THE ZAKHAROV EQUATIONS WITH ZERO HARMONIC AND MODULATION INSTABILITY

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It is shown that the new type of modulation instability of waves on a surface of the ideal fluid, which has been predicted recently by the author on the basis of a system of two equations of motion for the amplitude of an enveloping first harmonic and the non-oscillating component of a wave (the zero harmonic) within the method of multiple scales and the Euler equations of motion, can be reproduced with the help of the Zakharov equations for the Fourier amplitudes of the first and zero harmonics in the frame of the Hamiltonian formalism.

1. Introduction

Stokes [1] weakly nonlinear periodic solutions to the nonlinear equations describing the wave motion in conservative media are unstable to small harmonic long-wave perturbations. This instability was originally discovered for the waves on the surface of an ideal fluid in works [2–7]. Now it is known as the Benjamin–Feir modulation instability (BF MI). It was also found in many other nonlinear media and is a general physical phenomenon. As a result of works [3, 4], it became clear that the analogy between the behaviour of waves of small amplitudes in various media can be explained by a likeness of expansions of the Hamiltonian for waves of the different nature in a power series in a small nonlinearity and the further reformulation of the Euler equations of motion for various waves in the formally identical Hamilton equations. The Hamiltonian theory of waves on the surface of a fluid and in plasma became only the first examples the general program [3, 4] on the expansion of a Hamiltonian formalism of the nonlinear mechanics of particles onto the wave motion in a continuous medium: the searching for the pairs of canonical variables, the construction of a Hamiltonian of waves in the physical and Fourier spaces, the determination of the first nonlinear terms of its expansion in a series, and the following derivation of the simplified equations of motion for amplitudes of the lowest harmonics as the Hamilton equations obtained from the Hamiltonian expanded in a series with the truncated upper harmonics (Zakharov equations). In particular, the MI can be investigated with the use of the equations obtained in [3,4] and the coefficients calculated in the case of a fluid of infinite depth. Moreover, a more general type of MI was found on the basis of the interaction of N waves [8], e.g., type II MI [9] with five interacting waves.

The theory [3, 4] was also applied to the case of a fluid of finite depth [10]. Here, except for the strong complication of calculations due to the dependence of coefficients of the Hamiltonian on depth h, there is also the basic difference consisting in the appearance of a non-oscillating component (the zero harmonic which varies, by the terminology of the method of multiple scales, in slower time) among Fourier harmonics. Such component is equal to zero in the case of infinite depth in the considered order of precision. Upper harmonics are removed from the Hamiltonian and equations of motion by means of the reduction of the Hamiltonian. But to make the same with the zero harmonics is possible only at additional assumptions about the character of its time dependence. The elimination of the equation for the zero harmonic can lead to a decrease of the order of the dispersion equation and, thus, to losses of a part of its solutions. The necessity of an accurate treatment of the equation for the zero harmonic was also discussed outside of the Hamiltonian approach [11–14]. In [10], the reduction of a Hamiltonian was not executed and the equation for a zero harmonic was maintained, which would allow one to consider a wide spectrum of problems. However, the analytic evaluations involved only the zone of wave vectors of perturbations $\varkappa$ small in comparison with wave vectors of the first harmonic k_0. We may assume that, on the influence on the first harmonic of a perturbation with the wave vector $\varkappa \sim k_0$, the 0-harmonic with a wave vector of 0 will respond as a result of the nonlinear resonant interaction [8, 15] if it is possible in the system and the law of conservation of energy is realized.

Recently in works [16, 17] concerning the same problem as in [10] but on the basis of the system obtained [18, 19] from the Euler equations of motion and at the refusal from additional artificial assumptions about a character of the dependence of the zero harmonic on time, it is discovered that, at $\varkappa \simeq k_0$, there is really a band of MI. There arises a question whether this can be obtained in the Hamiltonian approach [10]. Work [10] is written is very shortly. We have checked up and reproduce all the results of work [10] in more direct way. At the same time, some little inaccuracies have been specified. Their elimination allows us to describe the type of MI indicated in [17] within the Hamiltonian method as well.

2. The Hamiltonian, Its Formal Expansion in an Integro-Power Series and Equations of Motion in the Fourier Representation

In the Hamilton formalism for potential nonlinear waves, the equation of motion for the "complex normal coordinate" $a(k,t)$ can be written in the form of the Hamilton equation

$$\frac{\partial a(\overrightarrow{k},t)}{\partial t} = -i\frac{\delta H}{\delta \overline{a}(\overrightarrow{k},t)} \tag{1}$$

and a complex conjugate equation. Here, $\overrightarrow{k}$ is a horizontal wave vector, and H the Hamiltonian of waves as a functional of $a(\overrightarrow{k})$ and $\overline{a}(\overrightarrow{k})$.

For waves on a surface of an ideal fluid, the profile of a wave (an increase and a decrease of the surface) $\eta(\overrightarrow{x},t)$ is related to $a(\overrightarrow{k},t)$ by the formula

$$\eta(\overrightarrow{x},t) = \frac{1}{2\pi}\int \sqrt{\frac{\omega(\overrightarrow{k})}{2g}}(a(\overrightarrow{k},t)+\overline{a}(-\overrightarrow{k},t))e^{i\overrightarrow{k}\,\overrightarrow{x}}d\overrightarrow{k},$$

$$\omega(\overrightarrow{k}) = \sqrt{g|\overrightarrow{k}|\tanh(|\overrightarrow{k}|h)},$$

where g is the gravitational acceleration, h the depth of a fluid, and $\overrightarrow{x} = (x,y)$ a vector of horizontal coordinates.

In the variables $a(k), \overline{a}(k)$, the Hamiltonian is expanded in a series in degrees of $a(k)$ and $\overline{a}(k)$ [10]:

$$H = \int\limits_{-\infty}^{\infty} \omega(k)\, a(k)\, \overline{a}(k)\, dk+$$

$$+\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} V(k,\, k_1,\, k_2)\ (\ \overline{a}(k)\, a(k_1)\, a(k_2)+$$

$$+a(k)\,\overline{a}(k_1)\,\overline{a}(k_2)\ )\,\delta(k-k_1-k_2)\,dk\,dk_1\,dk_2+$$

$$+\frac{1}{3}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} U(k,\, k_1,\, k_2)\ (\ a(k)\, a(k_1)\, a(k_2)+$$

$$+\overline{a}(k)\,\overline{a}(k_1)\,\overline{a}(k_2)\ )\ \delta(k+k_1+k_2)\,dk\,dk_1\,dk_2+$$

$$+\frac{1}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} W(k,k_1,k_2,k_3)\overline{a}(k)\overline{a}(k_1)a(k_2)a(k_3)\times$$

$$\times\delta(k+k_1-k_2-k_3)\,dk\,dk_1\,dk_2\,dk_3. \tag{2}$$

In the cases of infinite depth and finite arbitrary one, the expansion coefficients $V(k,\, k_1,\, k_2)$, $U(k,\, k_1,\, k_2)$, $W(k,\, k_1,\, k_2,\, k_3)$ were determined in [3, 4] and [10], respectively. We mention a number of works, for example [20–29], devoted to both these coefficients and the development of the approach. We will present the relevant expressions following to the notations in [22] for the further calculations and the establishment of a correspondence with the nonlinear coefficients obtained in [17] by the method of multiple scales:

$$V(k,\, k_1,\, k_2) = -V_0(-k,\, k_1,\, k_2)-$$

$$-V_0(-k,\, k_2,\, k_1) + V_0(k_1, k_2, -k\,),$$

$$U(k,\, k_1,\, k_2) = V_0(k,\, k_1,\, k_2)+$$

$$+V_0(k,\, k_2,\, k_1) + V_0(k_1, k_2, k\,),$$

$$V_0(k,\, k_1,\, k_2) = -N_0 N_1 M_2 E^{(3)}_{0,1}, \tag{3}$$

$$E^{(3)}_{0,1} = -\frac{1}{4\pi}\left((\overrightarrow{k}\cdot\overrightarrow{k}_1)+q_0 q_1\right),$$

$$W(k,\, k_1,\, k_2,\, k_3) =$$

$= W_0(-k, -k_1, k_2, k_3) + W_0(k_2, k_3, -k, -k_1)-$

$-W_0(-k, k_2, -k_1, k_3) - W_0(-k_1, k_2, -k, k_3)-$

$-W_0(-k, k_3, -k_1, k_2) - W_0(-k_1, k_3, -k, k_2),$

$W_0(k, k_1, k_2, k_3) = -2N_0 N_1 M_2 M_3 E^{(4)}_{0,1,2,3},$

$E^{(4)}_{0,1,2,3} = -\frac{1}{8(2\pi)^2}(2|\overrightarrow{k}|^2 q_1 + 2|\overrightarrow{k}_1|^2 q_0-$

$-q_0 q_1(q_{0+2} + q_{1+2} + q_{0+3} + q_{1+2})),$

$N(k) = \left(\frac{\omega(k)}{2q(k)}\right)^{1/2}, \quad M(k) = \left(\frac{q(k)}{2\omega(k)}\right)^{1/2},$

$q(k) = |\overrightarrow{k}| \tanh(|\overrightarrow{k}|h).$

Varying Hamiltonian (2), we obtain the equation of motion for $a(k, t)$ as the Hamilton equation unified in the approximation ε^3 for the standard Hamiltonian (2) as

$$\frac{\partial}{\partial t} a(k, t) + i[\omega(k)\, a(k)+$$

$$+\int_{-\infty}^{\infty} V(k, k-\xi, \xi) a(\xi)\, a(k-\xi)\, d\xi+$$

$$+2\int_{-\infty}^{\infty} V(k+\xi, k, \xi)\overline{a}(\xi)\; a(k+\xi)\, d\xi+$$

$$+\int_{-\infty}^{\infty} U(-k-\xi, k, \xi)\, \overline{a}(\xi)\, \overline{a}(-k-\xi)\, d\xi+$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\xi+\zeta-k, k, \xi, \zeta)\, \times$$

$$\times a(\xi)\, a(\zeta)\, \overline{a}(\xi+\zeta-k)\, d\xi d\zeta] = 0. \quad (4)$$

3. Equation of Motion for the System of Fourier Amplitudes of the First and Zero Harmonics

Let the wave field represent a pulse of oscillating waves with the central wave vector k_0. Then the Fourier amplitude of the first harmonic and its conjugate quantity are concentrated near the wave vector k_0,

$$a_1 = a_1(k, t)\, \delta(k - k_0). \quad (5)$$

Nonlinear terms of the equations of motion generate the non-oscillating component of a field (the zero harmonic) and the second and higher harmonics. Their account will be conducted by the expansion in the formal small parameter ε:

$$a = \varepsilon a_1 + \varepsilon^2(b + a_2). \quad (6)$$

The zero harmonic and its conjugate are concentrated near the wave vector $k = 0$,

$$b \;\to b(k, t)\, \delta(k), \qquad \overline{b} \to \overline{b}(k, t)\, \delta(k), \quad (7)$$

and the second harmonic

$$a_2 = a_{21} + a_{22} \quad (8)$$

consists of both a component of the wave field

$$a_{21} = a_{21}(k, t)\, \delta(k - 2\, k_0) \quad (9)$$

concentrated near the wave vector $2k_0$ and that

$$a_{22} = a_{22}(k, t)\, \delta(k + 2\, k_0), \quad (10)$$

concentrated near $-2k_0$.

With the purpose to construct the approximate equations of motion for the Fourier amplitudes of the lowest harmonics a_1 and b, we substitute (6) in (4) and we collect terms with the same degrees of ε.

3.1. The first order in ε

In the first order in ε, we obtain the equations of motion for the first harmonic in the linear approximation as

$$\frac{\partial}{\partial t} a_1(k) + i\, \omega(k)\, a_1(k) = 0. \quad (11)$$

3.2. The second order in ε

In the order of ε^2, the equations of motion for the zero and second harmonics look as

$$\frac{\partial}{\partial t} a_2(k) + i\,\omega(k)\,a_2(k) + \frac{\partial}{\partial t} b(k) + i\,\omega(k)\,b(k) +$$

$$+ i \int\limits_{-\infty}^{\infty} a_1(\xi)\,V(k,\,k-\xi,\,\xi)\,a_1(k-\xi)\,d\xi +$$

$$+ 2\,i \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,V(k+\xi,\,k,\,\xi)\,a_1(k+\xi)\,d\xi +$$

$$+ i \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,U(k,\,-k-\xi,\,\xi)\,\overline{a}_1(-k-\xi)\,d\xi = 0. \qquad (12)$$

In the first nonlinear term in (12), we consider that it includes the first harmonics $a_1(\xi)$ and $a_1(k-\xi)$ concentrated on the wave vector k_0 (5). Therefore, $\xi = k_0$ and $k - \xi = k_0$. Hence, the integration variable ξ is concentrated in a neighbourhood of k_0, and the wave vector k, for which this nonlinear term is different from zero, is $2\,k_0$. Thus, the first nonlinear term should be grouped together with the linear term $\frac{\partial}{\partial t} a_{21}(k) + i\omega(k)\,a_{21}(k)$, which is also concentrated on the wave vector $2\,k_0$. Thus, we obtain the evolutionary equation for the first component $a_{21}(k,\,t)$ of the Fourier amplitude of the second harmonic

$$2k_0: \quad \frac{\partial}{\partial t} a_{21}(k) + i\,\omega(k)\,a_{21}(k) +$$

$$+ i\,V(2\,k_0,\,k_0,\,k_0) \int\limits_{-\infty}^{\infty} a_1(\xi)\,a_1(k-\xi)\,d\xi = 0. \qquad (13)$$

Similarly, we obtain the evolutionary equation for $a_{22}(k,\,t)$ and $b(k,\,t)$:

$$-2k_0: \quad \frac{\partial}{\partial t} a_{22}(k) + i\,\omega(k)\,a_{22}(k) +$$

$$+ i\,U(-2\,k_0,\,k_0,\,k_0) \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,\overline{a}_1(-k-\xi)\,d\xi = 0, \qquad (14)$$

$$0: \quad \frac{\partial}{\partial t} b(k) + i\,\omega(k)\,b(k) +$$

$$+ 2\,i\,V(k_0,\,k_0,\,k) \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,a_1(k+\xi)\,d\xi = 0. \qquad (15)$$

Equations (13), (14) allow one to express the second harmonic through the first one in order to remove the second harmonic from all formulas in the approximation ε^3. For $a_{21}(k,\,t)$, in view of the time dependence $a_{21}(k,\,t) \sim e^{-2\,i\,\omega(k_0)\,t}$, Eq. (13) yields

$$a_{21}(k) = -\frac{V(2\,k_0,\,k_0,\,k_0)}{\omega(2\,k_0) - 2\,\omega(k_0)} \int\limits_{-\infty}^{\infty} a_1(\xi)\,a_1(k-\xi)\,d\xi. \qquad (16)$$

Taking the time dependence $a_{22}(k,\,t) \sim e^{2\,i\,\omega(k_0)\,t}$ into account, it follows from Eq. (14) that

$$a_{22}(k) = -\frac{U(-2\,k_0,\,k_0,\,k_0)}{\omega(-2\,k_0) + 2\,\omega(k_0)} \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,\overline{a}_1(-k-\xi)\,d\xi. \qquad (17)$$

We will not integrate Eq. (15) to avoid the additional assumptions about a character of the dependence $b(k)$ on time. Below, we will use (15) as the equation of motion for the 0-harmonic $b(k)$ and include it in the system with the equation for the first harmonic $a_1(k)$ which will be deduced in what follows. Let's remark that, in (15), $V(k_0,\,k,\,k_0)$ is changed by $V(k_0,\,k_0,\,k)$ taking into account a symmetry to permutations of the second and third arguments $V(k,\,k_1,\,k_2)$ in (3).

3.3. The third order in ε

Here, we obtain the equation of motion for the first harmonic $a_1(k)$ in the ε^3 approximation. Nonlinear terms look as

$$i \int\limits_{-\infty}^{\infty} a_2(\xi)\,V(k,\,k-\xi,\,\xi)\,a_1(k-\xi)\,d\xi +$$

$$+ 2\,i \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,V(k+\xi,\,k,\,\xi)\,a_2(k+\xi)\,d\xi +$$

$$+ i \int\limits_{-\infty}^{\infty} a_1(\xi)\,V(k,\,k-\xi,\,\xi)\,a_2(k-\xi)\,d\xi + \qquad (18)$$

$$+ i \int\limits_{-\infty}^{\infty} \overline{a}_1(\xi)\,U(k,\,-k-\xi,\,\xi)\,\overline{a}_2(-k-\xi)\,d\xi +$$

$$+ 2\,i \int\limits_{-\infty}^{\infty} \overline{a}_2(\xi)\,V(k+\xi,\,k,\,\xi)\,a_1(k+\xi)\,d\xi +$$

$$+i\int_{-\infty}^{\infty}\overline{a}_2(\xi)\,U(k,\,-k-\xi,\,\xi)\,\overline{a}_1(-k-\xi)\,d\xi+ \quad (19)$$

$$+i\int_{-\infty}^{\infty}a_1(\xi)\,V(k,\,k-\xi,\,\xi)\,b(k-\xi)\,d\xi+$$

$$+i\int_{-\infty}^{\infty}b(\xi)\,V(k,\,k-\xi,\,\xi)\,a_1(k-\xi)\,d\xi+$$

$$+2\,i\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,V(k+\xi,\,k,\,\xi)\,b(k+\xi)\,d\xi+$$

$$+2\,i\int_{-\infty}^{\infty}\overline{b}(\xi)\,V(k+\xi,\,k,\,\xi)\,a_1(k+\xi)\,d\xi+$$

$$+i\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,U(k,\,-k-\xi,\,\xi)\,\overline{b}(-k-\xi)\,d\xi+$$

$$+i\int_{-\infty}^{\infty}\overline{b}(\xi)\,U(k,-k-\,\xi,\,\xi)\,\overline{a}_1(-k-\xi)\,d\xi+$$

$$+i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(k,\,-k+\xi+\zeta,\,\xi,\,\zeta)\times$$

$$\times a_1(\zeta)\,a_1(\xi)\overline{a}_1(-k+\xi+\zeta)\,d\xi\,d\zeta. \quad (20)$$

We divide them into 5 groups.

1) In first three terms (18) which contain the second harmonic a_2 we consider that it consists of the component of a wave field a_{21} (9) concentrated in a neighbourhood of the wave vector $2k_0$ and the component of a wave field a_{22} (10), concentrated in the region of $-2k_0$, the amplitude of the first harmonic and its conjugate being concentrated on the wave vector k_0 (5). Arguing as in the derivation of (13), we can conclude that the kernels can be taken out of the integrals:

$$i\,V(3\,k_0,\,2\,k_0,\,k_0)\int_{-\infty}^{\infty}a_1(\xi)\,a_{21}(k-\xi)\,d\xi+$$

$$+2\,i\,V(2\,k_0,\,k_0,\,k_0)\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,a_{21}(k+\xi)\,d\xi+$$

$$+i\,V(3\,k_0,\,k_0,\,2\,k_0)\int_{-\infty}^{\infty}a_1(\xi)\,a_{21}(k-\xi)\,d\xi+$$

$$+i\,V(-k_0,\,-2\,k_0,\,k_0)\int_{-\infty}^{\infty}a_1(\xi)\,a_{22}(k-\xi)\,d\xi+$$

$$+2\,i\,V(-2\,k_0,\,-3\,k_0,\,k_0)\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,a_{22}(k+\xi)\,d\xi+$$

$$+i\,V(-k_0,\,k_0,\,-2\,k_0)\int_{-\infty}^{\infty}a_1(\xi)\,a_{22}(k-\xi)\,d\xi. \quad (21)$$

Moreover, these terms are concentrated at k equal to $3k_0$, k_0, $3k_0$, $-k_0$, $-3k_0$, and $-k_0$, respectively. Further, we retain only the second term (21) as essential, because it is concentrated on the wave vector k_0 of the first harmonic, for which we will construct an evolutionary equation of motion.

2) Analogously, in the following three terms (19) which contain the conjugate of the second harmonic $\overline{a}_2$, we take into account that it consists of the component of a wave field $\overline{a}_{21}$ concentrated in the region of the wave vector $2k_0$ ($\overline{a}_{21}=\overline{a}_{21}(k,\,t)\,\delta(k-2k_0)$) and the component $\overline{a}_{22}$ concentrated in a vicinity of $-2k_0$ ($\overline{a}_{22}=\overline{a}_{22}(k,\,t)\,\delta(k+2k_0)$). This allows us again to take out the kernels of the integrals:

$$i\,U(-3\,k_0,\,2\,k_0,\,k_0)\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,\overline{a}_{21}(-k-\xi)\,d\xi+$$

$$+2\,i\,V(k_0,\,-k_0,\,2\,k_0)\int_{-\infty}^{\infty}\overline{a}_{21}(-k-\xi)\,a_1(-\xi)\,d\xi+$$

$$+i\,U(-3\,k_0,\,k_0,\,2\,k_0)\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,\overline{a}_{21}(-k-\xi)\,d\xi+$$

$$+i\,U(k_0,\,-2\,k_0,\,k_0)\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,\overline{a}_{22}(-k-\xi)\,d\xi+$$

$$+2\,i\,V(k_0,\,3\,k_0,\,-2\,k_0)\int_{-\infty}^{\infty}\overline{a}_{22}(-k-\xi)\,a_1(-\xi)\,d\xi+$$

$$+i\,U(k_0,\,k_0,\,-2\,k_0)\int_{-\infty}^{\infty}\overline{a}_1(\xi)\,\overline{a}_{22}(-k-\xi)\,d\xi. \quad (22)$$

These terms are concentrated at k equal to $-3k_0$, $-k_0$, $-3k_0$, k_0, $3k_0$, and k_0, respectively. In (22), the fourth and sixth terms are essential as they are concentrated on the wave vector k_0 of the first harmonic.

3) In the following three terms containing the zero harmonic b, we consider that it is concentrated in a neighbourhood of the wave vector $k = 0$ (7), the amplitude of the first harmonic and its conjugate being concentrated on the wave vector k_0 (5). This allows us to partially fix the arguments of kernels:

$$i \int_{-\infty}^{\infty} a_1(\xi)\, V(k_0,\, k_0-\xi,\, k_0)\, b(k-\xi)\, d\xi +$$

$$+i \int_{-\infty}^{\infty} a_1(\xi)\; V(k_0,\, k_0,\, k_0-\xi) b(k-\xi)\, d\xi +$$

$$+2\,i \int_{-\infty}^{\infty} \overline{a}_1(\xi)\, V(k_0+\xi,\, k_0, -k_0)\, b(k+\xi)\, d\xi. \qquad (23)$$

Here, the first and second terms are concentrated at $k = k_0$, whereas the third one is concentrated at $k = -k_0$. So we keep only the first and second terms.

4) Analogously, in the following three nonlinear terms which contain the conjugate of the zero harmonic $\overline{b}$, we consider that it is concentrated on the wave vector $k = 0$ (5). This allows us to partially fix the arguments of kernels:

$$2\,i \int_{-\infty}^{\infty} \overline{b}(\xi-k)\, V(k_0,\, k_0,\, \xi-k_0)\, a_1(\xi)\, d\xi +$$

$$+i \int_{-\infty}^{\infty} \overline{b}(-k-\xi)\, U(k_0,\, -k_0-\xi,\, k_0) \overline{a}_1(\xi)\; d\xi +$$

$$+i \int_{-\infty}^{\infty} \overline{b}(-k-\xi)\, U(k_0,\, k_0,\, -k_0-\xi)\, \overline{a}_1(\xi)\, d\xi. \qquad (24)$$

Now the first term is concentrated at $k = k_0$, and second and third ones are concentrated at $k = -k_0$. Further, only the first term is kept as the main one.

5) As for the last nonlinear term (20), we consider that the amplitude of the first harmonic and its conjugate are concentrated on the wave vector k_0 (5). This allows us again to take out the kernels of the integrals:

$$i\, W(k_0,\, k_0,\, k_0,\, k_0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1(\xi) a_1(\zeta) \times$$

$$\times\, \overline{a}_1(-k+\xi+\zeta)\, d\xi\, d\zeta. \qquad (25)$$

We now construct the equation of motion for the first harmonic from the linear terms in the ε approximation (11) and from the above-mentioned basic nonlinear terms in the ε^3 approximation from (21)–(24):

$$2\,i\, V(2\,k_0,\, k_0,\, k_0) \int_{-\infty}^{\infty} \overline{a}_1(\xi)\, a_{21}(k+\xi)\, d\xi, \qquad (26)$$

$$2\,i\, U(-2\,k_0,\, k_0,\, k_0) \int_{-\infty}^{\infty} \overline{a}_1(\xi)\, \overline{a}_{22}(-k-\xi)\, d\xi, \qquad (27)$$

$$2\,i \int_{-\infty}^{\infty} V(k_0,\, k_0,\, k_0-\xi)\, b(k-\xi) a_1(\xi)\; d\xi,$$

$$2\,i \int_{-\infty}^{\infty} V(k_0,\, k_0,\, \xi-k_0)\, \overline{b}(\xi-k)\, a_1(\xi) d\xi,$$

and terms (25). In these equations, we took the symmetry of the coefficients $V(k_1,\, k_2,\, k_3)$, $U(k_1, k_2, k_3)$ relative to permutations of the arguments into account [10], [22].

Then we introduce the expressions for the components of the second harmonic a_{21} and a_{22} [(16), (17)] given in terms of the first harmonic obtained in the ε^2 approximation into (26), (27):

$$-\frac{2\,i\, V^2(2\,k_0,\, k_0,\, k_0)}{\omega(2\,k_0) - 2\,\omega(k_0)} \times$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{a}_1(\xi)\, a_1(\zeta)\, a_1(-\zeta+\xi+k)\, d\zeta\, d\xi,$$

$$-\frac{2\,i\, U^2(-2\,k_0,\, k_0,\, k_0)}{\omega(2\,k_0) + 2\,\omega(k_0)} \times$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{a}_1(\xi) a_1(\zeta)\, a_1(-\zeta+\xi+k)\, d\zeta d\xi.$$

In such a way, we obtain the equations of motion for the first harmonic:

$$\frac{\partial}{\partial t} a_1(k) + i\omega(k)\, a_1(k) -$$

$$-2\,i \left(\frac{V^2(2\,k_0,\, k_0,\, k_0)}{\omega(2\,k_0) - 2\,\omega(k_0)} + \frac{U^2(-2\,k_0,\, k_0,\, k_0)}{\omega(2\,k_0) + 2\,\omega(k_0)} \right) \times$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{a}_1(\xi) a_1(\zeta)\, a_1(-\zeta+\xi+k)\, d\zeta d\xi +$$
$$+2\,i \int_{-\infty}^{\infty} V(k_0,\, k_0,\, k_0-\xi)\, b(k-\xi) a_1(\xi)\ d\xi +$$
$$+2\,i \int_{-\infty}^{\infty} V(k_0,\, k_0,\, \xi-k_0)\, \overline{b}(\xi-k)\, a_1(\xi) d\xi +$$
$$+i\, W(k_0,\, k_0,\, k_0,\, k_0)\times$$
$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1(\zeta)\, a_1(\xi)\, \overline{a}_1(-k+\xi+\zeta)\, d\xi\, d\zeta = 0$$

or

$$\frac{\partial}{\partial t}\, a(k) + i\omega(k)\, a(k) +$$
$$+i \int_{-\infty}^{\infty} f(k_0-\xi)\, b(k-\xi) a(\xi)\, d\xi +$$
$$+i \int_{-\infty}^{\infty} f(\xi-k_0)\ \overline{b}(\xi-k)\, a(\xi) d\xi +$$
$$+i\lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\zeta)\, a(\xi)\, \overline{a}(\zeta+\xi-k)\, d\xi\, d\zeta = 0, \tag{28}$$

where we denote

$$f(k) = 2V(k_0,\, k_0,\, k), \tag{29}$$

$$\lambda = W(k_0,\, k_0,\, k_0,\, k_0) -$$
$$-\frac{2\,V^2(2\,k_0,\, k_0,\, k_0)}{\omega(2\,k_0) - 2\,\omega(k_0)} - \frac{2\,U^2(-2\,k_0,\, k_0,\, k_0)}{\omega(2\,k_0) + 2\,\omega(k_0)}, \tag{30}$$

and a_1 is designated as a. In the same notations, we rewrite the equations for the 0-harmonic (15) as

$$\frac{\partial}{\partial t}\, b(k) + i\,\omega(k)\, b(k) +$$
$$+i\, f\,(k) \int_{-\infty}^{\infty} \overline{a}(\xi)\, a(k+\xi)\, d\xi = 0. \tag{31}$$

Equations (28) and (31) coincide with Eqs. (19) and (20) in [10], though they are obtained by the somewhat different method, as compared with that in [10], of step-by-step account of approximations. But there is one difference. Expression (29) for $f(k)$ differs by the sequence of arguments from formula (18), $f(k) = 2V(k,\, k_0,\, k_0)$, in [10]. This difference cannot be removed by using properties of the symmetry of coefficients [22]. This can be seen from formula (3) for the coefficient $V(k,\, k_1,\, k_2)$.

4. Modulation Instability

We present the solution of the system of equations of motion (28), (31) which contains the correction on the nonlinearity as

$$a(k) = \mathcal{A}_0\, e^{-i\,t\,(\omega(k_0)+\lambda_1 \mathcal{A}_0^2)} \delta(k-k_0),$$

$$b(k) = \lambda_2 \mathcal{A}_0^2 \delta(k).$$

Substituting it in (28) and (31), we get

$$\lambda_1^{(1)} = \lambda, \quad \lambda_2^{(1)} = 0,$$

$$\lambda_1^{(2)} = \lambda - 2\frac{f^2(0)}{\omega(0)}, \quad \lambda_2^{(2)} = -\frac{f(0)}{\omega(0)}. \tag{32}$$

For waves on the surface of a fluid at small $\varkappa$, $\omega(\varkappa) \sim \varkappa$ and, as seen from (40), $f(\varkappa) \sim \sqrt{\varkappa}$. The expression for $\lambda_2^{(2)}$ diverges, therefore we choose the first variant.

We introduce a perturbation

$$a(k) = e^{-i\,t\,(\omega(k_0)+\lambda_1 \mathcal{A}_0^2)} [\,\mathcal{A}_0\, \delta(k-k_0) +$$
$$+\varepsilon\, \alpha(k)\, e^{-i\Omega\, t} \delta(k-k_0-\varkappa) +$$
$$+\varepsilon\, \alpha(k)\, e^{i\,\Omega\, t}\, \delta(k-k_0+\varkappa)], \tag{33}$$

$$b(k) = \lambda_2 \mathcal{A}_0^2 \delta(k) +$$
$$+\varepsilon\, \beta(k)\, e^{-i\,\Omega\, t}\, \delta(k-\varkappa) + \varepsilon\, \beta(k)\, e^{i\,\Omega\, t}\, \delta(k+\varkappa), \tag{34}$$

where $\alpha(k)$ and $\beta(k)$ are real quantities.

Let's explore a possibility of existence of the imaginary part of the frequency Ω for some wave vectors of a perturbation wave $\varkappa$ depending on the normalized depth of a fluid $k_0 h$, which will testify to the instability of a nonperturbed wave at such wave vectors of the perturbation. After the substitution of (33) and (34) in the linearized equations of motion (28), (31), we obtain a system of homogeneous equations for $\alpha(k_0+\varkappa)$, $\alpha(k_0-\varkappa)$, $\beta(\varkappa)$, and $\beta(-\varkappa)$:

$$\left(\Omega + \omega(k_0-\varkappa) - \omega(k_0) + \lambda\mathcal{A}_0^2\right)\alpha(k_0-\varkappa)+$$

$$+\lambda\mathcal{A}_0^2\,\alpha(k_0+\varkappa) + \mathcal{A}_0\left(f(-\varkappa)\,\beta(-\varkappa) + f(\varkappa)\,\beta(\varkappa)\right) = 0,$$

$$\left(\Omega - \omega(k_0+\varkappa) + \omega(k_0) - \lambda\,\mathcal{A}_0^2\right)\alpha(k_0+\varkappa)-$$

$$-\lambda\mathcal{A}_0^2\alpha(k_0-\varkappa) - \mathcal{A}_0\left(f(-\varkappa)\,\beta(-\varkappa) + f(\varkappa)\,\beta(\varkappa)\right) = 0,$$

$$\beta(-\varkappa)\,\left(\Omega + \omega(-\varkappa)\right)+$$

$$+\mathcal{A}_0 f(-\varkappa)\,\left(\alpha(k_0-\varkappa) + \alpha(k_0+\varkappa)\right) = 0,$$

$$\beta(\varkappa)\left(\Omega - \omega(\varkappa)\right)-$$

$$-\mathcal{A}_0\, f(\varkappa)\left(\alpha(k_0-\varkappa) + \alpha(k_0+\varkappa)\right) = 0.$$

Excepting $\beta(\varkappa)$ and $\beta(-\varkappa)$, we obtain

$$\left(\Omega + \omega(k_0-\varkappa) - \omega(k_0) - \lambda(\Omega)\,\mathcal{A}_0^2\right)\alpha(k_0-\varkappa)-$$

$$-\lambda(\Omega)\mathcal{A}_0^2\,\alpha(k_0+\varkappa) = 0,$$

$$\left(\Omega - \omega(k_0+\varkappa) + \omega(k_0) + \lambda(\Omega)\,\mathcal{A}_0^2\right)\alpha(k_0+\varkappa)+$$

$$+\lambda(\Omega)\mathcal{A}_0^2\,\alpha(k_0-\varkappa) = 0,$$

where

$$\lambda(\Omega) = -\lambda + \lambda^{(0)}(\Omega), \tag{35}$$

$$\lambda^{(0)}(\Omega) = \frac{f^2(-\varkappa)}{\omega(\varkappa)+\Omega} + \frac{f^2(\varkappa)}{\omega(\varkappa)-\Omega}. \tag{36}$$

The superscript in $\lambda^{(0)}(\Omega)$ underlines that it is the contribution to the nonlinear interaction from the 0-harmonic. Equating the determinant to zero gives the required equation for the perturbation frequency Ω

$$(\Omega-\delta)^2 = \Delta^2 - 2\lambda(\Omega)\mathcal{A}_0^2\,\Delta, \tag{37}$$

where

$$\Delta = \frac{1}{2}\left(\omega(k_0+\varkappa) + \omega(k_0-\varkappa)\right) - \omega(k_0),$$

$$\delta = \frac{1}{2}\left(\omega(k_0+\varkappa) - \omega(k_0-\varkappa)\right).$$

In the extended form, relation (37) looks like

$$\left(\Omega + \omega(k_0-\varkappa) - \omega(k_0)\right)\left(\Omega - \omega(k_0+\varkappa) + \omega(k_0)\right) =$$

$$= -2\lambda(\Omega)\mathcal{A}_0^2\,\Delta \tag{38}$$

and coincides with that in [10].

The first term in (35) is calculated from (30). For waves on the surface of a fluid of finite depth, we get

$$\lambda = \frac{k_0^3}{32\pi^2}\frac{9\sigma^4 - 10\sigma^2 + 9}{\sigma^3}, \quad \sigma = \tanh k_0 h. \tag{39}$$

Let's calculate the second term in (35). To derive $f(\varkappa) = 2V(k_0, k_0, \varkappa)$ according to (29), we simplify the coefficient $V(k, k_1, k_2)$ (3). We have

$$f(\varkappa) = \frac{k_0^{3/2}\omega_0^{1/2}}{4\sqrt{2}\pi\sqrt{\sigma}}\left(2\frac{\varkappa}{k_0}\sqrt{\frac{\omega_0}{\omega(\varkappa)}} + (1-\sigma^2)\sqrt{\frac{\omega(\varkappa)}{\omega_0}}\right). \tag{40}$$

Since our purpose is to investigate all four roots of Eq. (37), we do not approximate Ω in the denominator in (36), as it was made in [10]. According to (36), we get

$$\lambda^{(0)}(\Omega) = \frac{k_0^3}{16\pi^2\sigma}\times$$

$$\times\left(\frac{\varkappa^2}{\omega^2(\varkappa)-\Omega^2}\left(2\frac{\omega_0}{k_0} + (1-\sigma^2)\frac{\Omega}{\varkappa}\right)^2 + (1-\sigma^2)^2\right). \tag{41}$$

4.1. $\varkappa \ll k_0$*. Comparison with the known results*

In this case, we can approximate Ω in the denominator in (36). The asymptotes of four roots $\Omega(\varkappa)$ of Eq. (38) at small $\varkappa$ and $\mathcal{A}_0$ read

$$\Omega_{1,2} = c_g\varkappa \mp \frac{1}{6}\frac{\partial^3\omega(k_0)}{\partial k^3}\varkappa^3,$$

$$\Omega_{3,4} = \pm\sqrt{gh}\varkappa, \quad c_g = \frac{\omega_0}{2k_0}\left(1 + \frac{1-\sigma^2}{\sigma}k_0 h\right), \tag{42}$$

where c_g is the group velocity of linear waves. They are shown (after the normalization $\widehat{\Omega} = \frac{\Omega}{\omega_0}$, $\widehat{\varkappa} = \frac{\varkappa}{k_0}$) by dotted curves *1a, 2a, 3a, 4a* on the plots of the real part

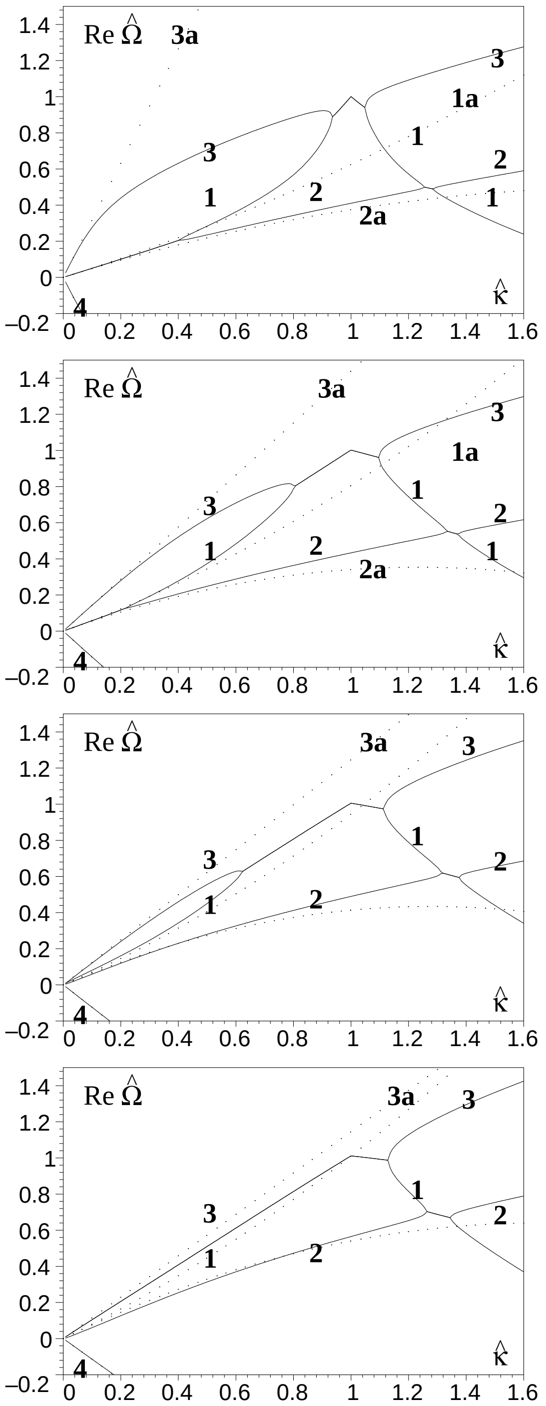

Fig. 1. Real part of the normed frequency $\widehat{\Omega}$ versus the normed wave vector $\widehat{\varkappa}$ for four roots of Eq. (37) for various depths (respectively, from the top down): $k_0h=$ 10; 2, 1.363, and 1. $k_0A_0=$ 0.2. The numbering of roots corresponds to that of their asymptotes at small $\varkappa$ (42). The asymptotes are drawn by dotted lines with a letter a near the number of a curve

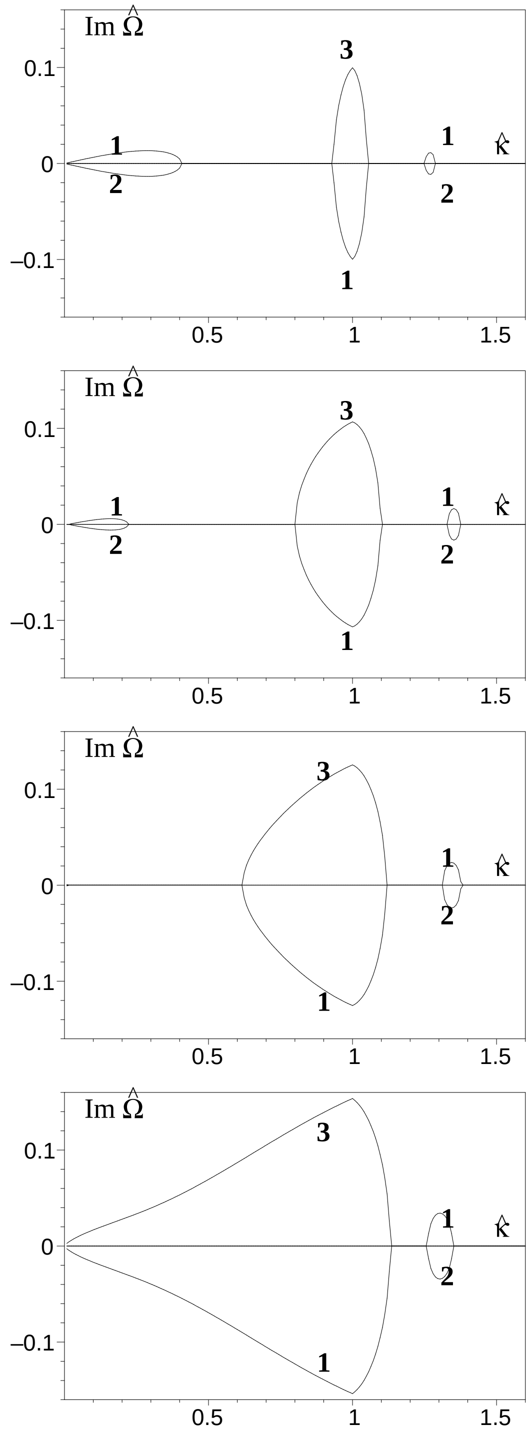

Fig. 2. The same as in Fig. 1 for the imaginary part of $\widehat{\Omega}$

$\mathrm{Re}\,\widehat{\Omega}$ in Fig. 1. Setting the purpose to determine the imaginary part of the first two roots in the next approximation in the case of $\varkappa \ll k_0$, we can use the asymptote $\Omega = \varkappa c_g$ in (37) (see also [25], [29]). We get

$$\lambda(\Omega)\,|_{\Omega=c_g\varkappa} = \frac{k_0^3}{16\pi^2\sigma}\Bigg(-\frac{9\sigma^4-10\sigma^2+9}{2\sigma^2}+ $$

$$+\frac{1}{gh-c_g^2}\left(2\frac{\omega_0}{k_0}+(1-\sigma^2)c_g\right)^2+(1-\sigma^2)^2\Bigg). \qquad (43)$$

Since $\Delta < 0$ for a convex function $\omega(k)$ according to the Jensen inequality, Eq. (37) can have complex roots, if $\lambda(\Omega) < 0$. Coefficient (43) [obtained at $f(\varkappa) = 2V(k_0, k_0, \varkappa)$] changes a sign at $k_0h = 1.363$ that coincides with the depth at which the Benjamin–Feir MI disappears. Some mismatch of (43) with formula (29) in [10] is related to the above-mentioned difference in the sequence of arguments in (29).

We now compare (38) with the corresponding equation

$$(\Omega+\omega(k_0-\varkappa)-\omega(k_0))(\Omega-\omega(k_0+\varkappa)+\omega(k_0)) =$$

$$= -2q(\Omega)A_0^2\,\Delta, \qquad (44)$$

obtained in [17] by the method of multiple scales from the Euler equations of motion. Here,

$$q(\Omega) = \widetilde{q} + q^{(0)}(\Omega), \qquad (45)$$

where

$$\widetilde{q} = \frac{\omega_0 k_0^2}{16\sigma^2}\left(-\frac{9\sigma^4-10\sigma^2+9}{\sigma^2}+2\left(\sigma^2-1\right)^2\right), \qquad (46)$$

$$q^{(0)}(\Omega) = \frac{\omega_0 k_0^2}{8\sigma^2}\frac{\varkappa^2}{\omega^2(\varkappa)-\Omega^2}\times$$

$$\times\left(2\frac{\omega_0}{k_0}+(1-\sigma^2)\frac{\Omega}{\varkappa}\right)\left(2\frac{\omega_0}{k_0}+(1-\sigma^2)c_g\right),$$

and $q^{(0)}(\Omega)$ is the contribution of the 0-harmonic to the nonlinear interaction. In the special case considered above, $\varkappa \ll k_0$, concerning two roots which correspond to the asymptote $\Omega = \varkappa c_g$, we have

$$q^{(0)}(\Omega)\,|_{\Omega=c_g\varkappa} = \frac{\omega_0 k_0^2}{8\sigma^2}\frac{1}{gh-c_g^2}\left(2\frac{\omega_0}{k_0}+(1-\sigma^2)c_g\right)^2. \qquad (47)$$

Taking into account the formula $A_0^2 = \frac{\sigma}{2\pi^2}\frac{k_0}{\omega_0}\mathcal{A}_0^2$ for the physical amplitude A_0 and the wave amplitude in the Fourier space $\mathcal{A}_0$, as well as relations (38) and (44), we should compare the coefficient $\lambda(\Omega)$ of the given paper with the coefficient $q(\Omega)$ in [17] multiplied by

$$\frac{\sigma}{2\pi^2}\frac{k_0}{\omega_0}. \qquad (48)$$

It is seen that expression (45) for $q(\Omega)$ as the sum of (46) and (47) with regard for (48) is identically equal to expression (43) for $\lambda(\Omega)$, which indicates the coincidence of results of the given work and [17] in the case of $\varkappa \ll k_0$.

4.2. $\varkappa \simeq k_0$. *New instability*

At arbitrary $\varkappa$, the numerical calculation of solutions of Eq. (37) is necessary. The equation of the fourth order obtained in [10] was not solved numerically and was reduced to a quadratic equation for a small deviation Ω from the resonance surface, $\omega(k_0+\varkappa) - \omega(k_0) = \omega(k_0) - \omega(k_0-\varkappa)$, for the analysis of the instability increment. The results of numerical tabulation of the dependence of the real and imaginary parts of $\widehat{\Omega} = \frac{\Omega}{\omega_0}$ on $\widehat{\varkappa} = \frac{\varkappa}{k_0}$ for four solutions of Eq. (37) for several values of k_0h for $k_0A_0 = 0.2$ are shown in Figs. 1 and 2. Indexing the roots corresponds to their asymptotes at small $\varkappa$ (4.). In Fig. 2, except for the known band of instability at $\varkappa \ll k_0$ (the Benjamin–Feir instability), we observe one more section of instability at $\varkappa \simeq k_0$. The third band is the right edge of the known "eight" of Phillips [8]. Unlike the BF instability which disappears at $k_0h = 1.363$, the additional band of instability exists at this and smaller depths. The essential role in the formation of this instability is played also by the first harmonic a_1 and the 0-harmonic b. Therefore the long-term evolution of the considered instability can lead to the formation of structures intermediate between solitons of the envelope of fast oscillations described by a nonlinear Schrödinger equation and solitary waves without a filling characteristic of shallow water. This type of MI was specified in [17] on the basis of a system of evolutionary equations for the zero and basic harmonics which was obtained by the method of multiple scales from the Euler equations of motion. The reproduction of this result by the Hamiltonian method indicates the validity of both approaches.

1. G.G. Stokes, Trans. Cambridge Philos. Soc. **8**, 441 (1849).
2. M.J. Lighthill, J. Inst. Math. Appl. **1**, 269 (1965).
3. V.E. Zakharov, Sov. Phys. JETP **24**, 740 (1967).
4. V.E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, N2, 190 (1968).

5. T.B. Benjamin, Proc. Roy. Soc. (London) **A299**, 59 (1967); J.E. Feir, *ibid.*, 54.
6. G.B.J. Whitham, Fluid Mech. **27**, 399 (1967).
7. H. Hasimoto and H. Ono, J. Phys. Soc. Jpn. **33**, 805 (1972).
8. O.M. Phillips, J. Fluid Mech. **9**, 193 (1960); Proc. Roy. Soc. (London) **299**, 104 (1967).
9. M. Francius and C. Kharif, Phys. Fluids **15**, 2445 (2003).
10. V.E. Zakharov and V.G. Kharitonov, J. Appl. Mech. Tech. Phys. **11**, N5, 741 (1970).
11. T. Colin, F. Dias, and J. Ghidaglia, Eur. J. Mech. Fluids **14**, 775 (1995).
12. E. Knobloch and and R.D. Pierce, Fluid Dynamics Research **22**, 61 (1998).
13. T.J. Bridges, Chaos **15**, 037113 (2005).
14. T.J. Bridges and N.M.Donaldson, J. Fluid Mech. **565**, 419 (2006).
15. V.E. Zakharov, F. Dias, and A. Pushkarev, Phys. Reports **398**, 1 (2004).
16. Yu.V. Sedletsky, Phys. Lett. A **243/4**, 293 (2005).
17. Yu.V. Sedletsky, J. Phys. A: Math. Gen. **39**, L529 (2006).
18. U. Brinch-Nielsen, I.G. Jonsson, Wave Motion **8**, 455 (1986).
19. Yu.V. Sedletsky, JETP **97**, 180 (2003).
20. V.E. Zakharov, Amer. Math. Soc. Transl. **182**, N 2, 167 (1998).
21. V.E. Zakharov, Eur. J. Mech. B/Fluids **18**, 327 (1999).
22. V.P. Krasitskii, J. Fluid Mech. **272**, 1 (1994).
23. V.P. Krasitskii and V.A. Kalmykov, Izv. Atmos. Ocean. Phys. **29**, 222 (1993).
24. H.C. Yuen and B.M. Lake, *Nonlinear Dynamics of Deep-Water Gravity Waves* (Academic Press, New York, 1982).
25. M. Stiassnie and L. Shemer, J. Fluid Mech. **143**, 47 (1984).
26. R.Q. Lin and W. Rerrie, J. Phys. Oceanogr. **27**, 1813, 2169 (1997).
27. S.Y. Annenkov and V.I. Shrira, Phys. Rev. Lett **96**, 204501 (2006).
28. P.A.E.M. Janssen, *The Interaction of Ocean Waves and Wind* (Cambridge University Press, Cambridge, 2004).
29. P.A.E.M. Janssen and M. Onorato, Tech. Memorandum N0. 464, European Centre for Medium-Range Weather Forecasts (2005).

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РІВНЯННЯ ЗАХАРОВА З НУЛЬОВОЮ ГАРМОНІКОЮ І МОДУЛЯЦІЙНА НЕСТІЙКІСТЬ

Ю.В. Седлецький

Р е з ю м е

Показано, що новий тип модуляційної нестійкості хвиль на поверхні ідеальної рідини, передбачений нещодавно з системи двох рівнянь руху, для амплітуди обвідної основної гармоніки і неосциллюючої компоненти хвилі (нульової гармоніки) в рамках методу багатьох масштабів і ейлерових рівнянь руху можно також описати, виходячи з системи двох рівнянь Захарова для фур'є-амлітуд першої і нульової гармонік, на основі гамільтонівського формалізму.