$$\xi_i\left(t;\xi\left(0\right)\right) = \left(e^{-at}\right)_{ij}\xi_i\left(0\right) + \int\limits_0^t d\tau\left(e^{-a\left(t-\tau\right)}\right)_{ij} Y_j\left(\tau\right), \tag{2}$$

where $a \equiv \|a_{ik}\|$ is the matrix composed of the elements of $a_{ik}$. The probability density function (PDF) of the values $\xi_i$ at the moment of time $t$ with the fixed $\xi\left(0\right)$ is evidently a multidimensional Dirac delta-function:

$$f\left(\xi,t;\xi\left(0\right)\right) = \delta\left(\xi - \xi\left(t,\xi\left(0\right)\right)\right) \equiv$$

$$\equiv \prod_i \delta\left(\xi_i - \xi_i\left(t,\xi\left(0\right)\right)\right). \tag{3}$$

If we do not know the exact $\xi\left(0\right)$ but possess their initial PDF $f\left(\xi\left(0\right),0\right)$, the PDF at an arbitrary moment of time $t$ has the form

$$f\left(\xi,t\right) = \int d\xi\left(0\right) f\left(\xi\left(0\right),0\right)\left\langle\delta\left(\xi - \xi\left(t,\xi\left(0\right)\right)\right)\right\rangle, \tag{4}$$

where $\langle ... \rangle$ stands for $\int d\tau p_Y\left(\tau\right) ...$; and $p_Y\left(\tau\right)$ is the noise's PDF. If we use the representation of the $n$-dimensional delta-function $\delta\left(\xi\right) = \left(2\pi\right)^{-n}\int dq \exp\left(iq\xi\right)$, then Eq. (2) yields

$$\left\langle\delta\left(\xi - \xi\left(t,\xi\left(0\right)\right)\right)\right\rangle = \left(2\pi\right)^{-n}\int dq G\left(q,t\right) \times$$

$$\times \exp\left[iq\left(\xi - e^{-at}\xi\left(0\right)\right)\right], \tag{5}$$

where

$$G\left(q,t\right) = \left\langle \exp\left\{-iq\int\limits_0^t d\tau e^{-a\left(t-\tau\right)} Y\left(\tau\right)\right\}\right\rangle. \tag{6}$$

Here, we use the matrix notation.

Then, representing the PDF in terms of a Fourier integral

$$f\left(\xi,t\right) = \left(2\pi\right)^{-n}\int dq e^{iq\xi} f\left(q,t\right) \tag{7}$$

and taking Eq. (4) into account, we get

$$f\left(q,t\right) = G\left(q,t\right) f\left(\left(e^{-at}\right)^T q, 0\right), \tag{8}$$

where $\left(e^{-at}\right)^T$ is the matrix transposed to $e^{-at}$.

Hereinafter we assume that the random process $Y_i\left(t\right)$ is a stationary Gaussian process, so that the following relations take place:

$$\left\langle Y_{i_1}\left(t_1\right) ... Y_{i_{2n+1}}\left(t_{2n+1}\right)\right\rangle = 0,$$

$$\left\langle Y_{i_1}\left(t_1\right) ... Y_{i_{2n}}\left(t_{2n}\right)\right\rangle =$$

$$= \sum g_{i_1 i_2}\left(t_1 - t_2\right) ... g_{i_{2n-1} i_{2n}}\left(t_{2n-1} - t_{2n}\right). \tag{9}$$

Here, the summation is executed over all possible pair compositions of $i_1, t_1; i_2, t_2; ...; i_{2n} t_{2n}$. The number of such pairs is $(2n-1)!! = 2n!/n!2^n$. $g_{i_1 i_2}\left(t_1 - t_2\right)$ is some function of the time difference.



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Using series expansion of the exponential function and Eq. (9) we get from Eq. (6)

$$G(q,t)=\sum_{n=0}^{\infty}\frac{(-i)^{2n}}{(2n)!}\frac{(2n)!}{n!2^n}\times$$
$$\times\left[\int_0^t dt_1\int_0^t dt_2 q_i\left(e^{-at_1}\right)_{ij} q_m\left(e^{-at_1}\right)_{ml} g_{jl}(t_1-t_2)\right]. \quad (10)$$

Here and below, we write, for simplicity, $g_{jl}(t_1-t_2)$ instead of $g_{j_1l_2}(t_1-t_2)$.

Introducing the quantity

$$M_{im}(t)=$$
$$=\frac{1}{2}\int_0^t dt_1\int_0^t dt_2\left(e^{-at_1}\right)_{ij}\left(e^{-at_2}\right)_{ml} g_{jl}(t_1-t_2), \quad (11)$$

we get

$$G(q,t)=\exp\left[-q_iq_mM_{im}(t)\right]. \quad (12)$$

## 3. Fokker–Planck Equation

It is easy to see that the quantity $G(q,t)$ obeys the equation

$$\frac{\partial G}{\partial t}+q_ia_{ik}\frac{\partial G}{\partial q_k}=-q_iq_mD_{im}(t)\,G(q,t), \quad (13)$$

where

$$D_{im}(t)=\frac{dM_{im}}{dt}+a_{ik}M_{km}(t)+a_{mk}M_{ik}(t). \quad (14)$$

In view of the obvious equality

$$\frac{\partial}{\partial t}f\left(\left(e^{-at}\right)^T q,0\right)=-qa\frac{\partial}{\partial q}f\left(\left(e^{-at}\right)^T q,0\right) \quad (15)$$

and Eq. (8), we can conclude that the function $f(q,t)$ obeys Eq. (13). Now, performing the inverse Fourier transformation, we obtain

$$\frac{\partial f(\xi,t)}{\partial t}=\frac{\partial}{\partial \xi_i}\left(a_{im}\xi_m f(\xi,t)\right)+D_{im}(t)\frac{\partial^2 f(\xi,t)}{\partial\xi_i\partial\xi_m}. \quad (16)$$

For the next step, let us simplify the expressions for $D_{im}(t)$ and $M_{im}(t)$. Since

$$a_{ik}\left(e^{-at}\right)_{kj}=\frac{\partial}{\partial t}\left(e^{-at}\right)_{ij},$$

$$\frac{\partial g_{jl}(t_1-t_2)}{\partial t_1}=-\frac{\partial g_{jl}(t_1-t_2)}{\partial t_2}$$

and

$$\frac{dM_{im}}{dt}=\frac{1}{2}\int_0^t dt_1\left(e^{-at_1}\right)_{ij}\left(e^{-at}\right)_{ml} g_{jl}(t_1-t)+$$
$$+\frac{1}{2}\int_0^t dt_2\left(e^{-at}\right)_{ij}\left(e^{-at_2}\right)_{ml} g_{jl}(t-t_2),$$

we arrive at the expression

$$D_{im}(t)=\frac{1}{2}\int_0^t dt_1\left[\left(e^{-at_1}\right)_{ij} g_{jm}(|t_1|)+\right.$$
$$\left.+\left(e^{-at_1}\right)_{mj} g_{ij}(|t_1|)\right]. \quad (17)$$

By determining $D_{im}(t)$ via a single integration, we can solve system (14) for the components of $M_{im}(t)$ and thus simplify the expression for the latter which is needed to construct the solution of the GFPE in Section 5.

## 4. Brownian Motion in a Linear Oscillator's Field

In case of the Brownian motion with a linear oscillator, the Langevin equations have the form

$$\frac{dx}{dt}=v,$$
$$\frac{dv}{dt}=-\gamma v-\omega^2x+Y(t), \quad (18)$$

where $x(t)$ is the particle's coordinate, $v(t)$ is the velocity, $\gamma$ is the friction constant, and $\omega$ is the oscillator's frequency. It is obvious that the coefficients $a_{ik}$ for system (18) are

$$a=\begin{pmatrix}0 & -1\\ \omega^2 & \gamma\end{pmatrix}, \quad (19)$$

and $g_{ij}(t_1-t_2)=\delta_{i2}\delta_{j2}g(t_1-t_2)$.

The Fokker–Planck equation (16) transforms into

$$\frac{\partial f(x,v,t)}{\partial t}=-\frac{\partial}{\partial x}(vf)+\frac{\partial}{\partial v}\left(\left(\omega^2x+\gamma v\right)f\right)+$$
$$+D_{11}(t)\frac{\partial^2 f}{\partial x^2}+\left(D_{12}(t)+\right.$$
$$\left.+D_{21}(t)\right)\frac{\partial^2 f}{\partial x\partial v}+D_{22}(t)\frac{\partial^2 f}{\partial v^2}. \quad (20)$$

To get the expression for $D_{im}(t)$, see Eq. (17), we have to know the elements of the matrix $e^{-at}$. To get them, we first note that the general solution of the homogeneous ($Y_i(t) = 0$) system (1) yields

$$\xi_i^{(0)}(t) = \left(e^{-at}\right)_{ij} \xi_j^{(0)}(0). \qquad (21)$$

On the other hand, a general solution of the homogeneous system from Eq. (18) is

$$v(t) = A_1 e^{-\gamma t/2} e^{\Omega t/2} + A_2 e^{-\gamma t/2} e^{-\Omega t/2},$$

$$x(t) = -\frac{1}{\omega^2}\left[\dot{v}(t) + \gamma v(t)\right] =$$

$$= -\frac{1}{2\omega^2}\Big[A_1 e^{-\gamma t/2} e^{\Omega t/2}(\gamma + \Omega) +$$

$$+ A_2 e^{-\gamma t/2} e^{-\Omega t/2}(\gamma - \Omega)\Big], \qquad (22)$$

where

$$\Omega = +\sqrt{\gamma^2 - 4\omega^2}. \qquad (23)$$

Substituting the initial conditions $x(0) = x_0$ and $v(0) = v_0$ into Eq. (22), we get the expressions for the integration constants $A_1$ and $A_2$ as

$$A_1 = -\frac{2\omega^2 x_0 + v_0(\gamma - \Omega)}{2\Omega},$$

$$A_2 = \frac{2\omega^2 x_0 + v_0(\gamma + \Omega)}{2\Omega}. \qquad (24)$$

Finally, collecting the terms with $x_0$ and $v_0$ in each expression in Eq. (22) and comparing it with Eq. (21), we obtain

$$e^{-at} = e^{-\gamma t/2}\times$$

$$\times\begin{pmatrix} \cosh\left(\frac{\Omega}{2}t\right) + \frac{\gamma}{\Omega}\sinh\left(\frac{\Omega}{2}t\right) & \frac{2}{\Omega}\sinh\left(\frac{\Omega}{2}t\right) \\ -\frac{2\omega^2}{\Omega}\sinh\left(\frac{\Omega}{2}t\right) & \cosh\left(\frac{\Omega}{2}t\right) - \frac{\gamma}{\Omega}\sinh\left(\frac{\Omega}{2}t\right)\end{pmatrix}. \qquad (25)$$

The substitution of the matrix elements of (25) into Eq. (17) yields

$$D_{11}(t) = 0,$$

$$D_{12}(t) = D_{21}(t) =$$

$$= \frac{1}{\Omega}\int_0^t dt_1 \sinh\left(\frac{\Omega}{2}t_1\right) e^{-\gamma t_1/2} g(|t_1|),$$

$$D_{22}(t) = \int_0^t dt_1 \Big[\cosh\left(\frac{\Omega}{2}t_1\right) -$$

$$- \frac{\gamma}{\Omega}\sinh\left(\frac{\Omega}{2}t_1\right)\Big] e^{-\gamma t_1/2} g(|t_1|). \qquad (26)$$

At this stage, we can compare these diffusion coefficients to those obtained in [12]. We consider only the case of the external driving noise (see Section 3.2 of the mentioned paper). To establish the connection with our GFPE and Eq. (W29) (here, the letter "W" indicates the reference to the equation from work [12]), let us substitute Eqs. (W54), (W55), and (W14) into Eq. (W35). Now we see that $\psi(t) \equiv 2D_{12}(t)$ and $\phi(t) \equiv D_{22}(t)$, i.e. we get a complete coincidence between our Eq. (20) and Wang's Eq. (W29).

System (14) transforms into

$$\begin{cases} D_{11}(t) = \dfrac{dM_{11}}{dt} - 2M_{12}(t); \\ D_{12}(t) = \dfrac{dM_{12}}{dt} + \omega^2 M_{11}(t) + \gamma M_{12}(t) - M_{22}(t); \\ D_{22}(t) = \dfrac{dM_{22}}{dt} + 2\omega^2 M_{12}(t) + 2\gamma M_{22}(t). \end{cases} \qquad (27)$$

Solving the latter system for $M_{ij}$, $(i, j = 1, 2)$ and taking Eq. (25) into account result in

$$M_{11}(t) = \int_0^t dt_1 \Bigg\{\frac{e^{-(t+t_1)\gamma/2}}{2\gamma\omega^2\Omega}\Big[\Omega e^{\gamma t/2} g(|t_1|)\times$$

$$\times\left(\Omega\cosh\left(\frac{\Omega}{2}t_1\right) + \gamma\sinh\left(\frac{\Omega}{2}t_1\right)\right) -$$

$$-g(|t - t_1|)\left(\Omega\cosh\left(\frac{\Omega}{2}t\right)\times\right.$$

$$\times\left(\Omega\cosh\left(\frac{\Omega}{2}t_1\right) + \gamma\sinh\left(\frac{\Omega}{2}t_1\right)\right) +$$

$$+ \sinh\left(\frac{\Omega}{2}t\right)\left(\gamma\Omega\cosh\left(\frac{\Omega}{2}t_1\right) + \right.$$

$$\left.\left.\left. + (\gamma^2 + 4\omega^2)\sinh\left(\frac{\Omega}{2}t_1\right)\right)\right)\Big]\Bigg\}; \qquad (28)$$

$$M_{12}(t) = M_{21}(t) = \frac{2}{\Omega^2}\int_0^t dt_1 \Big\{e^{-(t+t_1)\gamma/2}\times$$

$$\times\sinh\left(\frac{\Omega}{2}t\right)\sinh\left(\frac{\Omega}{2}t_1\right)g\left(|t-t_1|\right)\Bigg\};\qquad(29)$$

$$M_{22}(t)=\int\limits_0^t dt_1\left\{\frac{e^{-(t+t_1)\gamma/2}}{2\gamma\Omega^2}\left[\Omega e^{\gamma t/2}g\left(|t_1|\right)\times\right.\right.$$

$$\times\left(\Omega\cosh\left(\frac{\Omega}{2}t_1\right)-\gamma\sinh\left(\frac{\Omega}{2}t_1\right)\right)-$$

$$-g\left(|t-t_1|\right)\left(\Omega\cosh\left(\frac{\Omega}{2}t\right)\times\right.$$

$$\times\left(-\Omega\cosh\left(\frac{\Omega}{2}t_1\right)+\gamma\sinh\left(\frac{\Omega}{2}t_1\right)\right)+$$

$$+\ \sinh\left(\frac{\Omega}{2}t\right)\left(\gamma\Omega\cosh\left(\frac{\Omega}{2}t_1\right)\ -\right.$$

$$\left.\left.\left.\left.-\ \left(\gamma^2+4\omega^2\right)\sinh\left(\frac{\Omega}{2}t_1\right)\right)\right)\right]\right\}.\qquad(30)$$

Now we can obtain the exact form of $G(q,t)$, substituting these expressions into Eq. (12), and will proceed directly to the construction of the PDF.

## 5. Construction of the PDF

The Fourier transform of the function $f(x,v,t)$ can be written as

$$\hat{f}(q_1,q_2,t)=\int\limits_{-\infty}^{+\infty}dxdve^{-iq_1x}e^{-iq_2v}f(x,v,t).\qquad(31)$$

According to Eq. (8), we have

$$\hat{f}(q_1,q_2,t)=G(q_1,q_2,t)\hat{f}\left\{q_1\left(e^{-at}\right)_{11}+q_2\left(e^{-at}\right)_{21};\right.$$
$$\left.q_1\left(e^{-at}\right)_{12}+q_2\left(e^{-at}\right)_{22};0\right\}.\qquad(32)$$

Then, to obtain the PDF, we make the inverse Fourier transformation of Eq. (32):

$$f(x,v,t)=\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}dq_1dq_2e^{iq_1x}e^{iq_2v}\hat{f}(q_1,q_2,t).\qquad(33)$$

## 6. Space-homogeneous Initial Condition for Free Motion

Let us consider a space-homogeneous initial condition. For the PDF, it can be written as

$$f(x,v,0)=n\delta(v-v_0),\qquad(34)$$

where $n$ is the density of particles at the initial moment of time. The Fourier transform of Eq. (34) yields

$$\hat{f}(q_1,q_2,0)=2\pi n\delta(q_1)e^{-iq_2v_0}.\qquad(35)$$

Then, substituting the latter expression into Eq. (32) and taking Eq. (12) and Eqs. (28)–(30) into account, we arrive at

$$f(x,v,t)=\frac{n}{2\pi}\left|\left(e^{-at}\right)_{11}\right|^{-1}\int\limits_{-\infty}^{+\infty}dq_2\exp\left[-\sigma q_2^2+iuq_2\right]=$$
$$=\frac{1}{2\sqrt{\pi\sigma}}\exp\left[-u^2/4\sigma\right],\qquad(36)$$

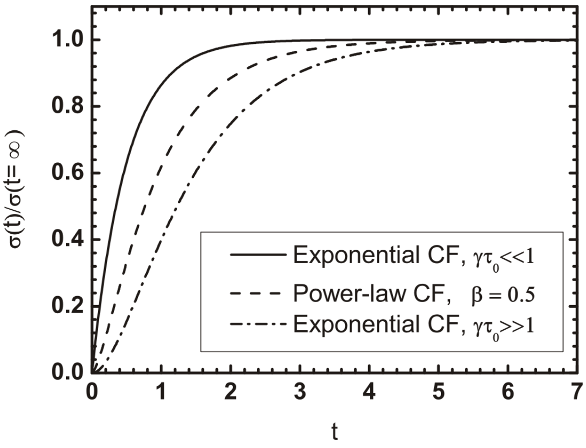
where

$$\sigma=\left(e^{-at}\right)_{11}^{-2}\left(e^{-at}\right)_{21}^{2}M_{11}(t)-$$
$$-2\left(e^{-at}\right)_{11}^{-1}\left(e^{-at}\right)_{21}M_{12}(t)+M_{22}(t);\qquad(37)$$
$$u=\left(e^{-at}\right)_{11}^{-1}\left(e^{-at}\right)_{12}\left(e^{-at}\right)_{21}v_0-$$
$$-\left(e^{-at}\right)_{22}v_0-\left(e^{-at}\right)_{11}^{-1}\left(e^{-at}\right)_{21}x+v,$$

$\sigma$ is the dispersion, and $u$ is the "effective" velocity. We note that, although the initial condition is space-homogeneous, there appears a heterogeneous state (the term with $x$ in the latter expression) due to the existence of the external potential. When the frequency $\omega=0$, this term vanishes. Because we are interested here in the velocity relaxation only, we consider the case where the external potential is zero and assume that $\omega=0$ in Eq. (37) and Eqs. (28)–(30):

$$\sigma=M_{22}(t)\left|_{\omega=0}\right.=\frac{e^{-\gamma t}}{2\gamma}\int\limits_0^t dt_1e^{-\gamma t_1}g\left(|t-t_1|\right)+$$
$$+\frac{1}{2\gamma}\int\limits_0^t dt_1e^{-\gamma t_1}g\left(|t_1|\right),\qquad(38)$$
$$u=-\left(e^{-at}\right)_{22}\left|_{\omega=0}\right.=-v_0e^{-\gamma t}+v.$$

Below, we present two illustrative examples for particular correlation functions.




Dispersion behaviour for the exponential and power-law correlation functions normalized by the corresponding equilibrium dispersion. Here, $\gamma = 1$

### *6.1. Exponential correlation function*

Substituting

$$g\left(|t|\right) = \frac{c}{\tau_0}\exp\left(-\left|t\right|/\tau_0\right) \tag{39}$$

into Eq. (38), we get

$$\sigma = \frac{c}{\gamma\left(1-\gamma^2\tau_0^2\right)}\left\{\gamma\tau_0\exp\left[-t\left(\gamma+1/\tau_0\right)\right]+ \right.$$
$$\left.+\frac{1}{2}\left[\left(1-\gamma\tau_0\right)-\left(1+\gamma\tau_0\right)e^{-2\gamma t}\right]\right\}, \tag{40}$$
$$u = -v_0 e^{-\gamma t} + v.$$

Now we can study two limiting cases where $\gamma\tau_0 \ll 1$ and $\gamma\tau_0 \gg 1$:

$$\sigma = \frac{c}{2\gamma}\left(1-e^{-2\gamma t}\right), \qquad \gamma\tau_0 \ll 1; \tag{41}$$

$$\sigma = \frac{c}{2\gamma^2\tau_0}\left(1-e^{-\gamma t}\right)^2, \qquad \gamma\tau_0 \gg 1. \tag{42}$$

As was expected, the first relation decays at large $t$'s to $c/2\gamma$ that fully coincides with that in the case of a delta-correlated Gaussian noise. Note that *both* relations decay to a constant with the characteristic time which is independent of the decay correlation time $\tau_0$ and dependent only on the friction constant $\gamma$: $\tau_{\rm rel} \sim \gamma^{-1}$.

### *6.2. Power-Law correlation function*

Here, we consider the relation

$$g\left(|t|\right) = \frac{c\left|t\right|^{-\beta}}{\Gamma(1-\beta)}, \tag{43}$$

where $\Gamma(1-\beta)$ is the gamma-function. We dwell only on $0<\beta<1$ corresponding to the most slowly decaying correlations (the detailed reasoning can be seen, e.g., in [12,17]). As $\beta \to 1$, Eq. (43) transforms into the delta-function (see the proof in [18]). Now Eq. (38) gives

$$\sigma = \frac{c}{2\gamma^{2-\beta}}\left\{\tilde{\gamma}\left(1-\beta,\gamma t\right)- \right.$$
$$\left.-\frac{\left(\gamma t\right)^{1-\beta}}{1-\beta}e^{-2\gamma t}M\left(1-\beta,2-\beta,\gamma t\right)\right\}, \tag{44}$$
$$u = -v_0 e^{-\gamma t} + v.$$

Here, $\tilde{\gamma}\left(1-\beta,\gamma t\right)$ is the incomplete gamma-function, and $M\left(1-\beta,2-\beta,\gamma t\right)$ is the Kummer function. Expression (44) has the following asymptotics at large time values:

$$\sigma \approx \frac{c}{2\gamma^{2-\beta}}\left(1-\frac{e^{-\gamma t}}{\left(\gamma t\right)^{\beta}\Gamma\left(1-\beta\right)}\right), \qquad \gamma t \gg 1. \tag{45}$$

The latter expression decays to a constant faster than Eq. (42), but slower than Eq. (41) (if we take the same $\gamma$'s). In Figure, one can see the comparison of the dispersions $\sigma$ for the power-law CF, Eq. (44), and the exponential CF from the previous subsection, Eqs. (41), (42).

## 7. Free Diffusion

Now let us consider a generalized free motion in the real space, rather than in the phase one. In such a case, Eq. (16) is simplified to the following:

$$\frac{\partial f}{\partial t} = D(t)\frac{\partial^2 f}{\partial x^2}. \tag{46}$$

In fact, it is a diffusion equation with the diffusion coefficient dependent on time:

$$D(t) = \int_0^t dt' g(t'). \tag{47}$$

The last equation can be easily solved via the substitution

$$\tau(t) = \int_0^t dt' D(t'). \tag{48}$$

Then

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial x^2}. \tag{49}$$



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Now it is easy to obtain the exact expressions for the diffusion coefficients and the substitutions for two above-considered correlation functions. For the correlation function given by Eq. (39), we get

$$D(t) = c\left(1 - e^{-|t|/\tau_0}\right), \tag{50}$$

$$\tau(t) = c\left\{t - \tau_0\left(1 - e^{-t/\tau_0}\right)\right\}; \tag{51}$$

and, for the correlation function given by Eq. (43), we have

$$D(t) = \frac{c}{\Gamma(2-\beta)} t^{1-\beta}, \tag{52}$$

$$\tau(t) = \frac{c}{\Gamma(3-\beta)} t^{2-\beta}. \tag{53}$$

### *7.1. Anomalous diffusion on an infinite axis*

The mean squared displacement $\langle x^2 \rangle$ naturally obeys Eq. (49):

$$\frac{\partial}{\partial \tau}\left\langle x^2\right\rangle = 2 \tag{54}$$

and

$$\left\langle x^2\right\rangle = 2\tau(t). \tag{55}$$

Then, for the exponential correlation function from Eq. (51), we have

$$\left\langle x^2\right\rangle = 2c\left\{t - \tau_0\left(1 - e^{-t/\tau_0}\right)\right\} \tag{56}$$

with the asympotics

$$\left\langle x^2\right\rangle \approx \begin{cases} \dfrac{c}{\tau_0}t^2, & t \ll \tau_0, \\ 2ct, & t \gg \tau_0. \end{cases} \tag{57}$$

In the normal diffusion regime, the mean squared displacement $\langle x^2 \rangle \sim t^1$. Here, we see that there exist the ballistic diffusion at small times (the time exponent is equal to 2) and the normal diffusion at large times.

For the power-law correlation function from Eq. (53), we obtain

$$\left\langle x^2\right\rangle = \frac{2c}{\Gamma(3-\beta)} t^{2-\beta}, \quad 0 < \beta < 1. \tag{58}$$

Here, we have a superdiffusion (the time exponent is greater, than 1) without ballistic regime, since there is no small-time limit in the model of CF given by Eq. (43). Next, we note that if we assume $\beta = 1$ in the last formula, we obtain the limiting case of the mean squared displacement for a delta-correlated noise: $\left\langle x^2\right\rangle = 2ct$.

### *7.2. Anomalous diffusion on a semiaxis: The first-passage problem*

In this subsection, we consider a particle starting from $x_0 > 0$ at $t = 0$ and moving on the right semiaxis until it reaches $x = 0$, where it is adsorbed. Let $\Phi(t;x_0) = \int_0^\infty dx\, f(x,t;x_0)$ be a probability that the particle will remain in $(0,\infty)$ till the moment $t$; then, $(1-\Phi)$ is the probability to be adsorbed till $t$, and

$$p(t;x_0) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_0^\infty dx f(x,t) \tag{59}$$

is the density having the meaning of the first-passage-time PDF. It can be seen that $p(t)$ is normalized,

$$\int_0^\infty dt p(t) = -\int_0^\infty dx f(x,t)\Bigg|_{t=0}^{t=\infty} =$$

$$= \int_0^\infty dx f(x,t=0) = \int_0^\infty dx \delta(x-x_0) = 1. \tag{60}$$

The mean first-passage time is

$$T = -\int_0^\infty dt t\frac{\partial \Phi}{\partial t} = \int_0^\infty \Phi(t;x_0) =$$

$$= \int_0^\infty dx \int_0^\infty dt\, f(x,t;x_0). \tag{61}$$

The solution of Eq.(49),

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0 \tag{62}$$

with the initial condition

$$f(x,0) = \delta\left(x - x_0\right), \quad x_0 > 0, \tag{63}$$

and the adsorbing boundary condition

$$f(0,t) = 0 \tag{64}$$

can be found by either the Fourier method or the method of images,

$$f(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\left(e^{-(x-x_0)^2/4\tau} - e^{-(x+x_0)^2/4\tau}\right). \tag{65}$$

For the power-law correlation function (see Eq. (43)), we consider firstly the PDF of the escape time in terms of $\tau$:

$$\Phi(\tau;x_0) = -\int_0^\infty dx\, f(x,\tau) = \operatorname{erf}\left(\frac{x_0}{\sqrt{4\tau}}\right), \qquad (66)$$

where $\operatorname{erf}(z) = 2\pi^{-1/2}\int_0^z d\xi \exp\left(-\xi^2\right)$, and

$$p(t) = -\left(\frac{d}{d\tau}\Phi(\tau;x_0)\right)\frac{d\tau}{dt} = \frac{x_0}{\sqrt{4\pi}}\frac{e^{-x_0^2/4\tau}}{\tau^{3/2}}\frac{d\tau}{dt}. \qquad (67)$$

Then, substituting Eq. (53) into the latter expression, we get

$$p(t) \underset{t\to\infty}{\approx} \frac{x_0}{\sqrt{4\pi}}\frac{1}{[\tau(t)]^{3/2}}\frac{d\tau}{dt} =$$

$$= \frac{x_0}{\sqrt{4\pi}}\sqrt{\frac{\Gamma(3-\beta)}{c}}\frac{1}{t^{2-\beta/2}}. \qquad (68)$$

Since $0 < \beta < 1$, we get a rapid adsorption phenomenon. When $\beta = 1$, we obtain the well-known result for the delta-correlated noise:

$$p(t) \approx \frac{x_0}{\sqrt{4\pi c}}t^{-3/2}.$$

## 8. Conclusions

In this paper, we have considered a linear stochastic system driven by coloured Gaussian noise. We generalized the derivation procedure of an "ordinary" FPE [16] to the case of a linear system with coloured Gaussian noise and gained the following results.

First, we derived the generalized Fokker–Planck equation for multidimensional linear systems driven by coloured Gaussian noise. We obtained an explicit form of the two-dimensional GFPE and calculated all the necessary coefficients.

Second, we constructed the PDF as the solution of this GFPE in the two-dimensional phase space without solving it directly. For space-homogeneous initial conditions and arbitrary correlation functions, we got a Gaussian PDF with the "effective" velocity and the dispersion dependent on time.

As the next result, by the examples of the exponential and slowly decaying power-law correlation functions, the explicit form of the PDFs was obtained. We studied the relaxation of these PDFs and found out the existence of the Maxwellian distribution. It was shown that the process with the power-law CF decays slower, than that with the exponential CF with a short correlation time, but faster than that with the long-correlated exponential CF. We proved that, in each case, the relaxation time to the equilibrium PDF depends only on the friction constant.

Finally, considering the diffusion on the infinite and semiinfinite axes with an adsorbing boundary, we have shown that the system with power-law correlations can give birth to the processes of anomalous diffusion in the real space.


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# КІНЕТИЧНЕ РІВНЯННЯ НЕМАРКІВСЬКИХ ГАУСОВИХ ПРОЦЕСІВ У ЛІНІЙНИХ БАГАТОВИМІРНИХ СИСТЕМАХ

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Р е з ю м е

Нами розроблено послідовний операторний формалізм виведення узагальненого рівняння Фоккера–Планка для лінійних багатовимірних стохастичних систем, що перебувають під дією кольорового гаусівського шуму. Вивчалася релаксація системи до максвелівського розподілу у фазовому просторі на прикладі двох кореляційних функцій (КФ): експоненціальної та степеневої; було встановлено, що за належного нормування релаксаційний процес останньої є повільнішим, ніж експоненціальної КФ з малим (порівняно з величиною, оберненою до сталої тертя) кореляційним часом, але швидшим за процес із експоненціальною КФ з великим кореляційним часом. Крім того, нами детально було досліджено наступну стадію релаксаційного процесу – дифузію у реальному просторі. Особливу увагу було приділено явищу супердифузії у випадку степеневої КФ на нескінченній осі та супердифузії із поглинанням на півосі.