# INFLUENCE OF THE ION THERMAL MOTION ON THE BOHM CRITERION

A.G. ZAGORODNY, I.V. ROGAL'

Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine
*(14b, Metrolohichna Str., Kyiv 03143, Ukraine; e-mail: azagorodny@bitp.kiev.ua)*

UDC 539.93


The Bohm criterion is one of the basic points in the treatment of the formation of a near-electrode plasma sheath. This criterion was formulated under the assumption that the plasma ions have zero temperature and electrons are described by the Boltzmann distribution. In the present paper, we study the influence of the ionic temperature on the screening of a plane electrode under the floating or fixed potential. It is shown that the thermal motion of ions leads to a decrease of the critical value of the ion velocity.

## 1. Introduction

The study of the formation of a near-electrode plasma sheath remains to be one of the important problems of physics for many years. The description of the modern state of this problem can be found, for example, in [1–3] and references therein. One of the general properties of the distributions of the density of particles and the potential near the surface is the presence of a near-wall sheath, in which the screening of the potential of the electrode occurs. The size of the sheath is usually significantly less than other characteristic spatial scales, in particular, than the free path of ions. In this case, the condition of screening is determined by the Bohm criterion [4], according to which the velocity of ion motion to the electrode must exceed the critical velocity $U_{\mathrm{cr}} = (T_e/m_i)^{1/2}$. If this criterion is satisfied, one can calculate the distributions of electrons, ions, and the electric potential from the Poisson equation and develop the asymptotic theory of a near-wall layer [1, 2, 5]. Such a theory is necessary for the construction of the solutions consistent with a distribution in the presheath region [1].

It is worth noting, however, that the Bohm criterion and the corresponding calculations of the potential of a screened electrode were obtained in the approximation of cold ions. It is obvious that such an assumption is valid not always. In particular, it is not true in the case of thermal plasma. In this connection, the question arises about how the thermal motion of ions can affect the Bohm criterion. The present work concerns an attempt to clear up this question.

The statement of the problem is formulated in Section 2. The distribution functions of electrons and ions are determined on the basis of the Vlasov stationary equation with boundary conditions which correspond to the absorption of plasma particles by the electrode (Section 3). It is assumed also that, at great distances from the electrode, the distribution function is Maxwellian, and ions move directedly to the electrode and possess the thermal distribution over velocities which is characterized by the relevant temperature. Just the presence of the thermal distribution of ions distinguishes the proposed statement of the problem from the traditional one. The analysis of the boundary-value problem for the self-consistent potential of the electric field is carried out in Section 4.

We applied different means for the construction of approximate solutions of the problem (Section 5). The solutions obtained are used for the numerical study of the problem. In particular, we calculate the distributions of electrons, ions, and the electric potential near an electrode absorbing plasma particles. The results of numerical analysis are discussed in Section 6.

## 2. Statement of the Problem and Input Equations

We will consider a two-component plasma composed of electrons and one-kind ions above the plane electrode ($x > 0$). The electrode absorbs charged particles of plasma and is under a floating potential. The field on the electrode surface is determined from the condition that the total current across the electrode surface is zero. Collisions between charged particles are not considered. We assume that electrons and ions possess different temperatures, and, in addition, ions as a whole are moving toward the electrode with some constant velocity. The temperature and the velocity of directional motion of ions are considered as parameters which can take relevant values. We assume also that, at a sufficiently great distance from the electrode surface, the condition of quasineutrality for plasma is satisfied. For the given temperature of ions ($T_i$), it is necessary to study, first, whether there exists the least (critical) drift velocity of ions ($U_{\rm cr}$) such that if it is exceeded, the electrode potential will be screened, and, second, how $U_{\rm cr}$ depends on $T_i$.

The input equations are composed from both the stationary kinetic equations for the distribution functions of electrons and ions and the Poisson equation for the potential of the self-consistent electric field. On the boundary of plasma ($x = 0$), the condition of absorption of charged particles by the electrode holds. The value of the potential on the "electrode–plasma" boundary ($x = 0$) is determined from the condition for the total current across the electrode surface to be zero. At $x = \infty$, i.e. for sufficiently great distances from the electrode, we assume that the field is absent.

Thus, it is necessary to find the functions $F_\sigma(x,v), \sigma = e, i$, and $\Phi(x)$ which satisfy the system of equations

$$v\frac{\partial F_\sigma}{\partial x} - \frac{e_\sigma}{m_\sigma}\frac{d\Phi}{dx}\frac{\partial F_\sigma}{\partial v} = 0,\ x > 0,\ v \in (-\infty, +\infty); \qquad (1)$$

$$\frac{d^2\Phi}{dx^2} = -4\pi\sum_\sigma e_\sigma n_\sigma(x),\ x > 0, \qquad (2)$$

with the boundary conditions

$$F_e(x,v)|_{x=0,v>0} = 0, \quad F_e(x,v)|_{x\to+\infty} =$$

$$= C_e e^{-m_e v^2/(2T_e)}\left[\theta(\sqrt{2e_e\Phi_0/m_e} - v)\theta(v) + \theta(-v)\right]; \qquad (3)$$

$$F_i(x,v)\Big|_{x=0,v>U\sqrt{1-\frac{2e_i\Phi_0}{m_iU^2}}} = 0, \quad F_i(x,v)\Big|_{x\to+\infty} =$$

$$= C_i\exp\left[-\frac{m_i}{2T_i}(v-U)^2\right]\theta(-v+U),\ U < 0; \qquad (4)$$

$$\Phi(0) = \Phi_0, \quad \Phi(+\infty) = 0. \qquad (5)$$

Here, $n_\sigma(x) = n_\sigma \int\limits_{-\infty}^{+\infty} F_\sigma(x,v)dv$, $n_\sigma = n_\sigma(+\infty)$ is the unperturbed density of particles of the kind $\sigma = e, i$, which satisfies the condition of quasineutrality

$$\sum_\sigma e_\sigma n_\sigma = 0, \qquad (6)$$

$e_\sigma$ is the charge of particles of the kind $\sigma$, $C_\sigma$ is the normalizing constant, $U$ is the velocity of a directional motion of ions, and

$$\theta(x) = \begin{cases} 1, & \text{if} \quad x \geq 0, \\ 0, & \text{if} \quad x < 0 \end{cases}$$

is the Heaviside function.

The boundary conditions (3), (4) mean that the surface has no electrons that move in the direction from it, and ions must have energies that exceed their energy at infinity. At great distances ($x \to +\infty$), only unabsorbed particles can be present. As for electrons, they are those moving to the surface or those which change the direction of motion without collision with the electrode. Ions can move only to the surface, and their velocity must exceed the drift velocity. In the case of collisionless plasma, these requirements are guaranteed by the corresponding limitations on the velocity of particles which are described with the help of the Heaviside $\theta$-function.

If the electrode is under a floating potential, then $\Phi_0$ is determined from the condition of compensation of the electron and ion currents

$$J_e(0) + J_i(0) = 0, \qquad (7)$$

where

$$J_\sigma(0) = J_\sigma(x)|_{x=0},\ J_\sigma(x) = e_\sigma n_\sigma \int\limits_{-\infty}^{+\infty} vF_\sigma(x,v)dv. \qquad (8)$$

While solving problem (1)–(7), it is expedient to pass to the dimensionless quantities

$$\bar{x} = x/\lambda_{\rm D},\ \bar{v} = (m_e/T_e)^{1/2}v;\ u = (m_e/T_e)^{1/2}U;\ \bar{z}_\sigma = \frac{e_\sigma}{e},$$

$$\bar{n}_\sigma = \frac{n_\sigma}{n_e},\ \bar{m}_\sigma = \frac{m_\sigma}{m_e},\ \bar{T}_\sigma = \frac{T_\sigma}{T_e};\ \varphi(\bar{x}) = \frac{e\Phi(x)}{T_e}|_{x=\bar{x}\lambda_{\rm D}},$$

$$f_\sigma(\bar{x},\bar{v}) = (T_e/m_e)^{1/2}F_\sigma(x,v)|_{x=\bar{x}\lambda_{\rm D},v=(T_e/m_e)^{1/2}\bar{v}};$$

$$\bar{n}_\sigma(\bar{x}) = \frac{n_\sigma(x)}{n_e}|_{x=\bar{x}\lambda_{\rm D}} = \bar{n}_\sigma\int\limits_{-\infty}^{+\infty} f_\sigma(\bar{x},\bar{v})d\bar{v};$$

$$j_\sigma(\bar{x}) = \frac{J_\sigma(x)}{J_{\rm th}}|_{x=\bar{x}\lambda_{\rm D}} = z_\sigma\bar{n}_\sigma\int\limits_{-\infty}^{+\infty}\bar{v}f_\sigma(\bar{x},\bar{v})d\bar{v},$$

$$J_{\rm th} = en_e\left(\frac{T_e}{m_e}\right)^{1/2},\quad \sigma = e,i. \qquad (9)$$

Here, $\lambda_{\rm D} = \left(\frac{T_e}{4\pi e^2 n_e}\right)^{1/2}$ is the Debye electron radius and $e = |e_e|$.

Below, we will use mainly the dimensionless quantities (9). The cases requiring to use other dimensionless or dimensional quantities will be indicated separately. To simplify the form of formulas, the bar over dimensionless quantities is dropped in what follows.

In terms of the dimensionless variables (9), problem (1)–(7) takes the following form:

$$v\frac{\partial f_\sigma}{\partial x} - \frac{z_\sigma}{m_\sigma}\frac{d\varphi}{dx}\frac{\partial f_\sigma}{\partial v} = 0,\ x > 0,\ v \in (-\infty,+\infty); \qquad (1')$$

$$\frac{d^2\varphi}{dx^2} = -\sum_\sigma z_\sigma n_\sigma(x),\ x > 0\ (\sigma = e,i), \qquad (2')$$

$$f_e(x,v)|_{x=0,v>0} = 0,\quad f_e(x,v)|_{x\to+\infty} =$$

$$= C_e e^{-v^2/2}\left[\theta\left(\sqrt{-2\varphi_0} - v\right)\theta(v) + \theta(-v)\right]; \qquad (3')$$

$$f_i(x,v)\Big|_{x=0,v>u\sqrt{1-\frac{2z_i\varphi_0}{m_iu^2}}} = 0,\quad f_i(x,v)\Big|_{x\to+\infty} =$$

$$= C_i\exp\left[-\frac{m_i}{2T_i}(v-u)^2\right]\theta(-v+u),\ u < 0; \qquad (4')$$

$$\varphi(0) = \varphi_0,\quad \varphi(+\infty) = 0; \qquad (5')$$

$$\sum_\sigma z_\sigma n_\sigma = 0. \qquad (6')$$

The quantity $\varphi_0$ is determined from the condition

$$j_e(0) + j_i(0) = 0. \qquad (7')$$

## 3. Stationary Distribution Functions and Self-Consistent Distributions of Currents and Charges

The solutions of Eq. (1′) with the boundary conditions (3′) and (4′) look as

$$f_e(x,v) = C_e e^{-v^2/2}e^{\varphi(x)}[\theta(\sqrt{2[\varphi(x)-\varphi(0)]} - v)\theta(v) +$$

$$+ \theta(-v)], \qquad (10a)$$

$$f_i(x,v) = C_i e^{-\frac{m_i}{2T_i}\ v\sqrt{1+\frac{2z_i\varphi}{m_iv^2}}-u\ ^2}\theta\left(-v\sqrt{1+\frac{2z_i\varphi}{m_iv^2}}+u\right) \qquad (10b)$$

or

$$f_i(x,v) = C_i e^{-\frac{m_i}{2T_i}\ v\sqrt{1+\frac{2z_i\varphi(x)}{m_iv^2}}-u\ ^2}\theta\left(-v-\sqrt{u^2-\frac{2z_i\varphi(x)}{m_i}}\right), \qquad (10c)$$

where $\theta(x)$ is the Heaviside function. In the construction of functions (10a)–(10c), we used the method of characteristics and the structure of the given functions at infinity, where the field is absent.

Using formulas (10a) and (10b), we can find $n_\sigma(x), \sigma = e,i$. In view of the condition of normalization

$$C_e = \sqrt{\frac{\pi}{2}}\left(1 + \operatorname{erf}\sqrt{-\varphi(0)}\right), \qquad (11)$$

we have

$$n_e(x) = e^{\varphi(x)}\frac{1 + \operatorname{erf}\sqrt{\varphi(x)-\varphi(0)}}{1 + \operatorname{erf}\sqrt{-\varphi(0)}},\ x \geq 0\quad (n_e = 1). \qquad (12)$$

 ISSN 0503-1265. Ukr. J. Phys. 2007. V. 52, N 12

Under the condition $|\varphi(x)| \ll |\varphi(0)|$, formula (12) describes the Boltzmann distribution of electrons usually used in similar problems.

For ions, respectively, we get

$$C_i = \int\limits_{-\infty}^{u} e^{-\frac{m_i}{2T_i}(v-u)^2} dv = \sqrt{\frac{\pi T_i}{2m_i}}, \tag{13}$$

$$n_i(x) = n_i \frac{2}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{(\xi - u/\sqrt{2}S_i)e^{-\xi^2}}{\sqrt{(\xi - u/\sqrt{2}S_i)^2 - z_i\varphi(x)/T_i}} d\xi,$$

$$S_i = \sqrt{\frac{T_i}{m_i}}. \tag{14}$$

We now consider the electron and ion currents $J_\sigma(x)$, $\sigma = e, i$, which are necessary for the determination of the potential on the region boundary. The calculations on the basis of formulas (10a), (10b), (11), and (13) give

$$J_e(0) = \sqrt{\frac{2}{\pi}} \frac{e^{\varphi_0}}{1 + \operatorname{erf}\sqrt{-\varphi_0}}. \tag{15}$$

$$J_i(0) = -z_i n_i \left( \sqrt{\frac{2}{\pi}} S_i - u \right). \tag{16}$$

If the electrode is under a floating potential, then the quantity $\varphi_0 = \varphi(0)$ can be determined from condition (7′). Using formulas (15) and (16), we obtain

$$\sqrt{\frac{2}{\pi}} \frac{e^{\varphi_0}}{1 + \operatorname{erf}\sqrt{-\varphi_0}} = z_i n_i \left( \sqrt{\frac{2}{\pi}} S_i - u \right)$$

or (with regard for the condition of quasineutrality (6′), $z_i n_i = 1$)

$$\frac{e^{\varphi_0}}{1 + \operatorname{erf}\sqrt{-\varphi_0}} = -\sqrt{\frac{\pi}{2}} u + S_i. \tag{17}$$

For the further consideration, it is convenient to introduce the notation

$$\tilde{U} = \frac{u T_i^{1/2}}{\sqrt{2} S_i} = u \sqrt{\frac{m_i}{2}}. \tag{18}$$

Then formula (14) and Eq. (17) take the form

$$n_i(\varphi; \tilde{U}, T_i) = n_i \frac{2}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{\left(\xi T_i^{1/2} - \tilde{U}\right) e^{-\xi^2} d\xi}{\sqrt{\left(\xi T_i^{1/2} - \tilde{U}\right)^2 - z_i \varphi}}, \tag{19}$$

$$\frac{e^{\varphi_0}}{1 + \operatorname{erf}\sqrt{-\varphi_0}} = -\tilde{U} \sqrt{\frac{\pi}{m_i}} + S_i,$$

$$\left(\tilde{U} \le 0;\ T_i \ge 0,\ \varphi_0 \le 0\right). \tag{20}$$

## 4. Boundary-Value Problem for the Self-Consistent Potential of the Electric Field and Its Analysis

The boundary-value problem (2′), (5′), (6′) can be written, with regard for the formulas obtained, in the following form:

$$\frac{d^2\varphi}{dx^2} = -\sum_\sigma z_\sigma n_\sigma(\varphi; \varphi_0, \tilde{U}, T_i),\ x > 0, \tag{21}$$

$$\varphi(0) = \varphi_0, \quad \varphi(+\infty) = 0. \tag{22}$$

Here, $n_\sigma(\varphi; \varphi_0, \tilde{U}, T_i)$ are determined, respectively, by formulas (12) and (19) and $\varphi_0$ can be found from Eq. (20) or can be given as an additional condition in the case of a fixed potential of the electrode.

We will write problem (21), (22) in a form more convenient for its study. In this case, we will use the condition of quasineutrality $z_i n_i = 1$ and, taking into account that the quantities $\varphi(x), u$ and $\tilde{U}$ take negative values, make substitution

$$\varphi(x) = -|\varphi(x)|, \quad u = -|u|, \quad \tilde{U} = -|\tilde{U}|. \tag{23}$$

Then formulas (12) and (19) and Eq. (20) become

$$n_e(|\varphi|, |\varphi_0|) = e^{-|\varphi|} \frac{1 + \operatorname{erf}\sqrt{|\varphi_0| - |\varphi|}}{1 + \operatorname{erf}\sqrt{|\varphi_0|}}. \tag{12′}$$

$$n_i(|\varphi|; |\tilde{U}|, T_i) = n_i \frac{2}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{(\xi T_i^{1/2} + |\tilde{U}|)e^{-\xi^2}}{\sqrt{(\xi T_i^{1/2} + |\tilde{U}|)^2 + z_i|\varphi|}} d\xi, \tag{19′}$$

$$\frac{e^{-|\varphi_0|}}{1 + \operatorname{erf}\sqrt{|\varphi_0|}} = |\tilde{U}| \sqrt{\frac{\pi}{m_i}} + S_i \quad (|\varphi_0| \ge 0). \tag{24}$$

Thus, we are faced with the necessity to solve the following nonlinear boundary-value problem (the modulus sign for the quantities $\varphi, \varphi_0, u, \tilde{U}$ is dropped):

$$\frac{d^2\varphi}{dx^2} = f(\varphi; \varphi_0, \tilde{U}, T_i),\ x > 0, \tag{25}$$

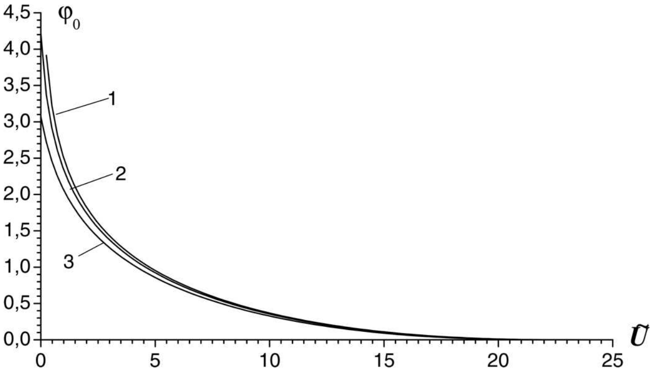


Fig. 1. Dependence of $\varphi_0$ on $\tilde{U}$, $\tilde{U} \in [0; \tilde{U}_{\max}(T_i)]$ for various values of $T_i$: *1* – $T_i = 0$; *2* – $T_i = 0.1$; *3* – $T_i = 1$

$$\varphi(0) = \varphi_0, \quad \varphi(+\infty) = 0. \tag{26}$$

Here, $\varphi_0 = \varphi_0(\tilde{U}, T_i)$ is determined from Eq. (24), and $f(\varphi; \varphi_0, \tilde{U}, T_i)$ looks as

$$f(\varphi; \varphi_0, \tilde{U}, T_i) = -e^{-\varphi} \frac{1 + \operatorname{erf}\sqrt{\varphi_0 - \varphi}}{1 + \operatorname{erf}\sqrt{\varphi_0}} +$$

$$+ \frac{2}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{\left(\xi T_i^{1/2} + \tilde{U}\right) e^{-\xi^2} d\xi}{\sqrt{\left(\xi T_i^{1/2} + \tilde{U}\right)^2 + z_i \varphi}}, \; T_i \geq 0, \; \tilde{U} \geq 0. \tag{27}$$

With regard for substitutions (23), the dimensionless velocity of ions $\tilde{U}$ and the potential $\varphi(x)$ in terms of dimensional quantities take the form

$$\tilde{U} = -U \sqrt{\frac{m_i}{2T_e}}, \; \varphi = -\frac{e\Phi}{T_e}. \tag{28}$$

Equation (24) from which we obtain $\varphi_0 = \varphi(0)$, is an important component of the statement of the problem, and, therefore, we consider it separately. In this case, the calculation algorithm of this quantity is as follows: we take some admissible value of the parameter $T_i$ which determines the interval $[0; \tilde{U}_{\max}(T_i)]$, in which the parameter $\tilde{U}$ is varied. Then, for every $\tilde{U}$ from the given interval, we determine $\varphi_0$ as the root of Eq. (24). Thus, $\varphi_0 = \varphi_0(\tilde{U}, T_i)$.

The form of Eq. (24) allows us to draw some general conclusions about its roots. In this case, we are interested only in nonnegative roots.

Let $f_1(\varphi_0) \equiv e^{-\varphi_0}/[1 + \operatorname{erf}(\sqrt{\varphi_0})]$, $\varphi_0 \geq 0$, and $f_2(\tilde{U}, T_i, m_i) \equiv \tilde{U}\sqrt{\frac{\pi}{m_i}} + S_i$, $\tilde{U} \geq 0$.

The functions $f_1(\varphi_0)$ and $f_2(\tilde{U}, T_i, m_i)$ have the following properties:

$$f_1(0) = 1, \quad f_1(+\infty) = 0;$$

$$f_2(\tilde{U}, 0, m_i) = \tilde{U}\sqrt{\pi/m_i}; \quad f_2(0, T_i, m_i) = S_i \geq 0,$$

$$\forall T_i \geq 0; \quad f_2(\infty, T_i, m_i) = \infty.$$

For every fixed value of $T_i \geq 0$, the function $f_2(\tilde{U}, T_i, m_i)$ is a monotonically increasing function of $\tilde{U}$, and Eq. (24) has a root only for those values of $\tilde{U}$, for which $f_2(\tilde{U}, T_i, m_i) \leq 1$.

For every $T_i \geq 0$, the quantity $\tilde{U}$ varies in the interval $[0; \tilde{U}_{\max}(T_i)]$, and

$$\tilde{U}_{\max}(T_i) = \sqrt{\frac{m_i}{\pi}} - \sqrt{\frac{T_i}{\pi}};$$

$$\max_{T_i \geq 0}[\tilde{U}_{\max}(T_i)] = \tilde{U}_{\max}(0) = \sqrt{\frac{m_i}{\pi}}. \tag{29}$$

In the case of hydrogen ions, $\tilde{U}_{\max}(0) = \sqrt{m_i/\pi} \approx \sqrt{1836/\pi} \approx 24.17$.

Thus, $\forall \tilde{U} \in [0; \tilde{U}_{\max}(T_i)]$, Eq. (24) has a single root $\varphi_0 = \varphi_0(\tilde{U}, T_i)$, and, for $\tilde{U} = \tilde{U}_{\max}(T_i)$, the root of this equation $\varphi_0 = 0$.

The plots of the function $\varphi_0 = \varphi_0(\tilde{U}, T_i)$ for various values of $T_i \geq 0$ are given in Fig. 1.

If we pass to the limit $T_i \to 0$ in Eq. (24), we obtain

$$\frac{e^{-|\varphi_0|}}{1 + \operatorname{erf}\sqrt{|\varphi_0|}} = \tilde{U}\sqrt{\frac{\pi}{m_i}}. \tag{24$'$}$$

Thus, there are no essential limitations on the existence of the solution of Eq. (24). But it is obvious that the boundary-value problem (25)–(27) has a solution not for each solution of Eq. (24). We are interested in: under which conditions there exist the nontrivial solutions of the given problem and how these conditions depend on parameters of the problem. We note that problem (24)–(27) has always the trivial solution $\varphi(x, \varphi_0) \equiv 0$ which corresponds to the case where $\tilde{U} = \tilde{U}_{\max}(T_i)$.

Let us study the function $f(\varphi; \varphi_0, \tilde{U}, T_i)$. We consider the general case where the quantity $T_i$ takes any values, i.e. $T_i \in [0; 1]$, and $\tilde{U} \in [0; \tilde{U}_{\max}(T_i)]$. We note that, for a given $T_i \geq 0$ and any $\tilde{U} \in [0; \tilde{U}_{\max}(T_i)]$, $\varphi$ varies from 0 to $\varphi_0(\tilde{U})$.

As seen, the function $f(\varphi;\varphi_0,\tilde U,T_i)$ has the following properties:

1) $f(0;\varphi_0,\tilde U,T_i)\equiv 0,$ (30)

2) $$f(\varphi;\varphi_0,\tilde U,0)=-e^{-\varphi}\frac{1+\mathrm{erf}\sqrt{\varphi_0-\varphi}}{1+\mathrm{erf}\sqrt{\varphi_0}}+\frac{\tilde U}{\sqrt{\tilde U^2+z_i\varphi}}. \quad (31)$$

Formula (31) is not defined at the point $(\varphi=0,\tilde U=0)$ which is the essentially singular point of function (31). Therefore, we take $\tilde U>0$.

We note that relation (31) corresponding to the case where $T_i=0$ differs from the charge distribution calculated for Boltzmann electrons [1] by the additional factor $\left(1+\mathrm{erf}\sqrt{\varphi_0-\varphi}\right)/\left(1+\mathrm{erf}\sqrt{\varphi_0}\right)$. It is obvious that, as $\varphi_0\gg\varphi(x)$, this factor tends to unity, and, as $x\to 0$ $(\varphi(x)\to\varphi_0)$, $\left(1+\mathrm{erf}\sqrt{\varphi_0-\varphi}\right)/\left(1+\mathrm{erf}\sqrt{\varphi_0}\right)\approx\frac{1}{2}$.

3) $$f(\varphi;\varphi_0(0),0,T_i)=-e^{-\varphi}\frac{1+\mathrm{erf}\sqrt{\varphi_0-\varphi}}{1+\mathrm{erf}\sqrt{\varphi_0}}+$$
$$+e^{z_i\varphi/T_i}\left(1-\mathrm{erf}\sqrt{\frac{z_i\varphi}{T_i}}\right),\ (T_i>0,\ \tilde U=0), \quad (32)$$

4) $$f(\varphi_0;\varphi_0,\tilde U,T_i)=-\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}}+$$
$$+\frac{2}{\sqrt\pi}\int\limits_0^{+\infty}\frac{(\xi T_i^{1/2}+\tilde U)e^{-\xi^2}d\xi}{\sqrt{(\xi T_i^{1/2}+\tilde U)^2+z_i\varphi_0}},$$

5) $$f(\varphi_0;\varphi_0,\tilde U,0)=-\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}}+\frac{\tilde U}{\sqrt{\tilde U^2+z_i\varphi_0}}$$

or, according to (24′),

$$f(\varphi_0;\varphi_0,\tilde U,0)=-\tilde U\sqrt{\frac{\pi}{m_i}}+\frac{\tilde U}{\sqrt{\tilde U^2+z_i\varphi_0}},$$

6) $$f'_\varphi(\varphi;\varphi_0,\tilde U,T_i)=e^{-\varphi}\frac{1+\mathrm{erf}\sqrt{\varphi_0-\varphi}}{1+\mathrm{erf}\sqrt{\varphi_0}}+$$
$$+\frac{e^{-\varphi}}{1+\mathrm{erf}\sqrt{\varphi_0}}\frac{1}{\sqrt{\pi(\varphi_0-\varphi)}}-$$
$$-\frac{z_i}{\sqrt\pi}\int\limits_0^{+\infty}\frac{\left(\xi T_i^{1/2}+\tilde U\right)e^{-\xi^2}d\xi}{\sqrt{\left[\left(\xi T_i^{1/2}+\tilde U\right)^2+z_i\varphi\right]^3}},\ T_i\ge 0, \quad (33)$$

7) $$f'_\varphi(\varphi;\varphi_0,\tilde U,0)=e^{-\varphi}\frac{1+\mathrm{erf}\sqrt{\varphi_0-\varphi}}{1+\mathrm{erf}\sqrt{\varphi_0}}+$$
$$+\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}}\frac{1}{\sqrt{\pi(\varphi_0-\varphi)}}-\frac{z_i}{2}\frac{\tilde U}{\sqrt{(\tilde U^2+z_i\varphi)^3}},\ \tilde U>0, \quad (34)$$

8) $$f'_\varphi(0;\varphi_0,\tilde U,T_i)=1+\frac{1}{\sqrt{\pi\varphi_0}}\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}}-$$
$$-\frac{z_i}{\sqrt\pi}\int\limits_0^{+\infty}\frac{e^{-\xi^2}d\xi}{\sqrt{\left(\xi T_i^{1/2}+\tilde U\right)^2}},\ T_i\ge 0, \quad (35)$$

9) $$f'_\varphi(0;\varphi_0,\tilde U,0)=1+\frac{1}{\sqrt{\pi\varphi_0}}\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}}-\frac{z_i}{2\tilde U^2}, \quad (36)$$

10) $$f'_\varphi(\varphi;\varphi_0(0),0,T_i)=e^{-\varphi}\frac{1+\mathrm{erf}\sqrt{\varphi_{0,\max}-\varphi}}{1+\mathrm{erf}\sqrt{\varphi_{0,\max}}}+$$
$$+\frac{e^{-\varphi_{0,\max}}}{1+\mathrm{erf}\sqrt{\varphi_{0,\max}}}\frac{1}{\sqrt{\pi\left(\varphi_{0,\max}-\varphi\right)}}+$$
$$+\frac{z_i}{T_i}e^{z_i\varphi/T_i}\left(1-\mathrm{erf}\sqrt{\frac{z_i\varphi}{T_i}}\right)-\frac{z_i}{T_i}\frac{1}{\sqrt\pi}\frac{1}{\sqrt{z_i\varphi/T_i}}, \quad (37)$$

where $T_i>0$, and $\varphi_{0,\max}=\max\limits_{\tilde U\ge 0}\varphi_0(\tilde U)=\varphi_0(0)$.

In Fig. 2($a$, $b$), we present the plots of the functions $y=f\left(\varphi;\varphi_0,\tilde U,T_i\right)$, $\varphi\in\left[0;\ \varphi_0(\tilde U)\right]$ for certain values of $T_i\in[0;\ 1]$ and $\tilde U\in\left[0;\tilde U_{\max}(T_i)\right]$. In the construction of these plots, we used Eq. (24) and formula (27).

Formulas (33)–(37) yield the following properties of the function $f'_\varphi\left(\varphi;\varphi_0,\tilde U,T_i\right)$:

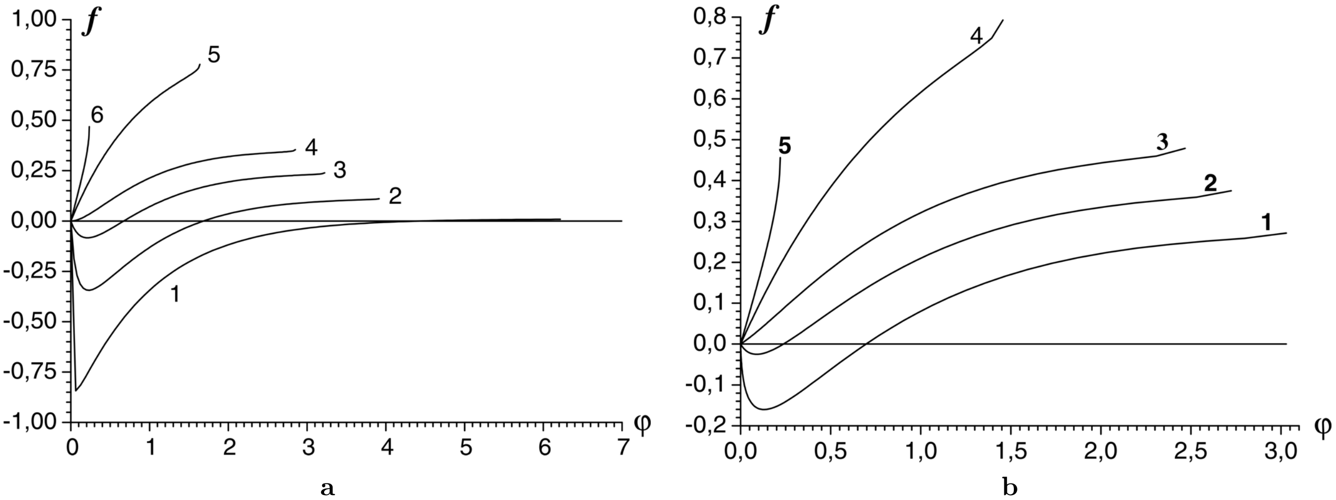


Fig. 2. Plots of the function $y = f\left(\varphi;\varphi_0,\tilde{U},T_i\right)$, $\varphi \in \left[0;\varphi_0(\tilde{U})\right]$, for various values of the parameters $T_i (0 \le T_i \le 1)$ and $\tilde{U} = k\tilde{U}_{\max}(T_i)$, $0.001 \le k \le 0.5$. (*a*): $T_i = 0$; *1* – 0.001; *2* – 0.01; *3* – 0.02; *4* – 0.029; *5* – 0.1; *6* – 0.5. (*b*): $T_i = 1$; *1* – 0.001; *2* – 0.01; *3* – 0.02; *4* – 0.1; *5* – 0.5

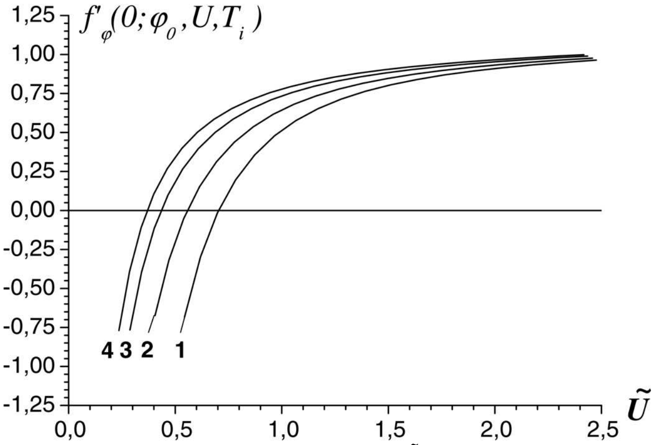


Fig. 3. Plots of the function $f'_\varphi\left(0;\varphi_0,\tilde{U},T_i\right)$, as a function of $\tilde{U}$, $\tilde{U} \in \left[0;\tilde{U}_{\max}(T_i)\right]$ for various values of the parameter $T_i$: *1* – 0; *2* – 0.1; *3* – 0.5; *4* – 1

1) For all admissible values of $T_i$ and $\tilde{U}$, the derivative $f'_\varphi\left(\varphi;\varphi_0,\tilde{U},T_i\right)$ does not exist for $\varphi = \varphi_0$, i.e. $\lim f'_\varphi\left(\varphi;\varphi_0,\tilde{U},T_i\right) = +\infty$ as $\varphi \to \varphi_0 - 0$.

2) If $\tilde{U} > 0$, the derivative $f'_\varphi\left(\varphi;\varphi_0,\tilde{U},0\right)$ exists for $\varphi = 0$ and is determined by formulas (35) and (36); for $\tilde{U} = 0$, the derivative for $\varphi = 0$ does not exist, according to (37).

3) For sufficiently small values of the parameter $\tilde{U}$, $f'_\varphi(\varphi;\varphi_0,...)$ changes the sign from the negative to positive one on the change of $\varphi$ from 0 to $\varphi_0(\tilde{U})$, and, hence, it has local minimum.

4) For each fixed value of $T_i \ge 0$, the function $f'_\varphi\left(0;\varphi_0,\tilde{U},T_i\right)$ changes the sign from the negative to positive on the change of $\tilde{U}$ from 0 to $\tilde{U}_{\max}(T_i)$; therefore, there exists such a (critical) value $\tilde{U} = \tilde{U}_{\rm cr}$, at which $f'_\varphi\left[0;\varphi_0(\tilde{U}_{\rm cr}),\tilde{U}_{\rm cr},T_i\right] = 0$. The presented properties of the function $f\left(\varphi;\varphi_0,\tilde{U},T_i\right)$ are used on the establishment of the conditions of the existence of nonzero solutions of problem (24)–(27).

In Fig. 3, we give the plots of the function $f'_\varphi\left(0;\varphi_0,\tilde{U},T_i\right)$ as a function of $\tilde{U}$, $\tilde{U} \in \left[0;\tilde{U}_{\max}(T_i)\right]$. These plots demonstrate the dependence of the behavior of the given function on the parameter $\tilde{U}$ for various values of $T_i \ge 0$ and the dependence of the point of the crossing of the axis $0\tilde{U}$ by the plots of the given function, $\tilde{U}_{\rm cr}$, on $T_i$. In the construction of these plots, we used Eq. (24) and formulas (35) and (36).

## 5. Construction of Approximate Solutions of the Problem

Equation (25) has fixed point $\varphi = 0, \forall T_i \ge 0, \forall \tilde{U} \in \left[0;\ \tilde{U}_{\max}(T_i)\right]$, and, therefore, problem (24)–(27) has trivial (zero) solution $\varphi(x;\varphi_0,\tilde{U}) \equiv 0$ which corresponds to the value of the parameter $\tilde{U} = \tilde{U}_{\max}(T_i)$. We are interested in nontrivial solutions of problem (24)–(27).

Consider the boundary-value problem

$$\varphi'' = f(\varphi),\ x > 0;\ \varphi(0) = \varphi_0,\ \varphi(+\infty) = 0, \tag{38}$$

where $\varphi_0 \geq 0$, and the function $f(\varphi)$ is continuous on the interval $[0;\varphi_0]$, continuously differentiable on the interval $[0;\varphi_0)$, and satisfies the condition $f(0) = 0$. We would like to know which (additional) conditions must be satisfied by the function $f(\varphi)$ in order that problem (38) have a nonzero solution $0 \leq \varphi(x,\varphi_0) \leq \varphi_0$ corresponding to the screened field. Let a solution of the given problem, $\varphi(x;\varphi_0)$, exist. Then the relations $\varphi''\varphi' = f(\varphi)\varphi'$, $d\varphi'^2 = 2f(\varphi)d\varphi$, $\varphi'^2(x) = = 2\int\limits_0^x f(\varphi(x))d\varphi + C_1 = 2\int\limits_{\varphi_0}^{\varphi} f(\xi)d\xi + C_1$, and $(\xi = \varphi(x))$ are valid. The conditions $\varphi(+\infty) = 0$ and $\varphi''(+\infty) = 0$ yield $\varphi'(+\infty) = 0$, $C_1 = -2\int\limits_0^{\varphi_0} f(\xi)d\xi$, and, therefore, $\varphi'^2(x) = 2\int\limits_0^{\varphi} f(\xi)d\xi$. The last relation yields the first necessary condition which must be satisfied by the function $f(\varphi)$:

$$\int\limits_0^{\varphi} f(\xi)d\xi \geq 0,\ \forall\varphi \geq 0. \tag{39}$$

For problem (24)–(27), where the function $f(\varphi;\varphi_0,\tilde{U},T_i)$ is determined by formula (27), it was shown that there exist the values of the parameter $\tilde{U}$ such that $f(\varphi;\varphi_0,\tilde{U},T_i) < 0$ if $\varphi \in [0;\varphi_c(\tilde{U})] \subset [0;\varphi_0(\tilde{U})]$. Therefore, for such $\tilde{U}$, problem (24)–(27) has no solution. We can show that the exact solution of problem (38) given implicitly looks as

$$\int\limits_{\varphi}^{\varphi_0} \frac{d\xi}{\sqrt{2\int\limits_0^{\xi} f(\zeta)d\zeta}} = x,\ x \geq 0. \tag{40}$$

The divergence of the integral on the left-hand side of Eq. (40) as $x \to +\infty$ (in this case, $\varphi(x) \to 0$) is the second necessary condition which must be satisfied by the function $f(\varphi)$. For example, in the case where $f(\varphi) = \varphi^n, n \geq 1$, both the first and second conditions are satisfied, and the boundary-value problem (38) has a solution. In the case where $f(\varphi) = \sqrt{\varphi}$, the first condition is satisfied, and the second one does not hold; therefore, problem (38) has no solution.

The presented conditions are the necessary ones for the existence of a solution of problem (38). If $f'_\varphi(0) \geq 0$ in this case, we will consider that boundary-value problem (38) has a solution. In particular, for $f'_\varphi(0) > 0$, the expansion $f(\varphi) = f'(0)\varphi + O(\varphi^2)$ is valid in a neighborhood of the point $\varphi = 0$, i.e. $f(\varphi) \approx f'(0)\varphi$. Therefore, integral (40) is divergent for the given function as $\varphi \to 0$, and the conditions of existence of a solution of problem (38) are satisfied. If $f'_\varphi(0) = 0$, then, in a neighborhood of $\varphi = 0$, the expansion $f(\varphi) = \frac{1}{2}f''(0)\varphi^2 + O(\varphi^3)$ holds, i.e. $f(\varphi) \approx a\varphi^2$, where $a = f''(0)/2$. Then, if $a > 0$, problem (38) has also a solution for the given function, namely,

$$\varphi(x;\varphi_0) = \frac{6}{a}\left(x + \sqrt{\frac{6}{a}}\varphi_0^{-1/2}\right)^{-2},\ x \geq 0;\ \varphi_0 \geq 0. \tag{41}$$

Formula (41) implies that $\varphi(x;\varphi_0) \to 0$, as $x \to +\infty$ by the power law; whereas, for $f(\varphi) = \varphi$, $\varphi(x;\varphi_0) \to 0$ by the exponential law as $x \to +\infty$.

Earlier, it has been shown that, for every fixed value of $T_i \geq 0$, there exists such (critical) value of $\tilde{U} = \tilde{U}_{\rm cr}(T_i)$, at which $f'_\varphi\left[0;\varphi_0(\tilde{U}_{\rm cr}),\tilde{U}_{\rm cr},T_i\right] = 0$. If $\tilde{U} > \tilde{U}_{\rm cr}(T_i)$, then $f'_\varphi\left[0;\varphi_0(\tilde{U}_{\rm cr}),\tilde{U}_{\rm cr},T_i\right] > 0$, and therefore $f'_\varphi\left(\varphi;\varphi_0,\tilde{U},T_i\right) > 0$ and $f\left(\varphi;\varphi_0,\tilde{U},T_i\right) > 0$ for all $\varphi \geq 0$. Thus, at $\tilde{U} > \tilde{U}_{\rm cr}$, problem (24)–(27) has a nonzero solution which can be presented in the form (40). In the case where $\tilde{U} = \tilde{U}_{\rm cr}$ in a neighbohood of the point $\varphi = 0$, the expansion $f\left(\varphi;\varphi_0,\tilde{U},T_i\right) = \frac{1}{2}f''\left(0;\varphi_0,\tilde{U},T_i\right)\varphi^2 + O(\varphi^3)$ is valid. That is, $f\left(\varphi;\varphi_0,\tilde{U},T_i\right) \approx a\varphi^2$, where $a = \frac{1}{2}f''\left(0;\varphi_0,\tilde{U},T_i\right)$. Therefore, if $a > 0$, then problem (24)–(27) also has a solution.

If we pass from Eq. (25) to the system of equations for $\varphi$ and $\varphi'$, then the point $\varphi = 0$, $\varphi' = 0$ in the phase plane $(\varphi,\ \varphi')$ is an isolated singular point of the given system of equations. Depending on a value of the parameter $\tilde{U}$, this point changes its type. Namely, we have a saddle singular point at $\tilde{U} > \tilde{U}_{\rm cr}$ and a center at $\tilde{U} < \tilde{U}_{\rm cr}$. At $\tilde{U} = \tilde{U}_{\rm cr}$, the point $\varphi = 0$, $\varphi' = 0$ is a nonelementary singular point. Thus, the parameter $\tilde{U}$ is a bifurcation parameter of the problem, and the value $\tilde{U} = \tilde{U}_{\rm cr}$ is critical or the bifurcation point for the given parameter.

In Fig. 4, we give the plot of $\tilde{U}_{\rm cr}$ versus $T_i$ (the reduced temperature of ions), and Fig. 5 presents the dependence of $\varphi_{0,\rm cr} \equiv \varphi_0\left(\tilde{U}_{\rm cr},T_i\right)$ on $T_i \in [0;1]$. In this case, we used that $\tilde{U}_{\rm cr}(T_i)$ is a solution of the following system of nonlinear equations:

$$1 + \frac{1}{\sqrt{\pi\varphi_0}}\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}} - -\frac{z_i}{\sqrt{\pi}}\int\limits_0^{+\infty}\frac{e^{-\xi^2}d\xi}{(\xi T_i^{1/2}+\tilde{U}_{\rm cr})^2} = 0,$$

$$\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}} = \tilde{U}_{\rm cr}\sqrt{\frac{\pi}{m_i}} + S_i, \tag{42}$$

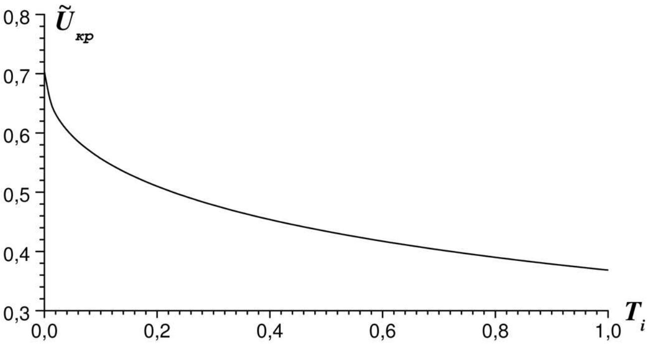

Fig. 4. Dependence of $\tilde{U}_{\rm cr}$ on $T_i$ (the reduced temperature of ions), $T_i \in [0;1]$

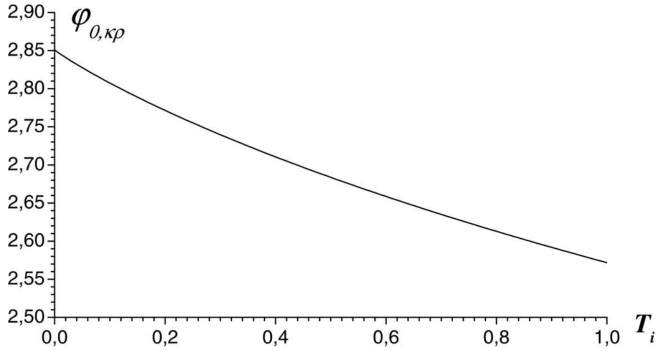

Fig. 5. Dependence of $\varphi_{0,\rm cr}$ on $T_i$, $T_i \in [0;1]$

where $S_i = (T_i/m_i)^{1/2}$, $\varphi_0 = \varphi_0(\tilde{U}_{\rm cr})$, $\tilde{U}_{\rm cr} = \tilde{U}_{\rm cr}(T_i)$.

If $\varphi_0$ is fixed, then $\tilde{U}_{\rm cr}(T_i)$ is determined by the first equation from the above-presented ones.

In the case of a floating potential, we get $1+\frac{1}{\sqrt{\pi\varphi_0}}\left(\tilde{U}_{\rm cr}\sqrt{\frac{\pi}{m_i}}+S_i\right)-\frac{z_i}{\sqrt{\pi}}\int\limits_0^{+\infty}\frac{e^{-\xi^2}d\xi}{(\xi T_i^{1/2}+\tilde{U})_{\rm cr})^2}=0$ from system (42), and this relation yields

$$\sqrt{\varphi_0} = \left(\tilde{U}_{\rm cr}\sqrt{\frac{\pi}{m_i}}+S_i\right) \Big/ \left[z_i\int\limits_0^{+\infty}\frac{e^{-\xi^2}d\xi}{(\xi T_i^{1/2}+\tilde{U}_{\rm cr})^2}-\sqrt{\pi}\right]. \tag{43}$$

By substituting (43) in (24), we get the equation for $\tilde{U}_{\rm cr}$:

$$\exp\left[-\frac{1}{m_i}\left(\frac{\tilde{U}_{\rm cr}+\sqrt{T_i/\pi}}{\frac{z_i}{\sqrt{\pi}}I(\tilde{U}_{\rm cr},T_i)-1}\right)^2\right] = \left(\tilde{U}_{\rm cr}\sqrt{\frac{\pi}{m_i}}+S_i\right)\times$$

$$\times\left[1+\mathrm{erf}\frac{\tilde{U}_{\rm cr}+\sqrt{T_i/\pi}}{\sqrt{m_i}\left(\frac{z_i}{\sqrt{\pi}}I(\tilde{U}_{\rm cr},T_i)-1\right)}\right]. \tag{44}$$

Here, $I(\tilde{U}_{\rm cr},T_i) = \int\limits_0^{+\infty}\frac{e^{-\xi^2}d\xi}{(\xi T_i^{1/2}+\tilde{U}_{\rm cr})^2}$; $T_i \geq 0$; and

$$\tilde{U}_{\rm cr} \in \left[0;\sqrt{\frac{m_i}{\pi}}-\sqrt{\frac{T_i}{m_i}}\right].$$

In the case where $T_i = 0$, Eqs. (42) look as

$$1+\frac{1}{\sqrt{\pi\varphi_0}}\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}} - \frac{z_i}{2\tilde{U}_{\rm cr}^2}=0,$$

$$\frac{e^{-\varphi_0}}{1+\mathrm{erf}\sqrt{\varphi_0}} = \tilde{U}_{\rm cr}\sqrt{\frac{\pi}{m_i}},\ \varphi_0=\varphi_0(\tilde{U}_{\rm cr}),$$

and we have $\sqrt{\varphi_0} = \frac{2\tilde{U}_{\rm cr}^3}{\sqrt{m_i}\left(z_i-2\tilde{U}_{\rm cr}^2\right)}$ instead of (43). The last formula implies that the inequality $z_i - 2\tilde{U}_{\rm cr}^2 > 0$ or $\tilde{U}_{\rm cr} < \sqrt{\frac{z_i}{2}} = \frac{\sqrt{2}}{2}\sqrt{z_i} \approx \frac{1.4142}{2}\sqrt{z_i} = 0.7071\sqrt{z_i}$, $(z_i = 1, 2, ...)$ must be satisfied.

We note that, in the case of the Boltzmann distribution of electrons at $T_i = 0$ [1, 2], the equations for $\tilde{U}_{\rm cr}$ take the form $1 - z_i/(2\tilde{U}_{\rm cr}^2) = 0$. Therefore, $\tilde{U}_{\rm cr} = \frac{\sqrt{2}}{2}\sqrt{z_i} \approx 0.7071\sqrt{z_i}$.

With regard for the deviation of a distribution of electrons from the Boltzmann one, we have, respectively,

$$\exp\left[-\frac{1}{m_i}\left(\frac{2\tilde{U}_{\rm cr}^3}{z_i-2\tilde{U}_{\rm cr}^2}\right)^2\right] = \tilde{U}_{\rm cr}\sqrt{\frac{\pi}{m_i}}\times$$

$$\times\left[1+\mathrm{erf}\frac{2\tilde{U}_{\rm cr}^3}{\sqrt{m_i}\left(z_i-2\tilde{U}_{\rm cr}^2\right)}\right],\quad \tilde{U}_{\rm cr}\in\left(0;\sqrt{\frac{z_i}{2}}\right]. \tag{45}$$

On the construction of an approximate solution of problem (24)–(27), we may use various methods [6–10]. In particular, the exact solution of the given problem is given implicitly in the form (40). With regard for the dependence of $x$ on $\varphi$ and parameters of the problem, formula (40) takes the form

$$x\left(\varphi;\varphi_0,\tilde{U},T_i\right) = \int\limits_{\varphi}^{\varphi_0}\frac{d\xi}{\sqrt{2\int_0^\xi f(\zeta;\tilde{U},T_i)d\zeta}} = \int\limits_{\varphi}^{\varphi_0}F(\xi)d\xi, \tag{46}$$

where $F(\xi) = 1/\sqrt{2\int_0^\xi f(\zeta;\tilde{U},T_i)d\zeta}$, $\xi\in[\varphi;\varphi_0]$, $\varphi\in[0;\ \varphi_0]$.



ISSN 0503-1265. Ukr. J. Phys. 2007. V. 52, N 12

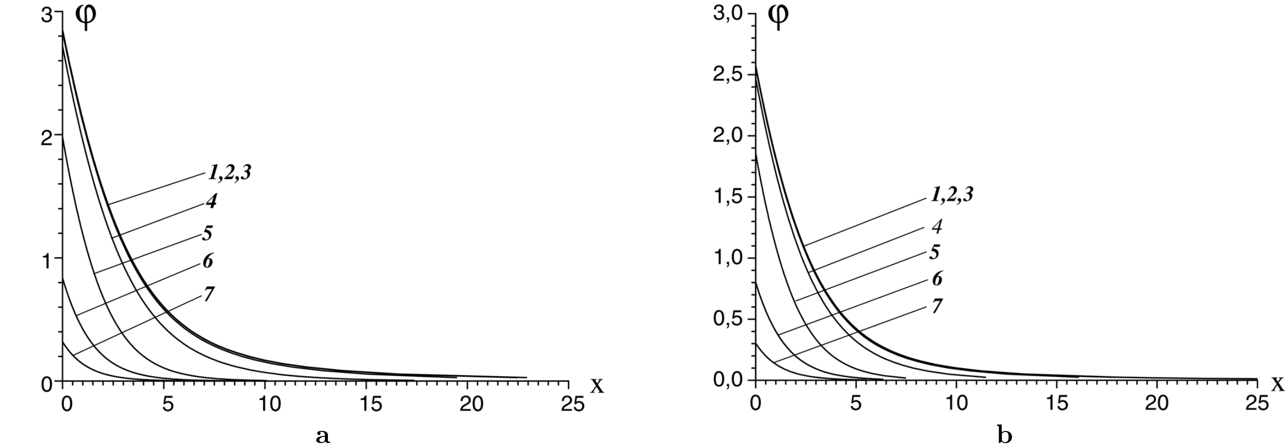


Fig. 6. Dependence of the potential $\varphi = \varphi\left(x;\varphi_0,\tilde{U},T_i\right)$ on the coordinate $x$ for various values of the parameters $T_i$ and $\tilde{U} = \tilde{U}_{\rm cr} + \delta$. ($a$): $T_i = 0$; ($b$): $T_i = 1$, *1* – $\delta = 10^{-4}$; *2* – $10^{-3}$; *3* – $10^{-2}$; *4* – $10^{-1}$; *5* – 1; *6* – 5; *7* – 10

Let $\varphi_n = nh_\varphi$, $h_\varphi = \varphi_0/N$, $n = 0,1,...,N$; $x_n(\varphi_0) = x(\varphi_n;\varphi_0) = \int\limits_{\varphi}^{\varphi_0} F(\xi)d\xi$, $n = 0,1,...,N$; $x_0(\varphi_0) = \int\limits_0^{\varphi_0} F(\xi)d\xi = +\infty$, $x_N(\varphi_0) = 0$, $x_n(\varphi_0) = x_{n+1}(\varphi_0) + \int\limits_{\varphi_n}^{\varphi_{n+1}} F(\xi)d\xi$, $n = N-1, N-2,..,1$.

Thus, for the calculation of $x_n$, we have the recurrence process

$$x_n(\varphi_0) = I_n(\varphi_0,\tilde{U},T_i) + x_{n+1}(\varphi_0),$$

$$n = N-1, N-2, ..., 1; \quad x_N(\varphi_0) = 0, \tag{47}$$

where $I_n(\varphi_0,\tilde{U},T_i) = \int\limits_{\varphi_n}^{\varphi_{n+1}} F(\xi;\varphi_0,\tilde{U},T_i)d\xi$, $n = 1,2,...,N-1$; $I_1 = \int\limits_{\varphi_1}^{\varphi_2} F(\xi)d\xi$, $I_{N-1} = \int\limits_{\varphi_{N-1}}^{\varphi_0} F(\xi)d\xi$; $I_n = \int\limits_{\varphi_n}^{\varphi_{n+1}} F(\xi)\,d\xi \approx \frac{\varphi_{n+1} - \varphi_n}{2} \sum_{k=1}^M A_k F\left(\xi_k^{(n)}\right) d\xi$,

$$\xi_k^{(n)} = \frac{\varphi_{n+1}-\varphi_n}{2}\eta_k + \frac{\varphi_{n+1}+\varphi_n}{2} = \frac{h_\varphi}{2}\eta_k + \varphi_n + \frac{h_\varphi}{2},$$

if $\varphi_{n+1} - \varphi_n = h_\varphi$, $\forall n = \overline{1,N-1}$; $\eta_k \in [-1;1]$, $k = \overline{1,M}$; $F\left(\xi_k^{(n)}\right) = 1/\sqrt{2\int\limits_0^{\xi_k^{(n)}} f\left(\zeta;\varphi_0,\tilde{U},T_i\right)d\zeta}$, $\int\limits_0^{\xi_k^{(n)}} f(\zeta)d\zeta \approx \frac{\xi_k^{(n)}}{2}\sum_{i=1}^M A_k f(\zeta_i)$, where $\zeta_i = \frac{\xi_k^{(n)}}{2}\eta_i + \frac{\xi_k^{(n)}}{2}$, $\eta_i \in [-1;1]$; $k = \overline{1,M}$; $n = \overline{1,N-1}$.

While solving problem (24)–(27) by the difference method, we put the given problem in correspondence with a nonlinear difference boundary-value problem which approximates the differential problem to within the order of the square of a step of discretization and is a system of nonlinear equations, whose solution is determined with the help of the Newton method.

On the solution of (24)–(27) by the method of its reduction to a Cauchy problem, we put the given boundary-value problem in correspondence with the Cauchy problem with the initial conditions

$$\varphi(0) = \varphi_0,\ \varphi'(0) = \varphi_0' = -\sqrt{2\int\limits_0^{\varphi_0} f\left(\xi;\varphi_0,\tilde{U},T_i\right)d\xi}\ . \tag{48}$$

We are interested in the solutions which are positive, monotonically decreasing, and tend to zero as $x \to +\infty$. Therefore, we take the "–" sign. An approximate solution of the posed Cauchy problem can be found also by one of the well-known methods, for example, by the Runge–Kutta method, by reducing it to an implicit difference scheme with the following use of one of the iteration methods, *etc.* [6–9].

Let $\tilde{U} < \tilde{U}_{\rm cr}$. Then the equation $f\left(\varphi;\varphi_0,\tilde{U},T_i\right) = 0$, $\varphi \in \left[0;\varphi_0(\tilde{U}\right]$, has the root $\varphi = \varphi_c(\tilde{U}) \neq 0$ in addition to the root $\varphi = 0$. That is, $f\left[\varphi_c(\tilde{U});\varphi_0(\tilde{U}),\tilde{U},T_i\right] = 0$. We can show that, in this case, $\varphi(x) \to \varphi_c$ as $x \to +\infty$, which does not correspond to the conditions of the boundary-value problem. If $\tilde{U} \to \tilde{U}_{\rm cr}$, then $\varphi_c \to 0$.

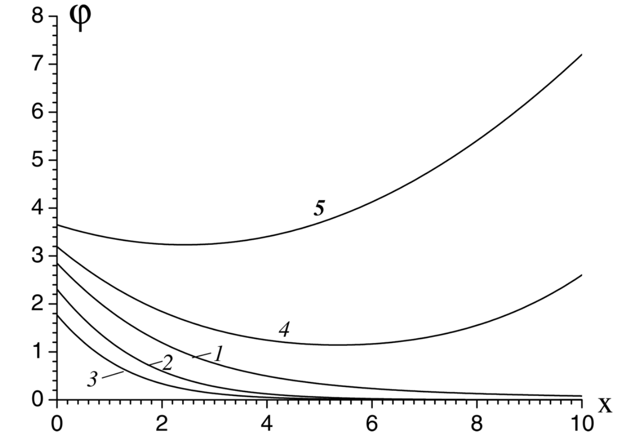


Fig. 7. Plots of the potentials $\varphi$ in the case where $T_i = 0$ and for various $\tilde{U}$. *1* – $\tilde{U} = \tilde{U}_{\rm cr} \approx 0.7037$; *2* – $\tilde{U} \approx 1.2188$; *3* – 2.1111; *4* – 0.4976; *5* – 0.3147

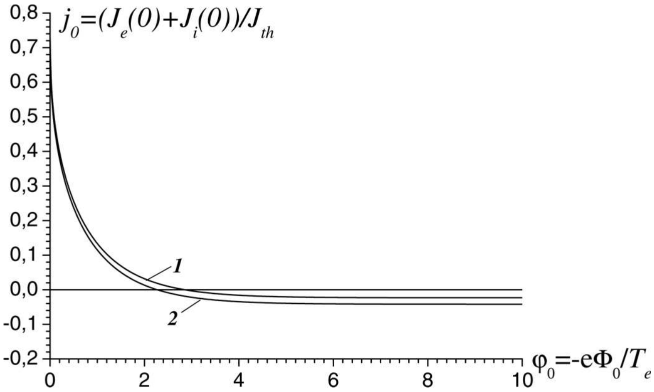


Fig. 9. Dependence of the total current ($j_0$) passing across the electrode surface on the field on the electrode surface ($\varphi_0$) for various values of the parameter $T_i$ and for $\tilde{U} = 0.70369 \approx \tilde{U}_{\rm cr}(0)$, *1* – 0; *2* – 1

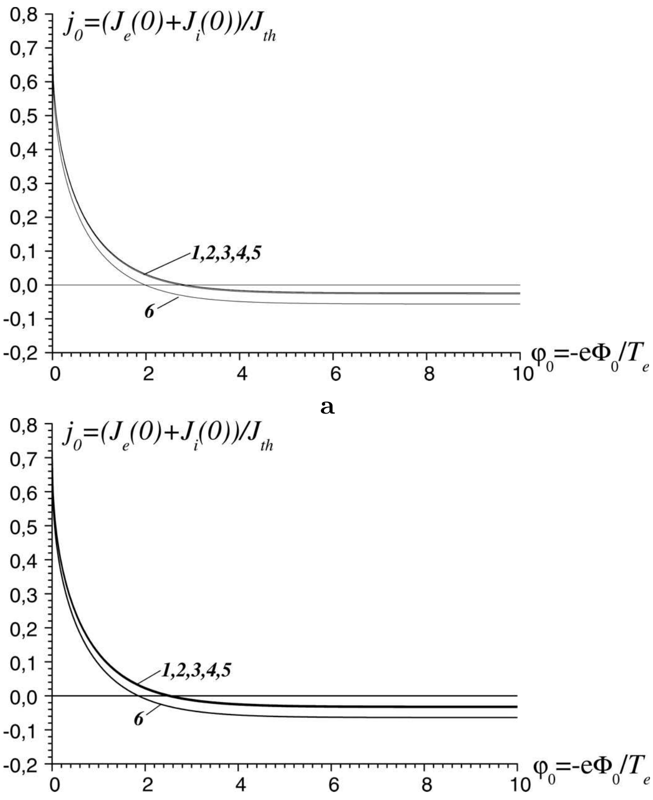


Fig. 8. Dependence of the total current ($j_0 = (J_e(0) + J_i(0))/J_{th}$) passing across the electrode surface on the field potential on the electrode surface ($\varphi_0 = -e\Phi/T_e$) for various values of the parameters $T_i$ and $\tilde{U} = \tilde{U}_{\rm cr} + \delta$. (*a*): $T_i = 0$; (*b*): $T_i = 1$, *1* – 0; *2* – $10^{-4}$; *3* – $10^{-3}$; *4* – $10^{-2}$; *5* – $10^{-1}$; *6* – 1

## 6. Numerical Analysis and Conclusions

On the execution of numerical calculations, we used solutions of problem (25)–(27) which were obtained by different methods. In this case, we have considered the cases of both a floating potential of the electrode [condition (24)] and a fixed potential. In the latter case, $\varphi_0$ was assumed to be given. The typical dependences of the potential for various values of the temperature of ions and the drift velocity are presented in Figs. 6, 7. Figures 6,*a,b* correspond to the floating potential of an electrode, and Fig. 7 does to a fixed one. The dependences of the current on the electrode potential are given in Figs. 6–9. The numerical results obtained on the basis of various calculation schemes which realize the obtained solutions coincide with a sufficiently high accuracy. This enhances the reliability of the obtained numerical results. Their analysis shows:

1. For every value of the temperature of ions $T_i$, there exists the own critical velocity of a drift of ions $\tilde{U}_{\rm cr}$ which is a lower bound for the drift velocity, beginning from which problem (25)–(27) has nonzero solutions satisfying the condition of monotonous tending to zero with increase in the distance from the electrode plane. The dependence of $\tilde{U}_{\rm cr}$ on the temperature of ions $T_i$ is shown in Fig. 4. If $\tilde{U} < \tilde{U}_{\rm cr}$, then, under condition (34) (a floating potential of the electrode) as $x \to +\infty$, the potential has a finite value or can be nonmonotonous in the case of a fixed potential of the electrode (Fig. 7).

2. A numerical value of the critical velocity of a drift of ions decreases with increase in the temperature of ions $T_i$. It reaches the greatest value at $T_i = 0$ (Fig. 4).

3. Near the critical velocity of a drift, the field potentials – solutions of problem (24)–(27) – coincide for different $T_i$ with a sufficiently high accuracy, i.e. they depend weakly on the temperature of ions. On the other hand, the field potentials for different $T_i$ and for the same drift velocity can significantly differ from one another.

4. Near the critical velocity of a drift, the total current ($j_0 = j(0) = j_e(0) + j_i(0)$), which passes across the electrode surface, depends on the field potential on the electrode surface $\varphi_0$, on the drift velocity of ions, and weakly depends on the temperature of ions $T_i$ (Fig. 8($a$,$b$)).


1. K.-U. Rieman, J. Phys. D: Appl. Phys. **24**, 493 (1991).
2. N. Hershkowitz, Phys. Plasmas **12**, 05502 (2005).
3. V.A. Godyak and N. Sternberg, Phys. Plasmas **9**, 4427 (2002).
4. D. Bohm, in *The Characteristics of Electrical Discharges in Magnetic Field*, edited by A. Guthrie and R.K. Wakerling (McGraw-Hill, New York, 1949), Chap. 3.
5. F.F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Plenum Press, New York, 1974), Chap. 8.
6. N.N. Kalitkin *Numerical Methods* (Nauka, Moscow, 1978) (in Russian).
7. I.I. Lyashko, V.L. Makarov, and A.A. Skorobagat'ko, *Methods of Calculations* (Vyshcha Shkola, Kyiv, 1977) (in Russian).
8. A.A. Samarsky, *Introduction to Numerical Methods* (Nauka, Moscow, 1987) (in Russian).
9. A.G. Zagorodny and I.V. Rogal', *The Distribution of Charged Particles in a Semibounded Dusty Plasma*, Preprint ITP-98-12U (Bogolyubov Institute for Theoretical Physics of the NASU, Kyiv, 1998) (in Ukrainian).
10. I.V. Rogal' and I.I. Yurtyn, in: *Papers of the Institute of Electrodynamics of the NASU, No. 1(4)* (Institute of Electrodynamics of the NASU, Kyiv, 2003), pp.11–15 (in Ukrainian).


Received 04.07.07.
Translated from Ukrainian by V.V. Kukhtin

## ПРО ВПЛИВ ТЕПЛОВОГО РУХУ ІОНІВ НА КРИТЕРІЙ БОМА

*А.Г. Загородній, І.В. Рогаль*

Р е з ю м е

Критерію Бома належить одне з основоположних місць при розгляді проблеми формування приелектродного шару плазми. Цей критерій сформульовано у припущенні, що іони плазми мають нульову температуру, а розподіл електронів є больцмановим. У цій роботі досліджено вплив теплового руху іонів на умову екранування плоского електрода, що перебуває як під плаваючим, так і під заданим потенціалом. Показано, що тепловий рух іонів приводить до зменшення критичногшо значення швидкості іонів.